Optimal rendezvous control for randomized communication topologies

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***Abstract*— In this paper we analyze randomized coordination control strategies for the rendezvous problem of multiple agents subject to input and measurement disturbances. The performance of these control strategies is measured in terms of three important metrics: average relative agents' distance, average input energy consumption, and number of packets per unit time that each agent can receive from the other agents. By adopting an LQ-like optimal control approach, we show how to numerically compute optimal feedback gains for randomized communication topologies. In particular we show that there is a trade off between these three metrics and that the optimal feedback is a sum of two terms: one that depends only the agents own positions and the other that depends only on relative distances between agents. We also show that randomly switching communication links allows for greater performance as compare to fixed communication topologies.**

I. INTRODUCTION

The need for coordination of multiple mobile vehicles appears in many applications such as search-and-rescue missions and pursuit evasion games [1][2][3][4]. Coordination among vehicles requires exchange of information between them. However, the amount of information that can be exchanged is limited by many factors such as channel bandwidth, radio antenna power, interference, and it is therefore desirable to devise coordination strategies that require the transmission of a limited number of messages among the agents [5][6][7]. However, limiting information exchange among agents negatively impacts the performance of the vehicles as a group in terms of other metrics such as energy consumption and time required to accomplish a task. The goal of this paper is to analyze the trade offs between these aspects within the framework of rendezvous control, i.e. convergence of all agents to a common location not necessarily specified.

Recent work has shown that the performance of rendezvous control is strongly dependent on the specific communication topology among the agents [8]. In particular, there has been a particular effort in estimating performance for specific fixed topology classes that exploit symmetries [9]. Most of previous work has been based on fixed communication topologies [10] or distance-dependent deterministic topologies [11]. In this paper we assume that each agent has a GPS-like sensor which provides its position with respect to some absolute coordinate frame. Also we consider a time-varying random communication topology, where every agent exchange messages with a small set of other agents that is selected at random among all agents. The rationale behind this communication scheme is that random selection allows agents to communicate with all other agents over time even if at any time step they can communicate only with a small number, therefore *on average* the agents communication graph is fully connected. This paper builds upon a previous paper [12] where it was shown that the total cost can be written as the sum of two terms which depend only on the initial agents' positions, under an LQ optimal control formulation of the penalty cost. In particular, one term depends on the relative distances and the other that depends on the center of mass of agents with respect to absolute coordinate system considered. Interestingly, it was shown that in order to minimize energy expenditure the agents do not necessarily converge towards their initial center of mass as one would expect. This is the case only if every agent communicates with all other agents at any time step. In general, the optimal gains that minimize the total cost cause the agents to move towards a point that is between the instantaneous center of mass and the a-priori expected position center of mass of all agents. In this paper, instead, we also include input disturbance and measurement noise and we reformulate the problem as an LQ stochastic optimal problem. Differently from [12], here we show that the optimal gains and the performance do not depend on the initial agents' positions, however the optimal input is still the sum of a feedback on each agent's position and on the relative distances with the other agents.

II. PROBLEM FORMULATION

Consider N identical agents whose dynamics is described by a scalar linear discrete time integrator:

$$x_i(t+1) = x_i(t) + u_i(t) + w_i, \quad i = 1, \dots, N$$

where $x_i \in \mathbb{R}$ represents agent position, $u_i \in \mathbb{R}$ the control input, and $w_i \in \mathbb{R}$ input disturbance. We assume that w_i are i.i.d. random variable with a zero-mean gaussian distribution, i.e. $w_i \sim \mathcal{N}(0, \sigma_w^2)$. We assume that each agent has a GPS-like sensor that provides its own position:

$$y_i = x_i + v_i$$

where $v_i \in \mathbb{R}$ represent the measurement noise. Also we assume that v_i are i.i.d. random variable with a zero-mean gaussian distribution, i.e. $v_i \sim \mathcal{N}(0, \sigma_v^2)$, and independent of w_i.

More compactly we can descrive the agents dynamics in vector form as follows:

$$x(t+1) = x(t) + u(t) + w(t) \quad (1)$$
$$y(t) = x(t) + v(t) \quad (2)$$

where $x = (x_1, x_2, \dots, x_N)^T$, $u = (u_1, u_2, \dots, u_N)^T$, $w = (w_1, w_2, \dots, w_N)^T$, $y = (y_1, y_2, \dots, y_N)^T$, and $v = (v_1, v_2, \dots, v_N)^T$. We also assume that the agents can transmit their current position to some other agents, independently

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of their relative distance, i.e. we assume they have infinite power antennas. This last assumption is rather unrealistic, but will allow us to derive close form solutions for the agents performance. We will come back to this assumption in the Conclusions section. The objective of rendezvous control is to devise a coordination scheme that forces the agents to converge to a common location, or, equivalently, that forces the relative distances among all agents to be null. A natural way to enforce this objective is to penalize relative distances among agents using a quadratic cost $c_x(x) : \mathbb{R}^N \to \mathbb{R}^+$ defined as follows:

$$c_x(x) = x^T Q x, \quad Q \geq 0, \quad Qx = 0 \Leftrightarrow x = \alpha \mathbf{1} \tag{3}$$

where $\mathbf{1} = (1, 1, \dots, 1)^T \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$. The requirement that the vector $\mathbf{1}$ is the unique eigenvector of Q relative to the zero-eigenvalue, is equivalent of saying that $x^T Q x = 0 \iff (x_i - x_j) = 0 \, \forall (i, j)$ thus implying that the cost $c(x)$ is null if and only if the agents are in the same location. Note that this definition does not forces the agents to stay in a fixed location, as the cost would be null even if $x(t) = \alpha(t)\mathbf{1}$. This definition is rather important as it allows independent design of trajectory following control and rendezvous control. We also want to penalize the total agents input effort in achieving rendezvousing by using a quadratic cost $c_u(x) : \mathbb{R}^N \to \mathbb{R}^+$, where:

$$c_u(u) = r||u||^2$$

where $r \in \mathbb{R}^+$. The goal is to obtain a (possibly time-varying) feedback control

$$u(t) = K(t)y(t) \tag{4}$$

where $K \in \mathbb{R}^{N \times N}$, which minimize the expected total cost given by:

$$J_T = \mathbb{E}\left[x^T(T)Qx(T) + \sum_{t=0}^{T-1}\left(x^T(t)Qx(t) + r||u(t)||^2\right)\right] \tag{5}$$

If we substitute Equation (4) into Equation (1), we get the closed loop dynamics given by:

$$x(t+1) = (\mathbb{I} + K(t))x(t) + K(t)v(t) + w(t) \tag{6}$$

where $\mathbb{I}$ is the identity matrix.

Despite its simple formulation, the previous problem is rather challenging since the communication graph among agents imposes some constraints on the choice of the matrix $K(t)$. In particular if at time t the agent j cannot transmit its position to agent i, then the ij-th entry of the matrix K must be null, i.e. $K_{ij}(t) = 0$, since $y_j(t)$ is not available to agent i. This can happen for different reasons such as unreliable communications links, interference, packet collision, limited communication range or simply because only a maximum number of packets can be transmitted per unit time. Therefore, it is useful to define the adjacency matrix $E \in \{0, 1\}^{N \times N}$ as follows:

$$E_{ij} = \begin{cases} 1 & \text{if agent } i \text{ receives packet from agent } j \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

The rendezvous control problem can be summarized as follows:

$$\begin{array}{ll} \min_{\{K(t)\}_{t=1}^{T-1}} J_T & \\ s.t. & x(t+1) = (\mathbb{I} + K(t))x(t) + K(t)v(t) + w(t) \\ & K_{ij}(t) = 0 \text{ if } E_{ij}(x(t), t) = 0 \end{array} \tag{8}$$

The second constraint makes the problem highly non-convex and time-varying in general. Note also that under the previous formulation the communication graph is directed, i.e. it is possible that node j can transmit its position to node i but not viceversa. Solving the previous problem in full generality is hopeless. Most of recent work on rendezvous control has concentrated on optimizing the rate of convergence with fixed communication topologies $K(x, t) = K$ where most of the off diagonal entries are null [10][9]. In particular, the goal was to analytically determine the rate of convergence based on some a priory constrains on the structure of K and to optimally design classes of communication topologies with limited communication requirements. These sets of problems are rather difficult and often lead to combinatorial optimization problems. Differently, in [11] the authors considered a the feedback matrix dependent on agents location $K(x, t) = K(x)$; in particular they assumed that the agents can communicate only with agents which are within a fixed communication range, i.e. $E(t) = E(x(t))$. This strategy reduces communication burden but cannot guarantee convergence of all agents to a common location. In [8] it was shown that the agents communication topology needs to form a fully connected graph within an arbitrary large but finite time interval in order for the agents to converge to a common location with probability one. In other words this means that there must exist a time interval $T \in \mathbb{N}$, the union of all adjacency matrices $\{E(t)\}_{t=t_i}^{t_{i+1}}$ must form a fully connected graph for all $i \in \mathbb{N}$, where $t_{i+1} - t_i \leq T$. Inspired by this result, we propose to consider a stochastic communication topology, i.e. a time-varying control feedback $K(t)$ where most of the off-diagonal entries are zeros, i.e. $K_{ij}(t) = 0$ for most of the indexes i, j, but *on average* they are not, i.e. $\mathbb{E}[K_{ij}(t)] \neq 0$. This is equivalent of saying that the communication topology forms on average a fully connected graph. Our strategy does not satisfy the condition stated in [8] as there is always a small probability that the communication topology graph is not connected for any arbitrary but finite time interval T. Instead we will say that the system is *rendezvous-stable* in mean square sense if the following condition is satisfied:

$$\mathbb{E}[|x_i(t) - x_j(t)|^2] \leq M, \ \forall t, \forall i, j$$

for some $M \in \mathbb{R}^+$. The following lemma links this definition of rendezvous-stability to the performance cost J_T.

Lemma 1: A rendezvous control strategy given by the sequence $\{K(t)\}_{t=0}^{\infty}$ is rendezvous-stabilizing if $\frac{1}{T} J_T \leq J_{max}, \forall T$. In particular, the previous condition is satisfied if the following limit exists and it is finite:

$$J_\infty = \lim_{T \to +\infty} \frac{1}{T} J_T = \lim_{t \to +\infty} \mathbb{E}[x^T(t)Qx(t) + r||u(t)||^2] \tag{9}$$

Proof: If $\frac{1}{T} J_T \leq J_{max}$ is bounded for all $T \geq 0$, then each term in the series of Equation (5) must be bounded, and in particular the term $\mathbb{E}[x^T(t)Qx(t)] < M$ for all

$t \geq 0$. Now, we want to show that $\mathbb{E}[x^T(t)Qx(t)] < M$ implies $\mathbb{E}[|x_i(t)-x_j(t)|^2] \leq M_2$ for all i,j. Without loss of generality, we just need to prove it for $i=1$ and $j=2$. From the properties of the matrix Q in Equation (3) it follows that $x^TQx = ||Lx||^2$ where $L \in \mathbb{R}^{(N-1)\times N}$ and $\ker(L) = <\mathbf{1}>$. Let us define the matrix $\widetilde{L} \in \mathbb{R}^{(N-1)\times N}$ such that $\widetilde{L} = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ . & . & . & \dots & . \\ 1 & 0 & 0 & \dots & -1 \end{bmatrix}$. Since $\ker(\widetilde{L}) = <\mathbf{1}>$, then there exists a invertible matrix $S \in \mathbb{R}^{(N-1)\times(N-1)}$ such that $L = S\widetilde{L}$. Therefore, we have $x^TQx = ||Lx||^2 = ||S\widetilde{L}x||^2 \geq |\lambda_{max}(S)|^2||\widetilde{L}x||^2 = |\lambda_{max}(S)|^2\sum_{j=2}^N(x_1 - x_j)^2 \geq |\lambda_{max}(S)|^2(x_1-x_2)^2$, where $|\lambda_{max}(S)| > 0$ is the largest eigenvalue of the matrix S. This implies that $\mathbb{E}[(x_1-x_2)^2] \leq \frac{1}{|\lambda_{max}(S)|^2}\mathbb{E}[x^TQx] < \frac{M}{|\lambda_{max}(S)|^2} = M_2$, which proves the first part of the lemma. Equation (9) follows from the fact that each term in the series of Equation (5) must converge for the limit $\lim_{T\to+\infty}\frac{1}{T}J_T = J_\infty$ to exist, therefore the limit of time average of series is equivalent to the limit of its terms. ∎

In our framework we quantify information exchange as number of messages received by each agent at any time step, which corresponds to the non-zero off-diagonal entries of the adjacency matrix $E(t)$, therefore our objective is to analyze performance of rendezvous control as a function of the number of messages exchanged among agents. To get some more insight about structure of rendezvous control feedback we study two interesting limiting cases. In the first scenario let us assume that each agent receives messages from all other agents, i.e. $E(t) = \mathbf{1}\mathbf{1}^T$. Also, we assume there is no input disturbance nor measurement noise, i.e. $v_t = w_t = 0$. Finally, the penalization term c_x is defined as the sum of the distances between each agents, i.e.

$$x^TQx = \sum_{i=1}^N\sum_{j=1}^N(x_i-x_j)^2 \Rightarrow Q = 2(N\mathbb{I} - \mathbf{1}\mathbf{1}^T) \quad (10)$$

Therefore the optimization problem of Equation (8) becomes:

$$\begin{aligned} &\min_{\{K(t)\}_{t=1}^{T-1}} \sum_{t=0}^{T-1}\Big(x^T(t)Qx(t) + r||u(t)||^2\Big) \\ &s.t. \quad x(t+1) = (\mathbb{I}+K(t))x(t) \end{aligned} \quad (11)$$

which is the classic LQ optimal control problem. It is well known that the optimal feedback gains $\{K(t)^*\}_{t=1}^{T-1}$ for $T \to +\infty$, i.e. infinite horizon LQ control, are static, i.e. $K^*(t) = K^*$, and K^* can be obtained from the solution of the following algebraic Ricatti equation:

$$\begin{aligned} P &= P + Q - P(P+r\mathbb{I})^{-1}P, \quad P \geq 0 \\ K^* &= -P(P+r\mathbb{I})^{-1} \end{aligned}$$

After some simple matrix manipulations it is possible to show that:

$$K^* = h^*(N\mathbb{I} - \mathbf{1}\mathbf{1}^T)$$

where $h^* \in \mathbb{R}$. The feedback control given by Equation (4) can be written as:

$$u = h^*(N\mathbb{I} - \mathbf{1}\mathbf{1}^T)x(t) \Longrightarrow u_i = h^*\sum_{j=1}^N(x_i - x_j).$$

This means that the optimal control of each agent when full communication is available is proportional to the sum of error distances from all the other agents. Note that this control feedback is independent of the reference frame. In the second scenario we use the same assumptions of the previous scenario, but now we assume that no communication is allowed among the agents, i.e. $E = \mathbb{I}$ which gives rise to the following optimization problem:

$$\begin{aligned} &\min_{\{K(t)\}_{t=1}^{T-1}} \sum_{t=0}^{T-1}\Big(x^T(t)Qx(t) + r||u(t)||^2\Big) \\ &s.t. \quad x(t+1) = (\mathbb{I}+K(t))x(t) \\ &\qquad\quad K(t) = \mathrm{diag}(k_1(t),\dots,k_N(t)) \end{aligned} \quad (12)$$

Using symmetry arguments it follows that optimal feedback gains $\{K(t)^*\}_{t=1}^{T-1}$ for $T \to +\infty$ is this scenario are constant and with the following structure $K^* = k^*\mathbb{I}, k^* \in \mathbb{R}$, therefore the control feedback can be written as:

$$u = K^*x(t) = k^*x \Longrightarrow u_i = k^*x_i$$

which means that the optimal input for each agent is a linear feedback on its own position with respect to the reference frame.

Based on these two scenarios and the discussion regarding the randomized communication topology with limited number of communication messages per uniti time, we propose a rendezvous control strategy where at any time step each agent receives the current location of other $\nu \in \{0,1,\dots,N-1\}$ distinct agents chosen at random. The control scheme is linear feedback with constant gains of its own position and the relative distance with the other agents:

$$u_i = -ky_i - h\sum_{j=1}^{\nu}e_{ij}(t)(y_i - y_j) \quad (13)$$

where $k,h \in \mathbb{R}$, $e_{ij} \in \{0,1\}$, $e_{ii} = 0$, and $\sum_{j=1}^N e_{i,j} = \nu$. The non-zero $e_{ij}(t)$'s correspond to the incoming communication links of agent i with the other agents at time step t. The control feedback is the sum of two terms: the first depends only on the origin system and requires no communication, while the second requires communication but is independent of the origin system. Therefore, by appropriately choosing k and h, it is possible to place more weight on one term or the other. More compactly, this control scheme can be written as:

$$\begin{aligned} u(t) &= \big(hE(t) - (k+\nu h)\mathbb{I}\big)y(t) \\ &= \big(hE(t) - (k+\nu h)\mathbb{I}\big)\big(x(t)+v(t)\big) \end{aligned} \quad (14)$$

where $E(t) \sim \mathcal{U}(\mathbf{E})$, i.e. the matrix E is uniformly sampled from set of matrices $\mathbf{E}$ defined as follows:

$$\mathbf{E} = \{E \in \{0,1\}^{N\times N} \mid E\mathbf{1} = \nu\mathbf{1}, E_{i,i} = 0\}.$$

It is important to remark that despite it is not possible to prove that the randomized control strategy is the optimal among all possible strategies having constraints on the maximum number of messages exchanged among agents, in the two extreme scenarios for $\nu = 0$ or $\nu = N-1$ with no disturbances, the previous control strategy does give the optimal solution.

Before continuing let us define the matrices Π and $\Pi_\perp$ as follows:

$$\Pi \triangleq \mathbb{I} - \frac{1}{N}\mathbf{1}\mathbf{1}^T, \quad \Pi_\perp \triangleq \frac{1}{N}\mathbf{1}\mathbf{1}^T \quad (15)$$

which have the following properties:

$$\begin{array}{l}\Pi = \Pi^T \geq 0, \quad \Pi_\perp = \Pi_\perp^T \geq 0, \quad \Pi = \Pi^2, \quad \Pi_\perp = \Pi_\perp^2 \\ \Pi + \Pi_\perp = \mathbb{I}, \quad \Pi\Pi_\perp = \Pi_\perp\Pi = 0\end{array} \tag{16}$$

According to the previous definition the matrix Q as defined in Equation (10) can be written as

$$Q = 2N\Pi$$

It is also possible to show that the matrix $E(t)$ uniformly randomly chosen from the set $\mathbb{E}$, satisfies the following properties:

$$\begin{array}{rcl}\mathbb{E}[E(t)] & = & \nu\Pi_\perp - \frac{\nu}{N-1}\Pi \\ \mathbb{E}[E^T(t)E(t)] & = & \nu^2\Pi_\perp + \frac{\nu(N-\nu)}{N-1}\Pi \\ \mathbb{E}[E^T(t)\Pi E(t)] & = & \nu\left(1-\nu\frac{N-2}{(N-1)^2}\right)\Pi \\ \mathbb{E}[E^T(t)\Pi_\perp E(t)] & = & \nu^2\Pi_\perp + \frac{\nu(N-\nu-1)}{N-1}\Pi\end{array} \tag{17}$$

Without loss of generality we rescale the cost J_T as follows:

$$J_T = \mathbb{E}\left[x^T(T)\Pi x(T) + \sum_{t=0}^{T-1}\left(x^T(t)\Pi x(t) + r||u(t)||^2\right)\right]$$

where the parameter $r \in [0, +\infty)$ tunes the tradeoff between small agents relative distances (r small) and small input control effort (r large).

We can now compute explicitly the cost function $J_T(k,h)$ using the standard dynamic programming approach based on the cost-to-go function $V_t(x)$ recursively defined as follows:

$$\begin{array}{rcl}V_T(x_T) & \triangleq & \mathbb{E}[x_T^T\Pi x_T \,|\, x_T] \\ V_t(x_t) & \triangleq & \mathbb{E}[x_t^T\Pi x_t + r||u_t||^2 + V_{t+1}(x_{t+1}) \,|\, x_t]\end{array}$$

where we used $x_t = x(t)$ to simplify notation. We claim that the cost-to-go function can be written as:

$$V_t(x_t) = s_t\mathbb{E}[x_t^T\Pi x_t \,|\, x_t] + s_t^\perp\mathbb{E}[x_t^T\Pi_\perp x_t \,|\, x_t] + d_t \tag{18}$$

where $s_t, s_t^\perp$ and d_t are appropiate positive scalars. The claim is clearly true for $t = T$, where $s_T = 1$, $s_T^\perp = 0$ and $d_T = 0$. We can prove our claim for all other time steps t by induction. Let us suppose that the claim is true for $t+1$, then we want to show that the claim is true also for time t.

After some tedious but straightforward calculations is possible to show that the claim is verified where the scalar $s_t, s_t^\perp$ and d_k can be obtained iteratively for $t = T, \ldots, 0$ as follows:

$$s_T = 1, \quad s_T^\perp = 0, \quad d_T = 0 \tag{19}$$

$$s_t = a_1(h,k)s_{t+1} + a_2(h)s_{t+1}^\perp + a_3(h,k) \tag{20}$$

$$s_t^\perp = a_4(k)s_{t+1}^\perp + a_5(k) \tag{21}$$

$$d_t = d_{t+1} + a_6(h,k)s_{t+1} + a_7(k)s_{t+1}^\perp + a_8(h,k) \tag{22}$$

where the coefficients $(a_1, \ldots, a_8)$ are positive quadratic functions of the gains k, h:

$$\begin{array}{l}a_1(h,k) = b_1h^2 + 2b_2(k-1)h + (k-1)^2 \\ a_2(h) = b_3h^2, \quad a_4(k) = (1-k)^2, \quad a_5(k) = rk^2 \\ a_3(h,k) = 1 + r\left(b_4h^2 + 2b_2kh + k^2\right) \\ a_6(h,k) = (N-1)\Big(\sigma_w^2 + \sigma_v^2\big(b_1h^2 + 2b_2kh + k^2\big)\Big) \\ a_7(k) = \sigma_w^2 + \sigma_v^2k^2 \\ a_8(h,k) = r\sigma_v^2\Big(k^2 + (N-1)\big(b_4h^2 + 2b_2kh + k^2\big)\Big)\end{array}$$

where we used $\mathbb{E}[w_t^T\Pi_\perp w_t] = \sigma_w^2$, $\mathbb{E}[w_t^T\Pi w_t] = (N-1)\sigma_w^2$ and similarly for v_t, and the coefficient $(b_1, \ldots, b_4)$ are functions of the number of agents N and the number of received messages ν:

$$\begin{array}{l}b_1 = \frac{N\nu^2 + \nu(\nu+1)(N-1)^2}{(N-1)^2} \\ b_2 = \frac{\nu N}{N-1} \\ b_3 = \frac{\nu(N-\nu-1)}{N-1} \\ b_4 = \frac{\nu N(\nu+1)}{N-1}\end{array}$$

We interested now in computing the averaged expected cost for the infinite horizon scenario for fixed gains k, h which can be computed as described in the following lemma:

Lemma 2: Consider the sequences $\{s_t\}_{t=T}^0$ and $\{s_t^\perp\}_{t=T}^0$ defined by Equations (19)-(21) for fixed gains (h,k). If the limits:

$$\lim_{T\to+\infty} s_0 = s_\infty, \quad \lim_{T\to+\infty} s_0^\perp = s_\infty^\perp$$

exist and are finite, then they are non-negative and we have:

$$\begin{array}{l}J_\infty = \lim_{T\to\infty}\frac{1}{T}J_T = a_6(h,k)s_\infty + a_7(k)s_\infty^\perp + a_8(h,k) \\ s_\infty = a_1(h,k)s_\infty + a_2(h)s_\infty^\perp + a_3(h,k) \\ s_\infty^\perp = a_4(k)s_\infty^\perp + a_5(k)\end{array}$$

Proof: For fixed (h,k) then the sequences $\{s_t\}_{t=T}^0$ and $\{s_t^\perp\}_{t=T}^0$ are monotonically non-decreasing with non-negative initial conditions, therefore if the limits exist then they must be non-negative. From the definition of cost-to-go function $V_t(x)$ we have that $V_0 = J_T$, therefore from Equations (18) and (22) we have:

$$\begin{array}{rl}\frac{1}{T}J_T = & \frac{1}{T}\Big(s_0\mathbb{E}[x_0^T\Pi x_0] + s_0^\perp\mathbb{E}[x_0^T\Pi_\perp x_0]\Big) + \\ & + \frac{1}{T}\sum_{t=0}^{T-1}\Big(a_6(h,k)s_t + a_7(k)s_t^\perp + a_8(h,k)\Big)\end{array}$$

If we take the limit for $T \to \infty$ the first two terms in the previous expression disappear regardless of the initial positions of the agents x_0, while the average of the series converges to the limit of the terms inside its parenthesis, which proves the lemma. ∎

III. OPTIMAL CONTROL DESIGN: THE GPS-LIKE SCENARIO

From the previous section we can now formulate the optimal control under the proposed randomized control strategy as the following optimization problem:

$$\begin{array}{ll}\min_{k,h} & J_\infty(k,h,\nu,N) = \frac{1}{N}\big(a_6(h,k)s + a_7(k)s^\perp + \\ & + a_8(h,k)\big) \\ s.t. & s = a_1(h,k)s + a_2(h)s^\perp + a_3(h,k), \;\; s > 0 \\ & s^\perp = a_4(k)s^\perp + a_5(k), \quad s^\perp > 0\end{array} \tag{23}$$

where we used with a little abuse of notation $s = s_\infty$ and $s^\perp = s_\infty^\perp$. The $J_\infty(k,h,\nu,N)$ represent the expected cost *per agent* for fixed gains k, h. This optimization problem is highly non-linear and cannot be solved analytically. However, numerical solution can be obtained by adding lagrange multipliers to remove constants and then applying gradient descent as proposed in [12].

Although the previous optimization problem cannot be solved in closed form, some interesting analytical results can be deduced and are summarized in the following theorem.

Theorem 1: Let us consider the total cost $J_\infty(k,h,\nu,N)$ as defined in Eqn. (23), the optimal gains $(k^*_\nu, h^*_\nu) = \text{argmin}_{k,h} J_\infty(k, h, \nu, N)$ and the minimum cost $J^*_\infty(\nu, N) = \text{argmin}_{k,h} J_\infty(k, h, \nu, N)$. Define also:

$$J^*_m = \min_{\kappa\in(0,2)} J_m(\kappa) = \frac{\sigma_w^2(1 + r\kappa^2) + \sigma_v^2\kappa^2(1 + 2\kappa r)}{\kappa(2 - \kappa)}$$

and the corresponding minimizer $\kappa^*_m = \text{argmin}_{\kappa\in(0,2)} J_m(\kappa)$, then we have:

(a) $J^*_\infty(\nu, N) \geq J^*_\infty(\nu + 1, N)$
(b) $k^*_\nu = 0, h^*_\nu = \frac{\kappa^*_m}{N}$ for $\nu = N-1$
(c) $J^*_\infty(N-1, N) = \frac{N-1}{N} J^*_m$
(d) $J^*_\infty(\nu, N) \leq J_\infty(\kappa^*_m, 0, \nu, N) \leq J^*_m$
(e) $\lim_{N\to\infty} J^*_\infty(\nu, N) = J^*_m, \quad \forall\nu$

In the interest of space, the proof is omitted and it will be included in a forthcoming technical report. Let us analyze the different claims of the theorem. Claim (a) states that the performance in terms of expected minimum cost per agent improves if more messages are exchanged between agents at any time step, as one would expect. The second claim (b) implies that when every agent receives the position from *all* other agents at any time step, then the optimal rendezvous strategy is to apply no feedback on the origin position, i.e. $k^*_\nu = 0$, and to move towards the instantaneous center of mass, as suggested in the introduction. Moreover in the full communication graph scenario claim (c) states that the minimum achievable average cost per agent converges to a constant value J^*_m that can be computed explicitly as the number of agent increases. Claim (d) instead says that in the absence of communication, i.e. $h^*_\nu = 0$, and feedback gain relative to the origin $k^*_\nu = \kappa^*_m$, then the average cost per agent is always bounded by J^*_m regardless of the number of agents. Finally, claim (e) combines the previous two claims showing that the performance difference between the full communication graph scenario and the no communication scenario disappears as the number of agents increases. This is quite a surprising result as it states that it is basically useless to communicate as the same performance can be obtained by doing a simple feedback on the agent's own position relative to the origin system. However, a carefully inspection of the problem formulation indicates that in reality there is an important prior information that all agents share, which is the measurement of their positions with respect to a common coordinate system. We call this scenario "GPS-like". There are many applications where agents can only rely on relative position estimation given by on-board sensors like cameras or range finders. This scenario is analyzed in the next session.

IV. Optimal control design: the GPS-free scenario

In this section, we consider the scenario where agents have no direct information about their own position relative to some fixed frame, as it would be in the case in the presence of GPS-like sensors onboard of agents. They can only have relative distance information. This scenario corresponds to setting $k = 0$ in the Eqn. (13), which leads an optimization problem similar to the one given in Eqn. (23), where we just have to substitute $s^\perp = 0$ and $k = 0$:

$$\begin{array}{ll} \min_h & C(h, \nu, N) = \frac{1}{N}\big(a_6(h, 0)s + a_8(h, 0)\big) \\ s.t. & s = a_1(h, 0)s + a_3(h, 0), \;\; s > 0 \end{array} \qquad (24)$$

where $C_\infty(h, \nu, N) = J_\infty(0, h, \nu, N)$ is the expected cost per agent in the GPS-free scenario. Although, it is possible to numerically compute the minimum cost and the optimal gain for the previous problem, there are few analytical results that be obtained and summarized in the following theorem:

Theorem 2: Let us consider the total cost $C_\infty(h, \nu, N)$ as defined in Eqn. (24), the optimal gain $h^*_\nu = \text{argmin}_h C_\infty(h, \nu, N)$ and the minimum cost $C^*_\infty(\nu, N) = \text{argmin}_h C_\infty(h, \nu, N)$. Define also:

$$C_m(\kappa, \nu) = \frac{\sigma_w^2\big(1+r\nu(\nu+1)\kappa^2\big)+\sigma_v^2\kappa^2(1+2r\nu\kappa)}{\kappa\nu\big(2 - (\nu+1)\kappa\big)}$$

$C^*_m(\nu) = \min_{\kappa\in\left(0,\frac{2}{\nu-1}\right)} C_m(\kappa, \nu)$ and the corresponding minimizer $\kappa^*_m = \text{argmin}_{\kappa\in\left(0,\frac{2}{\nu-1}\right)} C_m(\kappa, \nu)$. then we have:

(a) $C^*_\infty(\nu, N) \geq C^*_\infty(\nu+1, N)$
(b) $\lim_{N\to\infty} C^*_\infty(\nu, N) = C^*_m(\nu), \quad \forall\nu$
(c) $C^*_m(\nu) \geq C^*_m(\nu+1)$
(c) $\lim_{\nu\to\infty} C^*_m(\nu) = J^*_m$

Claim (a) states that performance increases as more messages are exchanged among agents, as we would expect. The second claim (b) it is rather important as it indicates that, for large number of agents N, the performance is independent of the number agents, and depends only on the number of exchanged messages ν. This is quite remarkable because it implies that by randomly switching the communication topology the performance scales nicely with the number of agents. This is in sharp contrast with fixed communication topologies, such as the cyclic topology, where performance degrades rapidly with the number of agents [9].

It is interesting to consider the case where the objective is only to minimize relative distances of agents where no penalization in placed on the input, i.e. $r = 0$, and there is no measurement noise, i.e. $\sigma_v = 0$. In this case we get that $C^*_m = \frac{\sigma_w^2(\nu+1)}{\nu}$ and $\kappa^*_m = \frac{1}{\nu+1}$. It is clear how the cost decreases as the number of messages increases, and the feedback gain decreases a number of messages exchanged.

V. Numerical examples

In this section we illustrate the optimal controller design and performance computed as described in the previous sections. We consider a systems with measurement noise and input disturbance covariance $\sigma_v^2 = \sigma_w^2 = 1$.

As proved in Theorem 1, Figure 1 shows that the performance gain given by full communication connectivity, i.e $\nu = N - 1$ relative to the scenario where no exchange of information is available, i.e. $\nu = 0$, decreases as the number of agents increases, but it becomes "flat" as N increases.

Figure 2 show the average cost for the GPS-like and the GPS-free scenarios for $N = 20$ vehicles. When full communication is available the performance of under GPS-like and GPS-free scenarios coincide. However, as the number of received messages ν decreases, in the GPS-free scenario the performance degrades considerably.

VI. Conclusions

In this paper we studied the trade off with respect to average agents's relative distance, energy consumption and number of information exchanged for a simple model of rendezvous of mobile vehicles. We approached the problem considering a randomized communication scheme where the

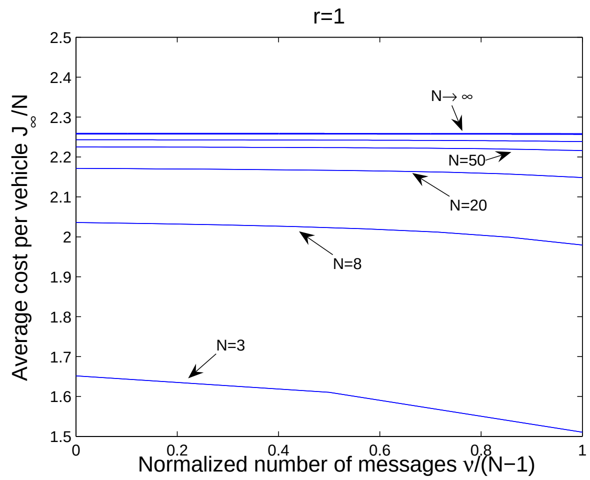

Fig. 1. Average performance cost per vehicles J_∞ versus the normalized number of messages received by each agents $\frac{\nu}{N-1}$ for different number of agents N.

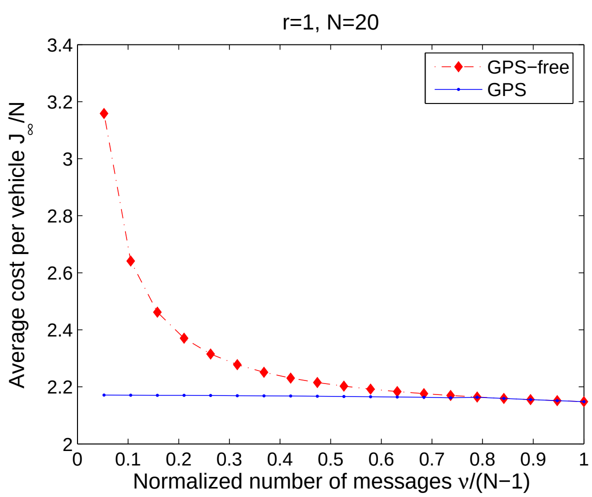

Fig. 2. Average performance cost per vehicles J_∞ versus the normalized number of messages received by each agents $\frac{\nu}{N-1}$ for GPS-like and GPS-free scenarios.

number of messages exchanged by each vehicle per unit time was fixed and we formulated the control problem as a stochastic linear quadratic optimization problem. We showed how increasing information exchange can improve performance, as one would expect. However, we also showed that if all agents have access to their own position such as in a GPS-like scenario, then the performance degradation is rather limited when a large number of agents is considered. Differently, when agents can only measure relative distances with other agents, i.e. in a GPS-free scenario, the degradation of performance is substantial.

Another important observation is that randomized communication schemes result in a fast information distribution among all agents and thus increasing performance. In other words, it seems that fixed or symmetric communication schemes might hamper performance, unless they are properly designed based on vehicles topology [9]. Differently, randomized communication schemes are easy to implement and have high performance. To be fair, however, it is important to remark that purely randomized schemes might not be possible in practice. In fact, communication among agents can be established only if they are sufficiently close to each other. For example, in [8] authors considered disk-communication, i.e. agents could communicate only if within a certain distance. A more realist model would be to consider a probabilistic communication model where the probability of successfully exchange a packet depends on the distance. This work is a preliminary attempt to understand the performance of randomized communication topologies, and many open questions remain.

Finally, there many similarities that the rendezvous problem has with the consensus agreement problem [13], therefore randomized communication topologies might be proven very effective also in that framework.

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