Proofs of Robust Distributed Estimation for Localization in Lossy SN

N. Bof * **M. Todescato** * **R. Carli** * **L. Schenato** *

* *Department of Information Engineering, Padova, 35131, Italy.*
(e-mail: bofnicol|todescat|carlirug|schenato @ dei.unipd.it).

Abstract: In this paper we address the problem of fault resilient estimation for large-scale systems, where the measurements are possibly corrupted due to low-cost sensors faults. As toy application, we consider the problem of localization in Sensor Networks (SN). We propose a distributed solution based on a recently developed generalized descent algorithm. To cope with real-world applications, the algorithm we propose is suitable for an asynchronous implementation and is numerically robust to non ideal communications, i.e., packet-losses. Under mild assumptions, theoretical convergence of the algorithm is shown. The algorithm is compared with a recently developed ADMM-based algorithm for robust state estimation.

Keywords: Multi-agent systems, Robust Estimation, Robust Statistics, Asynchronous Broadcast, Distributed Convex Optimization, Sensor Networks, WSN

1. INTRODUCTION

Nowadays, large-scale and distributed cyber-physical systems, consisting of a multitude of sensors and "smart agents" equipped with mild computational, communication and actuation capabilities, permeate our lives. Because of the size of the systems, low-cost sensors are tipically used. However the latter are more prone to random failures, and consequently, one important challenge to face is the systematic quantitative monitoring of the system. Indeed, by affecting the collected measurements, these failures eventually compromise the knowledge of the system' state, usually used for management and control. In order to avoid this issue, two strategies can be followed: (i) the development of suitable fault diagnosis algorithms (see Paradis and Han (2007) for a survey on the topic), consisting in detection, isolation and identification of the fault; (ii) the design of fault resilient state estimation procedures which are able to produce accurate outcomes by automatically filtering out the outliers. These two approaches, which may eventually complement each other, become necessary to implement reliable systems. However, the possibly large scale of these systems makes central monitoring strategies difficult and sometimes impossible to implement. Thus, distributed solutions must be addressed.

Fault detection and bad data analysis have been largely studied in the past. A lot of work has been done on the static analysis of faults. The main idea behind static analysis is to process the measurements residuals through suitable hypothesis tests in order to detect the source of the fault. In Chen et al. (2006) a distributed belief propagation approach is proposed for WSN. With specific applications to electrical power systems, in Korres (2011) a distributed bad data analysis and detection procedure is shown, which is based on the normalized residual test. Choi and Xie (2011) propose a reduced model for distributed wide area monitoring and a bad data analysis based on the χ^2-test. A more recent branch of research regards the development of fault diagnosis strategies for general networks of dynamical systems using sensors networks. In Franco et al. (2006) a distributed hypothesis testing method, based on a belief consensus technique to perform fault diagnosis, is presented. Consensus is exploited in Boem et al. (2011) as well, where the authors propose a distributed strategy which is based on the combination of local fault estimators to reach a common agreement on the fault detection. More recently, Boem et al. (2013) propose a method based on Pareto optimization. Finally, in Keliris et al. (2015) the authors present a distributed scheme for the detection of process and sensors faults for a certain class of nonlinear discrete-time systems.

Regarding distributed state estimation, a vast amount of literature can be found. However, historically, state estimation does not deal with the presence of outliers. In order to deal with bad data analysis, the standard approach consists of two iterative steps: first, state estimation is performed; second, hypothesis tests on the measurements residuals are applied as done in Korres (2011); Choi and Xie (2011). If a bad datum is detected, this is deleted from the data-set and state estimation is performed again. Hypothesis test on the new residuals can confirm or belie the detection. In this sense, this approach iteratively combines standard state estimation with static fault detection procedures, to eventually lead to a fault resilient state estimator.
A different approach is followed in Kekatos and Giannakis (2013), where the authors propose an iterative distributed strategy based on the classical ADMM algorithm to simultaneously solve the state estimation and the fault localization in power systems.

In this work we are interested in developing a fault resilient state estimator rather than a fault detection scheme. Conversely to what is done in Kekatos and Giannakis (2013), where the problem is solved using a least square approach with the introduction of an additional variable to take into account the presence of outliers, we exploit ideas coming from robust statistical analysis (Bloomfield and Steiger, 2012; Huber, 2011) to formulate a suitable convex problem. In particular the choice of a "1−norm"-based cost function let us automatically filter out potential outliers in the measurements caused by sensors faults. Inspired by the recent result in Todescato et al. (2015), we provide a distributed algorithm to solve the problem. Starting from a synchronous algorithm which assumes

perfect and ideal communications among sensor nodes, we modify it to deal with communication non idealities. This is an important aspect since, in real-world large-scale systems, ideal synchronous communications are not likely. The algorithm we propose is based on an asynchronous broadcast communication protocol. Numerically, the algorithm is shown to be robust to communication non-idealities. Under additional mild assumptions on the type of communication non-idealities and on the curvature of our prescribed cost function, convergence of the algorithm is theoretically proven.

We apply the proposed algorithm in the framework of sensors networks localization, even if the strategy applies to a more general setup. Because of the well known performance of the ADMM algorithm, we decide to compare the algorithm with the strategy recently proposed in Kekatos and Giannakis (2013). Since neither asynchronous nor robust implementation of the algorithm in Kekatos and Giannakis (2013) is provided, we suggest one. As shown by the numerical simulations, compared to the ADMM, our robust algorithm has the following features: (i) comparable steady state estimation accuracy; (ii) in scenarios of highly connected graphs, the algorithm is characterized by a faster behavior for both the asymptotic and the transient convergence rate; (iii) in general, conversely to the ADMM, the transient evolution of our algorithm is monotonically decreasing. Finally, from a theoretical point of view, (iv) the algorithm is provably convergent. To the best of our knowledge, a similar robust analysis for the ADMM is not available.

1.1 Mathematical Preliminaries

In this paper, $\mathcal{G}(\mathcal{V},\mathcal{E})$ denotes a directed graph, where $\mathcal{V}=\{1,\dots,N\}$ is the set of vertices and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the set of directed edges. More precisely the edge (i,j) is incident on node i and node j and is assumed to be directed away from i and directed toward j. The graph $\mathcal{G}$ is said to be bidirected if $(i,j)\in\mathcal{E}$ implies $(j,i)\in\mathcal{E}$. Given a directed graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, a directed path in $\mathcal{G}$ consists of a sequence of vertices $(i_1,i_2,\dots,i_r)$ such that $(i_j,i_j+1)\in\mathcal{E}$ for every $j\in\{1,\dots,r-1\}$. An undirected path in $\mathcal{G}$ consists of a sequence of vertices $(i_1,i_2,\dots,i_r)$ such that either $(i_j,i_{j+1})\in\mathcal{E}$ or $(i_{j+1},i_j)\in\mathcal{E}$ for every $j\in\{1,\dots,r-1\}$. The directed (resp. bidirected) graph $\mathcal{G}$ is said to be *strongly connected* (resp. *connected*) if for any pair of vertices (i,j) there exists a directed path (resp. an undirected path) connecting i to j. Given the directed graph $\mathcal{G}$, the set of neighbors of node i, denoted by $\mathcal{N}_i$, is given by $\mathcal{N}_i=\{j\in\mathcal{V}\,|\,(i,j)\in\mathcal{E}\}$. Moreover, $\mathcal{N}_i^+=\mathcal{N}_i\cup\{i\}$. Given a directed graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ with $|\mathcal{E}|=M$ let the *incidence matrix* $\mathcal{A}\in\mathbb{R}^{M\times N}$ of $\mathcal{G}$ be defined as $\mathcal{A}=[a_{ei}]$, where $a_{ei}=1,-1,0$, if edge e is incident on node i and directed away from it, is incident on node i and directed toward it, or is not incident on node i, respectively. Given a vector v with v^T we denote its transpose. The symbol I denotes the identity matrix of suitable dimension. Given a vector v, $diag(v)$ represents the diagonal matrix where the i-th diagonal element corresponds to the i-th element of the vector v.

2. PROBLEM FORMULATION

In the following, we consider a *localization-type* problem in Sensors Networks (Mao et al., 2007) where, starting from a set of noisy measurements, the agents' goal is to estimate their absolute positions. We want to develop a distributed strategy where the agents are allowed to exchange information locally, i.e., between neighbors. Moreover, for real-world applications, the algorithm must be robust to communication non idealities, e.g., packet dropouts, while being resilient to faulty measurements due to possible sensors failures.

Consider a set of N agents/sensors, labeled as $\mathcal{V}=\{1,\dots,N\}$, where each agent is described by a state vector $x_i\in\mathbb{R}^{n_i}$. For our purpose and for ease of notation, we restrict the analysis to the scalar case where $n_i=1,\ \forall i\in\mathcal{V}$.[1] By exploiting graph theoretical tools, we model the SN by means of a bidirected connected measurement graph $\mathcal{G}(\mathcal{V},\mathcal{E})$.

In the following we introduce the measurement model used and we formally state the problem at hand.

2.1 Measurement Model & Fault Resilient Estimation

Assume that each agent collects a certain number of measurements according to the measurement graph $\mathcal{G}$. More specifically, only two types of measurements can be collected. The first are noisy relative distance measurements with respect to neighboring agents, that is, for each $i\in\mathcal{V}$ and $j\in\mathcal{N}_i$, node i measures

$$b_{ij}=x_i-x_j+n_{ij}\,,\quad n_{ij}\sim\mathcal{N}(0,\sigma_{ij}^2)\ .$$

where σ_{ij}^2 denotes the relative measurement noise variance. The second type of measurements is a noisy absolute measurement of the form

$$b_i=x_i+n_i\,,\quad n_i\sim\mathcal{N}(0,\sigma_i^2)\ .$$

where σ_i^2 is the absolute measurement noise variance. By collecting all the state variables in the vector $\mathbf{x}:=[x_1,\dots,x_N]^T$ and by defining the measurement matrix H and the vectors of measurements, $\mathbf{b}$, and noises, $\mathbf{n}$, respectively as

$$H:=\begin{bmatrix}I\\ \mathcal{A}\end{bmatrix},\quad \mathbf{b}:=\begin{bmatrix}\{b_i\}_{i\in\mathcal{V}}\\ \{b_{ij}\}_{(i,j)\in\mathcal{E}}\end{bmatrix},\quad \mathbf{n}:=\begin{bmatrix}\{n_i\}_{i\in\mathcal{V}}\\ \{n_{ij}\}_{(i,j)\in\mathcal{E}}\end{bmatrix},$$

the overall measurement model[2] can be rewritten in compact form as

$$\mathbf{b}=H\mathbf{x}+\mathbf{n}\,,\quad \mathbf{n}\sim\mathcal{N}(0,R)\,, \tag{1}$$

where $R:=diag(\{\sigma_i^2\}_{i\in\mathcal{V}}\,,\,\{\sigma_{ij}^2\}_{(i,j)\in\mathcal{E}})$ denotes the noise variance matrix.

In presence of outliers, however, some of the measurements can be corrupted by an extra term, which has a probability distribution that highly differs from that of the expected gaussian noise. By collecting these outliers in the sparse vector $\mathbf{o}$, the measurement model (1) becomes

$$\mathbf{b}=H\mathbf{x}+\mathbf{n}+\mathbf{o}. \tag{2}$$

As above mentioned, we are willing to design a distributed state estimation procedure which is fault resilient, that is which is able to produce a reliable estimation by automatically filtering out the outliers. Conversely to classical least squares estimation where the objective is to minimize the weighted squared norm of the residuals, here we follow an approach which is inspired from robust statistical analysis (Bloomfield and Steiger, 2012; Huber, 2011), i.e., *least absolute estimation*. The main idea is to make use of suitable convex costs which, differently to the classical

[1] The more general case of multidimensional positions can be easily derived assuming independent measurements along each dimension.

[2] We underline the fact that we do not require all the nodes to collect absolute positioning measurements. However, for absolute positioning we require that at least one agents measures it. Conversely, only relative localization is performed.

quadratic costs, are locally quadratic only around the origin while they become linear away from it. Thanks to this, small residuals are weighted quadratically as in the classical least squares. On the contrary, big residuals, which usually identify the presence of sensors faults, are weighted linearly. Consequently, the estimator weights and "trusts" more the measurements corresponding to small residuals. Precisely, we consider a modified 1-norm defined as (Argaez et al., 2011)

$$\|\cdot\|_{1,\epsilon} : \mathbb{R}^n \mapsto \mathbb{R} : x \mapsto \|x\|_{1,\epsilon} := \sum_{i=1}^{n} \sqrt{x_i^2 + \epsilon}\,, \qquad (3)$$

where the parameter ϵ is used to tune the point where the function changes its behavior from quadratic to linear. Observe that, conversely to the standard 1-norm, $\|\cdot\|_{1,\epsilon}$ is differentiable and thus suitable to be used in gradient based approaches.

Remark 1. (On the choice of ϵ). The parameter ϵ regulates the switching behavior of the cost function from quadratic to linear and its value must be chosen in order to suitably filter out undesirable outliers. To do so, one possible choice, is to pick ϵ such that $\|\cdot\|_{1,\epsilon}$ behaves quadratically for values of residuals within $[-\alpha\sigma, +\alpha\sigma]$, $\alpha \in [0, +\infty)$, and linearly outside, being σ the standard deviation of the noise prior distribution.

By making use of the modified 1-norm (3), we are now ready to formulate our problem of interest. This reads as

$$\mathcal{P}_1 : \min_{\mathbf{x}\in\mathbb{R}^N} \underbrace{\|(\mathbf{b} - H\mathbf{x})\|_{1,\epsilon}}_{J(\mathbf{x})}. \qquad (4)$$

Observe that, in order to solve $\mathcal{P}_1$ efficient convex solvers might be used. However, these require global knowledge of the network model as well as of the measurements.

Before presenting the proposed procedure which is inspired on recent results in Todescato et al. (2015), in the next section we briefly recall the particular communication architecture exploited.

2.2 Partition-based Communication Architecture

We assume the network is partitioned into p non-overlapping areas A_h, where $h \in \mathcal{V}_c := \{1, \dots, p\}$. Each area contains a certain number s_h of agents $i \in \mathcal{V}$ and is monitored by a local master node which has complete knowledge of the intra-area communications and measurement model. This node collects all the intra-area states, can collect all the intra-area measurements, can process the data, and can communicate with the master nodes in charge to monitor the neighboring areas. According to this partitioning, it is possible to define a new bidirected communication graph $\mathcal{G}_c(\mathcal{V}_c, \mathcal{E}_c)$, where the edge set $\mathcal{E}_c \subseteq \mathcal{V}_c \times \mathcal{V}_c$ consists of the pairs (h, k) for which there exist agent $i \in A_h$ and agent $j \in A_k$ such that $(i, j) \in \mathcal{E}$. Moreover, it is possible to block partition the state, the measurement, the noise vectors as well as the measurement matrix as

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_p \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_p \end{bmatrix}, \ \mathbf{n} = \begin{bmatrix} \mathbf{n}_1 \\ \vdots \\ \mathbf{n}_p \end{bmatrix}, \ H = \begin{bmatrix} H_{11} & \cdots & H_{1p} \\ & \ddots & \\ H_{p1} & \cdots & H_{pp} \end{bmatrix},$$

where, for $h \in \mathcal{V}_c$, $\mathbf{x}_h \in \mathbb{R}^{s_h}$ is the intra-area state vector, while $\mathbf{b}_h, \mathbf{n}_h \in \mathbb{R}^{m_h}$ ($m_h = \sum_{i\in A_h} |\mathcal{N}_i^+|$) contain the measurements and noises of the agents owning to area A_h. Similarly, the block $H_{hk} \in \mathbb{R}^{m_h \times s_k}$ consists of the elements of the matrix H connecting the agents contained in area A_h with those contained in area A_k. Observe that $H_{hk} \neq 0$ if and only if $(h, k) \in \mathcal{E}_c$. Finally, the noise variance matrix becomes $R = blkdiag(R_1, \dots, R_p)$, $R_h \in \mathbb{R}^{m_h \times m_h}$.

This area partitioning is extremely flexible in large-scale systems where, for privacy issues, master nodes are built to have access only to intra-areas knowledge while they can perform inter-area communications. Observe that, according to the partition-based architecture, it is possible to rewrite Problem $\mathcal{P}_1$, highlights the separability structure of the cost function which is now written as sum of "local" costs, as

$$\mathcal{P}_1 : \min_{\mathbf{x}_1,\dots,\mathbf{x}_p} \sum_{h\in\mathcal{V}_c} \underbrace{\|\mathbf{b}_h - \sum_{k\in\mathcal{N}_h^+} H_{hk}\mathbf{x}_k\|_{1,\epsilon}}_{J_h(\mathbf{x}_h, \{\mathbf{x}_k\}_{k\in\mathcal{N}_h})}. \qquad (5)$$

3. DISTRIBUTED SOLUTIONS

3.1 Synchronous Fault Resilient Estimation

Here, we present a distributed solution for Problem (5) which exploits the partition-based communciation architecture of Section 2.2. This algorithm represents the starting point for an asynchronous robust version which will be presented in the next section. The proposed algorithm, which we refer to as *Distributed Fault Resilient Estimation* algorithm (hereafter denoted as DFRE), is inspired on the *Blok Jacobi Algorithm* recently developed in Todescato et al. (2015). The algorithm is essentially a generalized gradient descent of the type

$$\mathbf{x}(t+1) = \mathbf{x}(t) - \rho D^{-1}(t)\nabla J(t)\,,$$

where, at iteration t, $\nabla J(t)$ is the gradient of the cost function evaluated in the current state estimate $\mathbf{x}(t)$, $D(t)$ is a block diagonal matrix which is used to accelerate the convergence of the algorithm and ρ is the gradient step size. In particular, from the cost in (5), we have that the h-th block of the gradient vector, computed by the master node of area A_h is equal to

$$[\nabla J(t)]_h = -\sum_{k\in\mathcal{N}_h^+} \underbrace{H_{kh}^T \left((diag(\mathbf{z}_k(t)))^2 + \epsilon I \right)^{-1/2} \mathbf{z}_k(t)}_{\mathbf{g}_k(t)}\,, \qquad (6)$$

where $\mathbf{z}_h(t) := \mathbf{b}_h - \sum_{k\in\mathcal{N}_h^+} H_{hk}\mathbf{x}_k(t)$ represents the vector of current estimation residuals. Observe that to compute $\mathbf{z}_h(t)$, the master node h has to receive the states $\mathbf{x}_k$ from nodes $k \in \mathcal{N}_h$, and then to compute (6) it needs to receive the vector $\mathbf{g}_k(t)$ from $k \in \mathcal{N}_h$. Regarding the $D(t)$ matrix, as done in Todescato et al. (2015), the idea is to use the second order information of the cost. In particular, we set the h-th diagonal block of $D(t)$ equal to the h-th diagonal block of the cost Hessian. In the aforementioned paper, the adoption of a quadratic cost function led to a constant second derivative term, that is $D(t) = \bar{D}\,\forall t \geq 0$. However, in our case, $D(t)$ changes over time and is state dependent. In particular, the h-th block of $D(t)$ is equal to

$$D_h(t) = \epsilon H_{hh}^T \left((diag(\mathbf{z}_h(t)))^2 + \epsilon I \right)^{-3/2} H_{hh}\,, \qquad (7)$$

which can be computed by the master node h using only intra-area information. Finally, each master node can update its current estimate as

$$\mathbf{x}_h(t+1) = \mathbf{x}_h(t) - \rho D_h^{-1}(t)[\nabla J(t)]_h\,. \qquad (8)$$

Algorithm 1 formally describes the DRFE algorithm. Note that, in order to compute the $\mathbf{g}$'s and update the $\mathbf{x}$'s, Algorithm 1 requires two communication rounds per iteration.

Algorithm 1 DFRE

for $t \in \mathbb{N}$ each $h \in \mathcal{V}_c$ **do**
  sends $\mathbf{x}_h(t-1)$ to $k \in \mathcal{N}_h$;
  receives $\mathbf{x}_k(t-1)$ from $j \in \mathcal{N}_k$;
  computes $\mathbf{g}_h(t)$ as defined in (6);
  sends $\mathbf{g}_h(t)$ to $k \in \mathcal{N}_h$;
  receives $\mathbf{g}_k(t)$ from $k \in \mathcal{N}_k$;
  computes $\mathbf{x}_h(t)$ by using (8);
end for

Remark 2. (On the gradient computation).
Observe that to compute the gradient and, in particular, $\mathbf{z}_h$, $h \in \mathcal{V}_c$, the exchange of $\{\mathbf{x}_k\}_{k\in\mathcal{N}_h}$ among neighbors is required. However, it is worth noticing that only the entries of $\mathbf{x}_k$ corresponding to columns of H_{hk} which are different from the vector of all zeros, are needed. Consequently, the information exchange can be reduced.

Remark 3. (On the number of communication rounds).
Observe that in the particular case of SN localization, the communication rounds per iteration needed to implement Algorithm 1 can be reduced to one. Indeed, by assuming that at the beginning of the algorithm neighboring areas exchange their inter-area relative measurements then, only the communication of the states $\mathbf{x}$'s is required. This is because in localization-type problems the measurement matrix has a particular structure being composed only of entries equal to 1 and -1. However, for the sake of completeness we presented the more general case where we assume neither model nor measurements exchange among neighbors. Moreover, in Section 5 we compare the DRFE algorithm with the ADMM-based algorithm proposed in Kekatos and Giannakis (2013) where no measurements exchange is implemented.

3.2 Asynchronous, Robust, Fault Resilient Estimation

Algorithm 1 is designed for the scenario of synchronous ideal communications where neither packet losses nor delays occur. Here, inspired on the *Robust Block Jacobi* algorithm proposed in Todescato et al. (2015), we eventually generalize Algorithm 1 to the case of asynchronous and non ideal lossy communications. We refer to this modified version as *Robust Distributed Fault Resilient Estimation* algorithm (denoted hereafter as r-DFRE). In particular, we exploit an *asynchronous broadcast communication protocol* where one master node, say $h \in \mathcal{V}_c$, wakes up, updates its variables and, at the end of the computations, sends them to all its neighbors. Observe that the protocol requires only one communication round per iteration since, conversely to DFRE, we can send at the same time the state $\mathbf{x}_h$ and its gradient-related variable $\mathbf{g}_h$. However, in order to perform its local updates, it is necessary that each node stores in its local memory the following variables:

- $\mathbf{x}_h$: estimate of its state;
- $\mathbf{x}_k^{(h)}$, $k \in \mathcal{N}_h$: local estimate of the state of the neighboring area k (note that $\mathbf{x}_h^{(h)} \equiv \mathbf{x}_h$);
- $\mathbf{g}_h$: gradient-related variable regarding intra-area h information;
- $\mathbf{g}_k^{(h)}$, $k \in \mathcal{N}_h$: gradient-related variable regarding intra-area k information (note that $\mathbf{g}_h^{(h)} \equiv \mathbf{g}_h$);

Thanks to this additional memory not only we can use an asynchronous communication protocol, but we are also able to deal with packet losses in the communication. The latter can be conveniently modeled using the indicator function

$$\gamma_k^{(h)}(t) = \begin{cases} 1 \text{ if } h \text{ received the packet from } k \\ 0 \text{ otherwise} \end{cases}$$

as done in Todescato et al. (2015). Indeed, if node $k \in \mathcal{N}_h$ does not receive the packet that node h sends to it, then it simply does not update its memory. Namely,

$$\begin{aligned} \mathbf{x}_h^{(k)}(t) &= \begin{cases} \mathbf{x}_h(t) & \text{if } \gamma_h^{(k)} = 1 \\ \mathbf{x}_h^{(k)}(t-1) & \text{otherwise} \end{cases} \\ \mathbf{g}_h^{(k)}(t) &= \begin{cases} \mathbf{g}_h(t) & \text{if } \gamma_h^{(k)} = 1 \\ \mathbf{g}_h^{(k)}(t-1) & \text{otherwise} \end{cases} \end{aligned} \tag{9}$$

Consequently, when node k will wake up, it will use information regarding node h which are possibly out of date. Specifically Eq.(6) becomes

$$[\nabla J(t)]_h = -\sum_{k\in\mathcal{N}_h^+} \mathbf{g}_k^{(h)}(t)\,. \tag{10}$$

This algorithm, which uses memory, is our r-DFRE algorithm, whose formal description can be found in Algorithm 2.

Algorithm 2 r-DFRE

for $t \in \mathbb{N}$ **do**
  assume node $h \in \mathcal{V}_c$ wakes up;
  using (7) and (10), updates $\mathbf{x}_h(t)$ as in (8);
  computes $\mathbf{g}_h(t)$;
  sends $\mathbf{x}_h(t), \mathbf{g}_h(t)$ to $k \in \mathcal{N}_h$;
  if $\gamma_h^{(k)}(t) = 1$ **then** node k
    receives $\mathbf{x}_h(t)$ and $\mathbf{g}_h(t)$;
    updates $\mathbf{x}_h^{(k)}(t)$ and $\mathbf{g}_h^{(k)}(t)$ as in (9);
  end if
end for

3.3 Convergence Analysis of the r-DFRE Alogirthm

Here we present a convergence result regarding the r-DFRE algorithm described in Algorithm 2. In order to state our result, which is based on the convergence analysis of *partially asynchronous algorithms* provided in Bertsekas and Tsitsiklis (1989), we introduce the following assumptions characterizing the type of communications non idealities.

Assumption 4. (Persistent activation).
There exists a positive integer D such that each master node $h \in \mathcal{V}_c$ wakes up and performs its update at least once within any interval $[t, t+D]$.

Assumption 5. (Bounded packet-losses).
There exists a positive integer L such that the number of consecutive communication failures between every pair of neighboring nodes in the communication graph $\mathcal{G}_c$ is less than L.

Finally, we need a technical assumption on the curvature of the prescribed cost function which force us to consider a slightly modified cost.

Assumption 6. (Curvature of Cost Function).
Consider the following modified version of $\|\cdot\|_{1,\epsilon}$

$$[\|x\|_{1,\epsilon}]_i := \begin{cases} \sum_i \sqrt{x_i^2 + \epsilon} & \text{if } |x_i| \le \overline{x} \\ \alpha x_i^2 + \gamma & \text{if } |x_i| > \overline{x} \end{cases}$$

where $\overline{x}$ is a "saturation" point and the parameter α and γ can be used to extend twice continuous differentiability of the cost in correspondence of $\overline{x}$.

Basically, Assumption 6 ensures that the curvature, i.e., the second derivative of the cost function, is bounded below by a strictly positive value. Observe that, from a practical point of view, the saturation value $\overline{x}$ can be any arbitrarily large finite value. Thus, this modification does not practically influence the r-DFRE algorithm.
The next proposition characterizes the convergence [3] of Algorithm 2.

Proposition 7. (Convergence of r-DFRE algorithm).
Consider Problem 5 and the r-DFRE Algorithm 2. Let Assumptions 4 and 5 and the technical Assumption 6 hold. Then, there exists $\overline{\rho}$ such that, for all $0 < \rho < \overline{\rho}$, it holds $\lim_{t\to\infty} \nabla J(x(t)) = 0$.

The proof can be found in Appendix A.

4. ADMM ALGORITHM

Here, we briefly introduce the distributed ADMM-based algorithm proposed in Kekatos and Giannakis (2013) (which, for brevity, in this paper we refer to as ADMM algorithm) to perform state estimation in smart electric grid in the presence of possible faulty measurements. The idea behind the proposed algorithm is that of simultaneously estimating the state vector as well as a sparse vector of outliers, which is used to perform fault resilient estimation and fault detection. More specifically, assuming a measurement model of the form (2), the aim is to solve

$$\mathcal{P}_2 \ : \ \min_{\mathbf{x},\mathbf{o}} \frac{1}{2}\|\mathbf{b} - H\mathbf{x} - \mathbf{o}\|_2^2 + \lambda\|\mathbf{o}\|_1,$$

where $\mathbf{o}$ is the vector of outliers. The $1-$norm, used to regularize the problem, serves as convex approximation of the $0-$norm, i.e., the cardinality function and it is used to enforce sparsity of $\mathbf{o}$. Finally λ is a user-defined parameter used to tune the regularization.
As for the cost function of problem $\mathcal{P}_1$, also the one of $\mathcal{P}_2$ can be decomposed exploiting a partition-based communication architecture similar to the one of Section 2.2. This architecture allows to solve the problem in a distributed fashion using the ADMM algorithm. We refer the interested reader to Kekatos and Giannakis (2013) for an exhaustive explanation.
Interestingly, the ADMM algorithm simultaneously returns an estimate of the state as well as of the vector of outliers. The latter can be used to perform fault detection. However, similar to the DFRE but conversely to the r-DFRE, the ADMM algorithm assumes synchronous and reliable communications and the authors do not offer any robust implementation in the presence of communication non idealities. In order to compare the proposed r-DFRE algorithm with the ADMM, we implement a possible extension of the algorithm in Kekatos and Giannakis (2013) which can be exploited in the case of non ideal communications. In particular we assume an asynchronous broadcast communication protocol and we assume that each master node can store in memory the last received information from its neighbors, similarly to what is done in Section 3.2.

5. SIMULATIONS

In this section we compare the r-DFRE algorithm with a "robustified" version of ADMM. We consider a one dimensional environment of length $L = 200$[m] equally partitioned in $p = 20$ areas, each of them supervised by a master node. In each area the number of agents varies from 5 to 10. Inside each area, we assume the nodes are positioned and connected according to a bidirected *random geometric* measurement graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with connectivity radius r. Moreover we assume that each node is able to measure its absolute position as well.
Regarding the inter-area connections, since we are working on a line, we assume for $h \in \{2, \dots, p-1\}$ that $\mathcal{N}_h = \{h-1, h+1\}$, while $\mathcal{N}_1 = \{2\}$ and $\mathcal{N}_p = \{p-1\}$. In order to enforce connectivity among adjacent areas, for each pair of adjacent areas we take m relative measurements corresponding to the m closest pairs of nodes. The possibility to manually chose the value of m let us highlight some interesting behavior depending on the level of inter-area connectivity.
We assume the measurements are characterized by the same prior distribution. In particular, we set $\sigma_i = \sigma_{ij} = \sigma = 0.1$[m], $\forall i \in \mathcal{V}$, $\forall (i,j) \in \mathcal{E}$. Concerning the outliers, we assume that 10% of the measurements are corrupted by an additive noise, whose absolute amplitude is uniformly distributed in the range $[1.5, 2]$[m]. For consistency, in case the outlier corrupts a relative distance measure among agents of the same area, the measurement is saturated to the communication radius r characterizing the intra-areas random geometric graph $\mathcal{G}$. In case the outlier corrupts a relative inter-area measurement, the saturation is set to a value proportional to their communication radius.
Concerning ϵ, we have to take into account that, as above mentioned, its value controls the switching behavior of $\|\cdot\|_{1,\epsilon}$ from a quadratic to a linear trend. Thus, it influences the robustness of $\|\cdot\|_{1,\epsilon}$ to the presence of outliers and eventually the accuracy of the solution. Nevertheless, ϵ controls the smoothness of the cost and consequently, the rate of convergence of our proposed gradient-based algorithm. In the next simulations, we heuristically set $\epsilon = \frac{1}{4}\sigma^{8/3}$, which enforces the cost function $\|\cdot\|_{1,\epsilon}$ to behave quadratically within $[-\sigma, \sigma]$. Preliminary simulations showed that this value represents a good trade-off between accuracy and rate of convergence.
For the choice of λ, following Kekatos and Giannakis (2013), we set $\lambda = 1.34\sigma$. [4]

We compare the two algorithms in terms of averaged root mean squared error ($ARMSE$). Given M Monte Carlo runs for different graph realizations, denote with $\mathbf{x}_{\{i\}}(t)$ the estimate at time t given by one of the two algorithms in the $i-$th Monte Carlo run. Then, $ARMSE$ reads as

$$ARMSE(t) := \frac{1}{M}\sum_{i=1}^{M} RMSE(\mathbf{x}_{\{i\}}(t)) \ , \qquad (11)$$

where

$$RMSE(\mathbf{x}) := \frac{1}{\sqrt{N}}\|\mathbf{x} - \mathbf{x}_{\text{true}}\|$$

represents the root mean squared error between the true nodes positions, $\mathbf{x}_{\text{true}}$, and the estimate, $\mathbf{x}$. We first use ARMSE to compare the steady state accuracy of the estimates, solutions of problems $\mathcal{P}_1$ and $\mathcal{P}_2$. This corresponds to the evaluation of the $ARMSE$, computed using the estimates given by the r-DFRE and the ADMM, as $t \to \infty$, since the accuracy of the technique is given by the final value of the estimate. Moreover, we also use classical least squares estimation (LS) to verify if the estimation technique given by the solution of problem $\mathcal{P}_1$ is truly able

[3] Interestingly, assuming the presence of bounded delays in the communication, the r-DFRE algorithm remains provably convergent. Indeed, this is true since the presence of bounded delays, together with the specific broadcast communication protocol chosen, implies the persistent activation of each node.

[4] In Kekatos and Giannakis (2013), $\lambda = 1.34$ since the measurements are normalized by their standard deviation. Equivalently $\sigma = 1$.

$ARMSE(\infty)$ $\mathcal{P}_1$	$ARMSE(\infty)$ $\mathcal{P}_2$	$ARMSE(\infty)$ LS
0.0612	0.0594	0.2290

Table 1. Steady state estimation accuracy. Values of $ARMSE$, computed over $M = 1000$ Monte Carlo runs, for the solutions of problems $\mathcal{P}_1$, $\mathcal{P}_2$ and of LS, respectively, for fixed values of $r = 3$[m] and $m = 5$.

to deal with the presence of outliers. Table 1 allows us to carry out a comparison.
Note that LS returns completely inaccurate estimates due to the presence of the outliers. Conversely, the other two approaches are fault resilient and comparable, even though $\mathcal{P}_2$ gives, in general, slightly better results. Even if the results have been obtained for specific values of r and m, the same behavior holds for different values of the parameters.

Once verified the goodness of the solution given by problem $\mathcal{P}_1$, we compare the r-DFRE and the ADMM with respect to their convergence rate [5] . We recall that in order to effectively implement both the algorithms, a preliminary step concerns with the choice of the parameters ρ for the r-DFRE and c for the ADMM (which represents the penalty in the augmented Lagrangian). We follow the common approach of selecting the values $\rho = \rho^*$ and $c = c^*$, respectively, corresponding to the fastest asymptotic convergence rate [6] . This is done in order to minimize the number of iterations to converge toward to optimal solution of the corresponding problem.
Figure 1 shows the evolution of the ARMSE between the solution of the algorithms at time t, $\mathbf{x}(t)$, and the final solution of the respective problem in the upper panel, and the ARMSE between $\mathbf{x}(t)$ and the true positions $\mathbf{x}_{\text{true}}$ in the lower panel. In particular, for the r-DFRE, we report the evolution corresponding to $\rho = \rho^*$ while, for the ADMM, two different values of c are reported, namely, $c = c^*$ and $c = c^*/10$. This choice is due to the different convergence behavior of the ADMM. Indeed, from the upper panel of Figure 1 it can be seen that, by choosing $c = c^*$, the ADMM is characterized by an undesirable transient but converges to its corresponding optimum faster than the r-DFRE. Conversely, by choosing $c = c^*/10$, the ADMM shows a better transient but slows down its asymptotic behavior becoming even slower than the r-DFRE. However, by comparing the algorithms in terms of estimation accuracy with respect to the true positions, as shown in the lower panel of Figure 1, a different behavior emerges. In particular, for $c = c^*$ the ADMM is much slower than the r-DFRE to reach the same level of accuracy. Conversely, by choosing $c = c^*/10$, the ADMM and the r-DFRE are characterized by a similar behavior both in terms of transient and asymptotic evolution. A final remarkable feature of the r-DFRE compared to the ADMM, is that during the transient evolution, the evolution is always monotonically decreasing.
We recall that the possible discrepancy between transient and asymptotic behavior is a known fact, see Fagnani and Zampieri (2008). As highlighted by the previous analysis, this translates in a non trivial procedure to find the optimal value of c for the ADMM. Conversely, for the r-DFRE, the extensive simulations performed suggest the fact that the optimal ρ leads to an optimal transient and asymptotic behavior simultaneously.
As final remark, notice that for both the algorithms the number of iterations required to reach a good estimate of the agents' positions is quite large. However, it is necessary to take into account that, in the asynchronous broadcast communication protocol we are considering, at each time instant only one (randomly chosen) node is activated, so the number of iterations to achieve a particular precision should be averaged over the number of master node in the network.

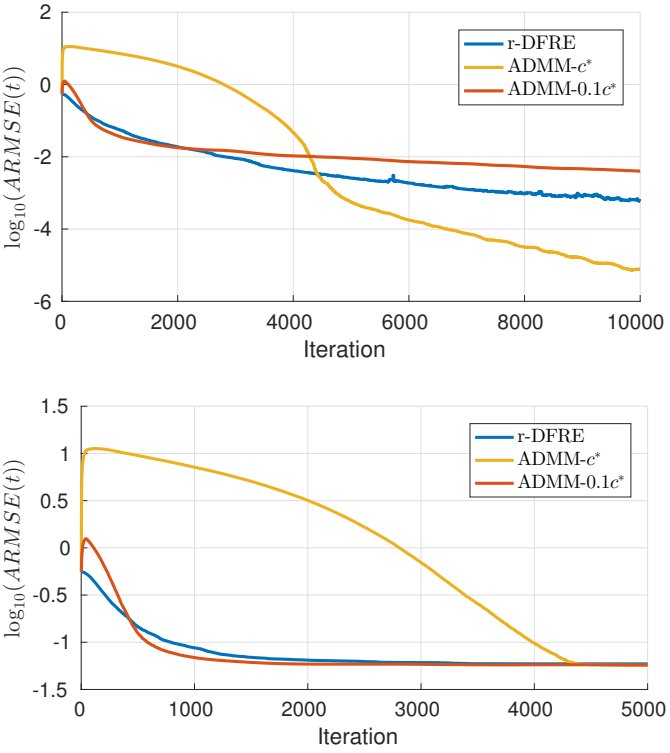

Fig. 1. Upper panel: ARMSE, over $M = 100$ Monte Carlo runs, with respect to the optimal solution of the respective problem. Lower panel: ARMSE with respect to the true positions. The parameter used for the simulations are $m = 5$ and $r = 3$ [m].

To highlight the dependency of the algorithms on the sensors network connectivity, we perform simulations for different values of inter-area connections m and intra-area communication radius r. Each panel of Table 2 shows, for a given number of inter area connections m, the number of iterations needed to reach the 95% accuracy from the optimal solution of the corresponding problem for increasing values of the communication radius r. Observe that, for low m and increasing r (Table 2 upper panel), the r-DFRE highly improves, the ADMM corresponding to $c = c^*/10$ slightly improves while for $c = c^*$ moderately degrades. For a larger value of m (Table 2 lower panel), for both c's the ADMM deteriorates, while the r-DFRE keeps improving for increasing r. The analysis just performed suggests that the connectivity of the underlying graph affects the behavior of the algorithm and in general it is a quantity that must be taken into account. Remarkably, even if in a totally different scenario, the connection between highly connected graphs and deteriorating performance of ADMM has been shown in the recent Bof et al. (2016) as well.

Finally, Figure 2 shows a simulation in the presence of packet losses. In particular a packet loss probability of 50% is chosen. The plot shows that both algorithms are robust to packet losses. However, as stated in Proposition 7, convergence of the r-DFRE is theoretically proven.

[5] In all these simulations, both algorithms are initialized to the same initial conditions, which correspond to the absolute measurements.
[6] To given a quantitative idea, in all the simulation performed, the order of magnitude of ρ^* and c^* are -4 and -2, respectively.

m = 5	$r = 3$[m]	$r = 8$[m]
r-DFRE	2593	811
ADMM $0.1c^*$	1643	1526
ADMM c^*	4361	4896

m = 10	$r = 3$[m]	$r = 8$[m]
r-DFRE	2186	933
ADMM $0.1c^*$	2130	2673
ADMM c^*	5777	6278

Table 2. Number of iterations, averaged over $M = 100$ runs, required to reach a 95% accuracy from the optimal solution for increasing intra-area communication radius r and for different values of inter-area connections m.

Conversely, a similar rigorous result for the ADMM is not available.

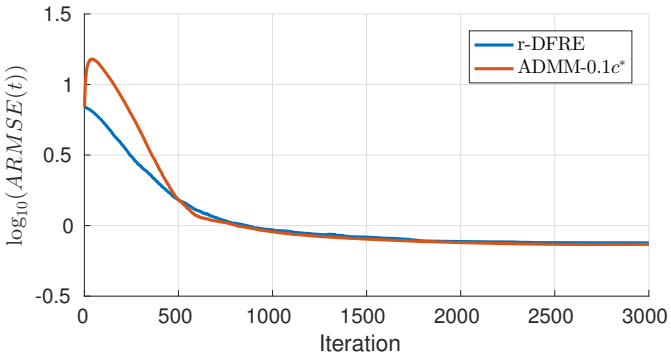

Fig. 2. ARMSE, averaged over $M = 100$ runs, between the estimate of the positions of the agents and their true positions, with a packet loss probability of 50%. The parameter used for the simulations are $m = 5$ and $r = 3$ [m].

6. CONCLUSIONS AND FUTURE DIRECTIONS

In this work we presented an asynchronous and robust algorithm which can be used to perform fault resilient estimation in presence of outliers. We applied it to a localization-type problem and showed that its results, in terms of estimation accuracy and convergence behavior, are comparable with those of a (slightly modified) ADMM algorithm recently proposed by Kekatos and Giannakis (2013) to perform estimation in presence of corrupted measurements. Through simulations, we showed that, for the ADMM, different parameter choices might lead to completely opposite convergence behaviors. Conversely, the proposed solution behaves consistently during the transient and the steady state evolution, thus alleviating the parameter tuning phase. Moreover, we showed that the graph connectivity can play a fundamental role in the convergence behavior of the algorithms. Finally, we observed that both algorithms give good results even in presence of packet losses. However, under mild assumptions, the proposed solution is thoretically provably convergent while a similar result for the ADMM is not available.

As future directions, we are interested in applying the r-DFRE to power system state estimation, and to better characterize its performance in presence of packet losses and communication non idealities. Another interesting research avenue regards a better understanding of the parameters tuning phase and their dependency on the underlying graph related quantities. Finally, in this paper we did not considered fault detection. As future work, therefore, we will study how to implement a distributed fault detector rather than only a distributed fault resilient estimator.

Appendix A. PROOF OF PROPOSITION 7

To prove Proposition 7 we leverage tools for the class of *partially asynchronous algorithms*. In particular, we resort to Proposition 5.2 of Bertsekas and Tsitsiklis (1989) about the convergence of the generalized block-gradient algorithm, which we recall next with all the necessary assumptions

Proposition 8. (Convergence of the block-gradient). Consider the generalized block-gradient algorithm

$$x_i(t+1) = x_i(t) + \rho s_i(t)\,.$$

where x_i represents the state of the i-th processor, ρ is a step size and $s_i(t)$ is the prescribed descent direction.

Let $J : \mathbb{R}^n \mapsto \mathbb{R}$ be the cost function to be minimized satisfying:

(i) J is twice continuously differentiable;
(ii) $J(x) \geq 0$, $\forall x \in \mathbb{R}^n$;
(iii) $\nabla J(x)$ is Lipschitz continuous with constant L.

Let T^i be the set containing the instants when the variable x_i is updated. Moreover, let $\tau_j^i(t)$ be the most recent time instant when processor i has received information from processor j. Assume that there exists a positive integer B such that:

(iv) for every $i \in \{1, \dots, n\}$ and $t \geq 0$, at least one element of $\{t, t+1, \dots, t+B-1\}$ belongs to T^i;
(v) $\forall i, j$ and $t \geq 0$, it holds that:

$$\max\{0, t-B+1\} \leq \tau_j^i(t) \leq t$$

Finally, assume there exist positive constants K_1 and K_2 such that, $\forall i$ and $\forall t \in T^i$ it holds:

(vi) $s_i(t)^T \nabla_i J(x^i(t)) \leq -\|s_i(t)\|^2 / K_1$;
(vii) $\|s_i(t)\| \geq K_2 \|\nabla_i J(x^i(t))\|$,

where $x^i(t) := (x_i(\tau_1^i(t)), \dots, x_n(\tau_n^i(t)))$, i.e. the memory accessible to processor i at time t.

Then, there exists $\bar{\rho} > 0$ such that for all $0 < \rho < \bar{\rho}$ it holds $\lim_{t\to\infty} \nabla J(x(t)) = 0$ □

In order to prove convergence of our r-DFRE algorithm using Proposition 8 we need to check if all the necessary assumptions hold.
Firstly, thanks to Assumptions 4 and 5, conditions (iv) and (v) on the existence of a positive integer B characterizing the nature of non ideal communication and information exchange are satisfied. Secondly, regarding the cost $\|\cdot\|_{1,\epsilon}$, it is easy to see that (i) and (ii) are satisfied, that is, it is twice continuously differentiable and positive. Moreover, regarding (iii), its Jacobian is Lipschitz continuous with constant L which is a function of the parameter ϵ and, thanks to the technical Assumption 6 it holds that $L > 0$. Finally, what remains is to prove the existence of K_1 and K_2 for which (vi) and (vii) are satisfied. According to our notation $x^i(t) \equiv \{\mathbf{x}_j^{(i)}(t)\}_{j\in\mathcal{N}_i}$ and $\nabla_i J(x^i(t)) \equiv [\nabla J(t)]_i$. Then, we have

$$s_i(t) = -D_i^{-1}(t)[\nabla J(t)]_i, \quad i \in \mathcal{V}_c\,,$$

and, concerning condition (vi), we can write

$$-[\nabla J(t)]_i^T D_i(t)[\nabla J(t)]_i \leq -[\nabla J(t)]_i^T D_i^2(t)[\nabla J(t)]_i/K_1$$

which, if $D_i(t)$ is symmetric positive definite, is equivalent to

$$I - \frac{D_i(t)}{K_1} \geq 0\,.$$

The latter holds if $K_1 \geq \lambda_{\max}(D_i(t))$, being $\lambda_{\max}(D_i(t))$ the largest eigenvalue of the matrix $D_i(t)$.
Regarding the second inequality (vii) we can write

$$[\nabla J(t)]_i^T D_i^2(t)[\nabla J(t)]_i \geq K_2^2[\nabla J(t)]_i^T[\nabla J(t)]_i$$

which is equivalent to

$$D_i^2(t) - K_2^2 I \geq 0$$

and holds if $K_2 \leq +\sqrt{\lambda_{\min}(D_i^2(t))} = |\lambda_{\min}(D_i(t))| = \lambda_{\min}(D_i(t))$, where the last equality holds since, thanks to Assumption 6, $D(t)$ is positive definite. Then, by choosing

$$K_1 \geq \sup_i\{\lambda_{\max}(D_i(t))\}\ ,\ K_2 \leq \inf_i\{\lambda_{\min}(D_i(t))\}\,, \quad \text{(A.1)}$$

the inequalities hold for every $i \in \mathcal{V}_c$.
Observe that it is always possible to find K_1 and K_2 satisfying (vi) and (vii) as shown by Eq.(A.1) since $D(t)$ is positive definite. It is important to notice that, without the additional Assumption 6, this is no more true since our prescribed cost $\|\cdot\|_{1,\epsilon}$ approximates the standard $1-$norm and its second derivative tends to zero for increasing values of $\mathbf{x}$ leading to $K_2 = 0$, which is not admissible for Proposition 8.

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