On the performance of randomized communication topologies for rendezvous control of multiple vehicles

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***Abstract*— In this paper we analyze randomized coordination control strategies for the rendezvous problem of multiple agents with unknown initial positions. The performance of these control strategies is measured in terms of three important metrics: average relative agents' distance, total input energy consumption, and number of packets per unit time that each agent can receive from the other agents. By adopting an LQ-like optimal control approach, we show how to numerically compute optimal feedback gains for randomized communication topologies. In particular we show that there is a trade off between these three metrics and that the optimal feedback is a sum of two terms: one that depends only the on agents own positions and the other that depends only on relative distances between agents.**

***Keywords*— Rendezvous control, networked control system, consensus agreement, optimal control, randomized topology**

I. INTRODUCTION

The need for coordination of multiple mobile vehicles appears in many applications such as search-and-rescue missions and pursuit evasion games. Coordination among vehicles requires exchange of information between them. However, the amount of information that can be exchanged is limited by many factors such as channel bandwidth, radio antenna power, interference, and it is therefore desirable to devise coordination strategies that require the transmission of a limited number of messages among the agents. However, limiting information exchange among agents negatively impacts the performance of the vehicles as a group in terms of other metrics such as energy consumption and time required to accomplish a task. The goal of this paper is to analyze the trade offs between these aspects within the framework of rendezvous control, i.e. convergence of all agents to a common location not necessarily specified. Recent work has shown that the performance of rendezvous control is strongly dependent on the specific communication topology among the agents. In particular, there has been a particular effort in estimating performance for specific fixed topology classes that exploit symmetries. Most of previous work has been based on fixed communication topologies. In this paper we consider a time-varying random communication topology, where every agent exchange messages with a small set of other agents that is selected at random among all agents. The rationale behind this communication scheme is that random selection allows agents to communicate with all other agents over time even if at any time step they can communicate only with a small number. Under the assumption that the feedback structure is a linear function of agents own positions and relative distances with the other agents and considering a cost function defined as a quadratic penalty on relative distances between the agents and the sum of all agents energy expenditure, we show that the total cost can be written as the sum of two terms which depend only on the initial positions. In particular, one term depends on the relative distances and the other that depends on the center of mass of agents with respect to absolute coordinate system considered. Interestingly, we show that in order to minimize energy expenditure the agents do not necessarily converge towards their initial center of mass as one would expect. This is the case only if every agent communicates with all other agents at any time step. In general, the optimal gains that minimize the total cost cause the agents to move towards a point that is between the instantaneous center of mass and the a-priori expected position center of mass of all agents.

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II. PROBLEM FORMULATION

Consider N identical agents whose dynamics is described by a scalar linear discrete time integrator:

$$x_i(t+1) = x_i(t) + u_i(t), \quad i = 1, \dots, N$$

where $x_i \in \mathbb{R}$ represent agent position, and $u_i \in \mathbb{R}$. More compactly we can descrive the dynamics in vector form as follows:

$$x(t+1) = x(t) + u(t)$$

where $x = (x_1, x_2, \dots, x_N)^T$ and $u = (u_1, u_2, \dots, u_N)^T$. We assume that each agent can measure its own position perfectly and there is no external disturbance in the agent dynamics. We assume that the agent can transmit their current position to some other agents, independently of their relative distance, i.e. we assume they have infinite power antennas. The goal is to obtain a (possibly time-varying) feedback control

$$u = K(t)x, \quad K \in \mathbb{R}^{N \times N}$$

yielding the rendezvous, i.e. the all agents must converge to a common location. More formally, we want the closed loop system:

$$x(t+1) = (I + K(t))x(t)$$

where I is the identity matrix, is such that, for all initial conditions $x(0) \in \mathbb{R}^N$, it holds

$$\lim_{t \to \infty} x(t) = \alpha(t)\mathbf{1}$$

where $\mathbf{1} = (1,1,\ldots,1)^T \in \mathbb{R}^N$, and $\alpha \in \mathbb{R}$. A more restrictive definition of rendezvous control would be $\lim_{t\to\infty} x(t) = \alpha\mathbf{1}$ which forces all agents to converge to a fixed location.

The problem as stated above seems trivial, since the choice $K(t) = -I$ would make all agent to converge to the origin of the coordinate system in one step and does not need any exchange of information among agents. However, this scheme might lead to a very high energy expenditure, i.e. $||u||^2$, since the origin might be very far from all agents. On the other hand, if all agents knew all other agents positions, the best strategy from an energy perspective, is to converge to the common center of mass $x_b(0) = \frac{1}{N}\sum_{i=1}^N x_i(0) = \frac{1}{N}\mathbf{1}^T x(0)$.

This strategy, however requires the exchange of a number of messages while the agents do not move. On the other hand, from a performance point of view it is desired that agents converge to a common point as fast as possible, therefore waiting for the computation of the exact center of mass can reduce overall rate of convergence. This is a common problem in control, often known as exploration-exploitation dilemma, in fact very often it is unclear weather it is better to wait to gather more information or use current incomplete information to accomplish the desired task. The goal of this paper is to study communication topologies for rendezvous control which require only a limited information exchange, and to highlight the performance tradeoff in terms of energy consumption, rate of convergence and quantity of information exchanged.

In our framework we quantify information exchange as number of messages received by each agent at any time step, which corresponds to the non-zero off-diagonal entries of the feedback matrix $K(t)$. Most of recent work on rendezvous control has concentrated on fixed communication topologies $K(t) = K$ where most of the off diagonal entries are null [1][2]. In particular, the goal was to analytically determine the rate of convergence based on some a priory constrains on the structure of K and to optimally design classes of communication topologies with limited communication requirements. This sets of problems are rather difficult and often lead to the solution of a combinatorial problems. Another approach has considered feedback matrix which depended on agents location $K(t) = K(x)$ [3]; in particular it was assumed that agents can communicate only with agents which are within a fixed communication range. This strategy reduces communication burden but cannot guarantee convergence of all agents to a common location. In [4] it was shown that the agents communication topology needs to form a fully connected graph within an arbitrary large but finite time interval in order for the agent to converge to a common location. In other words this means that there must exist a time interval $T \in \mathbb{N}$, such that the non-zero entries of the union of all matrices $\{K(t)\}_{t=t_i}^{t_{i+1}}$ must form a fully connected graph for all $i \in \mathbb{N}$, where $t_{i+1} - t_i \leq T$. Inspired by this result, we propose to consider a stochastic communication topology, i.e. a time-varying control feedback $K(t)$, whose off-diagonal entries mostly zeros instantaneously, i.e. $K_{i,j}(t) = 0$ for most most of the indexes i, j, but *on average* they are not, i.e. $\mathbb{E}[K_{i,j}(t)] \neq 0$. This is equivalent of saying that the communication topology forms on average a fully connected graph. Our strategy does not satisfy the condition of [4] as there is always a small probability that the communication topology graph is not connected for any arbitrary but finite time interval T, therefore it is not possible to guarantee that $\lim_{t\to\infty} x(t) = \alpha(t)\mathbf{1}$. Since we are considering probabilistic control strategies, we will consider the following condition of rendezvous control:

$$\lim_{t\to\infty} \mathbb{E}[||x(t) - \alpha(t)\mathbf{1}||^2] = 0$$

i.e. $x(t) \to \alpha(t)\mathbf{1}$ in mean square sense. Also, we will limit our control feedback to symmetric schemes, i.e. every agent must use the same control strategy. In particular we assume that at any time step each agent receives the current location of other ν distinct agents chosen at random. The control scheme is linear feedback with constant gains of its own position with respect to a fixed point α and the relative distance with the other agents:

$$u_i = -k(x_i - \alpha) + h\sum_{j=1}^{\nu} e_{i,j}(t)(x_j - x_i)$$

where $k, h, \alpha \in \mathbb{R}$, $e_{i,j} \in \{0,1\}$, $e_{ii} = 0$, and $\sum_{j=1}^N e_{i,j} = \nu$. The non-zero $e_{i,j}(t)$s correspond to the communication links of agent i with the other agents at time step t. The control feedback is the sum of two terms: the first depends only on the origin system and requires no communication, while the second requires communication but is independent of the origin system. Therefore, by appropriately choosing k and h, it is possible to place more weight on one term or the other. As it will be shown later, the parameter α can be used to improve performance when some prior information about distribution of vehicles is known. More compactly, this control scheme can be written as:

$$u(t) = -\big((k+\nu h)I + hE(t)\big)(x(t) - \alpha\mathbf{1}) \qquad (1)$$

where $E \in \mathcal{E}$ and the set $\mathcal{E}$ is defined as follows:

$$\mathcal{E} = \{E \in \{0,1\}^{N\times N} \mid E\mathbf{1} = \nu\mathbf{1}, E_{i,i} = 0\}.$$

The performance of control feedback for fixed gains k, h is based on a quadratic functional that penalizes relative distances between agents as well as energy consumptions, i.e.

$$J = \mathbb{E}\left[x^T(T)Qx(T) + \sum_{t=1}^{T-1}\left(x^T(t)Qx(t) + r||u(t)||^2\right)\right]$$

where $r \in \mathbb{R}^+$ and $Q \geq 0$ has the following property:

$$x^TQx = 0 \iff (x_i - x_j) = 0\,\forall(i,j) \qquad (2)$$

This condition penalizes a scenario where all agents are not in the same location and is independent system origin. One possible choice is to penalize the sum of the square distances of agents from their instantaneous center of mass

$x_b(t)$. With this regard let us define the distance of agent i from the instantaneous center of mass $y_i(t) = x_i(t) - x_b(t)$, or more compactly:

$$y = x - \mathbf{1}x_b = x - \mathbf{1}\frac{1}{N}\mathbf{1}^T x = \left(I - \frac{1}{N}\mathbf{1}\mathbf{1}^T\right)x = Yx$$

Therefore, since we want to penalize $||y||^2$, we can chose $Q = qY^TY = qY$, where $q \in \mathbb{R}^+$. This choice for the cost weight Q satisfies Equation (2). Although this choice is somewhat arbitrary, it has the advantage to simplify derivations, and still it allows us to balance the tradeoff between rate of convergence and power consumption by changing the values of the weights q and r. Therefore from now on we will consider the following cost:

$$\begin{aligned} &J(k,h,\alpha,x_0) = \\ &= \mathbb{E}\left[q||y(T)|| + \textstyle\sum_{t=1}^{T-1}\left(q||y(t)||^2 + r||u(t)||^2\right) \;\middle|\; x_0\right] \end{aligned} \tag{3}$$

subject to control defined in Equation (1).

III. Optimal control gains design

In this section we compute explicitly the total rendezvous control J for fixed control gains k, h, α. Then we propose a numerical algorithm to find the optimal gains k^*, h^*, α^* which minimize the cost J. Before continuing we point out some properties of the matrix Y:

$$\begin{aligned} Y &\triangleq I - \frac{1}{N}\mathbf{1}\mathbf{1}^T, \quad Y_\perp \triangleq \frac{1}{N}\mathbf{1}\mathbf{1}^T \\ Y &= Y^T \geq 0, \quad Y = Y^2, \quad Y + Y_\perp = I, \\ YY_\perp &= Y_\perp Y = 0, \quad Y_\perp = Y_\perp^T \geq 0, \quad Y_\perp = Y_\perp^2 \end{aligned}$$

The random matrix $E(t)$ is uniformly chosen from the set $\mathcal{E}$ and satisfies the following properties:

$$\begin{aligned} \mathbb{E}[E(t)] &= \nu Y_\perp - \frac{\nu}{N-1}Y \\ \mathbb{E}[E^T(t)E(t)] &= \nu^2 Y_\perp + \frac{\nu(N-\nu)}{N-1}Y \\ \mathbb{E}[E^T(t)YE(t)] &= \nu\left(1 - \nu\frac{N-2}{(N-1)^2}\right)Y \\ \mathbb{E}[E^T(t)Y_\perp E(t)] &= \nu^2 Y_\perp + \frac{\nu(N-\nu-1)}{N-1}Y \end{aligned}$$

We can now compute explicitly the cost function $J(k,h,x_0)$ using the standard dynamic programming approach based on the cost-to-go function $V_t(x)$ recursively defined as follows:

$$\begin{aligned} V_T(x_T) &\triangleq \mathbb{E}[q||y_T||^2] \\ V_t(x_t) &\triangleq \mathbb{E}[q||y_t||^2 + r||u_t||^2 + V_{t+1}(x_{t+1})] \end{aligned}$$

where we used $x_t = x(t)$ to simplify notation. We claim that the cost-to-go function can be written as

$$V_t(x_t) = s_t\mathbb{E}[x_t^T Y x_t] + s_t^\perp \mathbb{E}[(x_t - \alpha\mathbf{1})^T Y_\perp (x_t - \alpha\mathbf{1})]$$

where s_t and $s_t^\perp$ are appropiate positive scalars. The claim is clearly true for $t = T$, where $s_T = q$ and $s_T^\perp = 0$, since by definition $\mathbb{E}[x_T^T Y x_T] = \mathbb{E}[||y_T||^2]$. We can prove our claim for all other time steps t by induction. To simplify notation we define the following change of coordinate system:

$$z = x - \alpha\mathbf{1}$$

therefore the dynamics of the system and the cost-to-go can be written as:

$$\begin{aligned} u_t &= (-(k+\nu h)I + hE_t)z_t \\ z_{t+1} &= ((1-k-\nu h)I + hE_t)z_t \\ V_t(z_t) &= s_t\mathbb{E}[(z_t+\alpha\mathbf{1})^T Y (z_t + \alpha\mathbf{1})] + s_t^\perp\mathbb{E}[z_t^T Y_\perp z_t] \\ &= s_t\mathbb{E}[z_t^T Y z_t] + s_t^\perp\mathbb{E}[z_t^T Y_\perp z_t] \end{aligned}$$

where we used the fact that $Y\mathbf{1} = 0$.

Let us suppose that the claim is true for $t+1$, then we want to show that the claim is true also for time t. After some tedious but straightforward calculations we can show that the claim is verified where the scalar s_t and $s_t^\perp$ can be obtained iteratively as follows:

$$\begin{aligned} s_T &= q, \quad s_T^\perp = 0 \\ s_t &= q + r\left(\frac{\nu N(\nu+1)}{N-1}h^2 + 2\frac{\nu N}{N-1}kh + k^2\right) + \\ &\quad + \frac{\nu(N-\nu-1)}{N-1}h^2 s_{t+1}^\perp + \\ &+ \Big(\frac{N\nu^2 + \nu(\nu+1)(N-1)^2}{(N-1)^2}h^2 - \frac{2\nu N}{N-1}(1-k)h + \\ &\quad + (1-k)^2\Big)s_{t+1} \\ &= a_1(k,h) + a_2(h)s_{t+1}^\perp + a_3(k,h)s_{t+1} \\ s_t^\perp &= rk^2 + (1-k)^2 s_{t+1}^\perp \\ &= a_4(k) + a_4(k)s_{t+1}^\perp, \qquad t = T-1,\ldots,0 \end{aligned}$$

where the scalar coefficients $a_i(\cdot)$ are quadratic functions of the control gains k, h. The total cost J_T is then equivalent to the cost-to-go from time step $t = 0$ given by:

$$\begin{aligned} &J_T(k,h,\alpha,x_0) = V_0(x_0) = V_0(z_0) = \\ &= s_0\mathbb{E}[z_0^T Y z_0] + s_0^\perp\mathbb{E}[z_0^T Y_\perp z_0] \\ &= s_0\mathbb{E}[(x_0-\alpha\mathbf{1})^T Y (x_0-\alpha\mathbf{1})] + \\ &+ s_0^\perp\mathbb{E}[(x_0-\alpha\mathbf{1})^T Y_\perp (x_0-\alpha\mathbf{1})] \\ &= s_0\mathbb{E}[x_0^T Y x_0] + s_0^\perp\mathbb{E}[(x_0-\alpha\mathbf{1})^T Y_\perp (x_0-\alpha\mathbf{1})] \\ &= s_0\mathbb{E}[x_0^T Y x_0] + s_0^\perp\left(\mathbb{E}[x_0 Y_\perp x_0] - 2\alpha\mathbb{E}[\mathbf{1}^T x_0] + N\alpha^2\right) \\ &= s_0\mathbb{E}[x_0^T Y x_0] + s_0^\perp\left(N\mathbb{E}[x_b(0)^2] - 2N\alpha\mathbb{E}[x_b(0)] + N\alpha^2\right) \end{aligned}$$

The previous equations fully determine the cost function in terms of the initial position distribution of agents x_0, feedback gains (k,h), and feedback position α.

We also define the infinite horizon cost, if it exists and it is finite, as

$$J_\infty(k,h,\alpha,x_0) = \lim_{T\to\infty} J_T(k,h,\alpha,x_0)$$

Clearly such cost exists if and only if the limits $\lim_{T\to\infty} s_0 = s_\infty$ and $\lim_{T\to\infty} s_0^\perp = s_\infty^\perp$ exist and are finite. At this point we can compute the minimum cost as the solution of the following optimization problem:

$$\begin{aligned} \min_{k,h,\alpha} \quad & J_\infty(k,h,\alpha,x_0) = s_\infty\mathbb{E}[x_0^T Y x_0] + \\ & s_\infty^\perp\mathbb{E}[(x_0-\alpha\mathbf{1})^T Y_\perp (x_0-\alpha\mathbf{1})] \\ s.t. \quad & s_\infty = a_1(k,h) + a_2(h)s_\infty^\perp + a_3(k,h)s_\infty \\ & s_\infty^\perp = a_4(k) + a_4(k)s_\infty^\perp \\ & |a_3(k,h)| < 1, \quad |a_4(k)| < 1 \end{aligned}$$

Obviously the previous problem to be well defined requires, in general, the knowledge of some statistics of the initial distribution of vehicles positions. In particular, the total cost J_∞ can be rewritten as follows:

$$J_\infty(k,h,\alpha,x_0) = s_\infty\left(\sum_{i=1}^N \mathbb{E}[(x_i(0)-x_b(0))^2]\right) + \\ + s_\infty^\perp\left(N\mathbb{E}[x_b(0)^2] - 2N\alpha\mathbb{E}[x_b(0)] + N\alpha^2\right)$$

where the term that multiplies s_∞ depends only on distances of agents from their initial center of mass, while the second only on the distance of the center of mass from the origin. Since s_∞ and $s_\infty^\perp$ do not depend on the variable α, the optimal choice for this variable is the one that minimizes the second term and it is given by:

$$\alpha^* = \mathbb{E}[x_b(0)]$$

which corresponds to the a-priori knowledge of agents' center of mass. This means that each agents moves towards a combination of the expected initial center of mass and the other agents actual positions. If we substitute back into the original optimization problem we have

$$\begin{aligned} \min_{k,h} \quad & J_\infty(k,h,x_0) = s_\infty\left(\sum_{i=1}^N \mathbb{E}[(x_i(0)-x_b(0))^2]\right) + \\ & + s_\infty^\perp\left(N\mathrm{var}[x_b(0)]\right) \\ s.t. \quad & s_\infty = a_1(k,h) + a_2(h)s_\infty^\perp + a_3(k,h)s_\infty \\ & s_\infty^\perp = a_4(k) + a_4(k)s_\infty^\perp \\ & |a_3(k,h)| < 1, \quad |a_4(k)| < 1 \end{aligned}$$

where $\mathrm{var}[x_b(0)] = \mathbb{E}[x_b^2(0)] - (\mathbb{E}[x_b(0)])^2$. According to this analysis the optimization problem is well defined as long as the following three statistics are known:

$$\begin{aligned} b_1 &= \mathbb{E}[x_b(0)] \\ b_2 &= \sum_{i=1}^N \mathbb{E}[(x_i(0)-x_b(0))^2] \\ b_3 &= N\mathrm{var}[x_b(0)] \end{aligned}$$

which correspond to the a-priori knowledge about the system to solve the LQ problem as set up in this paper.

However, there are three special cases for which the optimal gains h, k are independent of the vehicles distribution.

A. Case 1: cheap control ($r = 0$)

This is the case where the most important thing is fast convergence regardless of energy consumption. According to this choice of weight we have:

$$\begin{aligned} s_t^\perp &= 0, \forall t \\ s_t &= q + \Big(\frac{N\nu^2 + \nu(\nu+1)(N-1)^2}{(N-1)^2}h^2 - \\ & -\frac{2\nu N}{N-1}(1-k)h + (1-k)^2\Big)s_{t+1} \end{aligned}$$

It should be clear that the optimal gains to minimize $J_T(k,h,\alpha,x_0)$ are $h^* = 0$ and $k^* = 1$. This would make all agents to converge to the point α in one step and $s_t = q$ for all t, therefore the minimal cost is given by

$$J_T^*(x_0,\alpha) = q\mathbb{E}[x_0^T Y x_0], \quad h^* = 0, k^* = 1$$

independently of the choice of the variable α.

B. Case 2: fully connected graph ($\nu = N-1$)

This is the case when all agents communicate with all agents, therefore they have perfect exact information of all agents positions. The equations then become:

$$\begin{aligned} s_T &= q, \quad s_T^\perp = 0 \\ s_t &= q + r\,(k+Nh)^2 + (1-k-Nh)^2\, s_{t+1} \\ s_t^\perp &= rk^2 + k^2\, s_{t+1}^\perp, \qquad t = T-1,\ldots,0 \end{aligned}$$

It should be clear that the optimal choice for the gain k is given by:

$$k^* = 0.$$

For this choice the minimizer h^* can be obtained by computing the positive solution of the following standard algebraic Riccati Equation:

$$\begin{aligned} s_{are} &= q + s_{are} - \frac{s_{are}^2}{s_{are}+r}, \quad s_{are} > 0 \\ h_{are} &= \frac{1}{N}\frac{s_{are}}{s_{are}+r} \end{aligned}$$

The optimal cost is given by

$$J_T^*(x_0,\alpha) = s_{are}\mathbb{E}[x_0^T Y x_0], \quad h^* = h_{are}, k = 0$$

which is independent of the choice of the variable α since the gain $k^* = 0$. For this case corresponding to a fully connected graph, it is not difficult to show that the agents converge to the center of mass of initial agents positions. It is also possible to show that this choice of gains is the one that minimizes the total cost J among all possible control strategies.

C. Case 3: GPS-free agents ($k = 0$)

This case models a scenario where no global positioning sensor is available to agents which can only measure relative distances with other agents. This scenario is relevant in indoor missions or scenarios when GPS-like systems are not feasible. In this case the cost equations reduces to:

$$\begin{aligned} s_\infty^\perp &= 0 \\ s_\infty &= q + \frac{\nu N(\nu+1)}{N-1}h^2 r + \\ &+ \left(\frac{N\nu^2 + \nu(\nu+1)(N-1)^2}{(N-1)^2}h^2 - \frac{2\nu N}{N-1}h + 1\right)s_\infty \end{aligned}$$

The optimal gain h^* can by first minimizing the right hand side of the previous equation for h and then by solving the corresponding Riccati-like equation. The minimum cost for a Gaussian distribution of agents' initial conditions is shown in Fig. 1 and it is compared to a scenario for which GPS information and knowledge about statistics of initial distribution is exploited. While both curves are decreasing function of the number of communication links ν and coincide for $\nu = N-1$, it is clear that prior information about agents distribution can be effectively used to reduce degradation of performance. Although the cost curves might be different for different choices of initial distributions, it is important to remark that prior knowledge of agents position distribution can be effectively used to reduce performance degradation when only little information exchange among agents is available. This aspect can be seen also for the cheep control scenario, i.e.

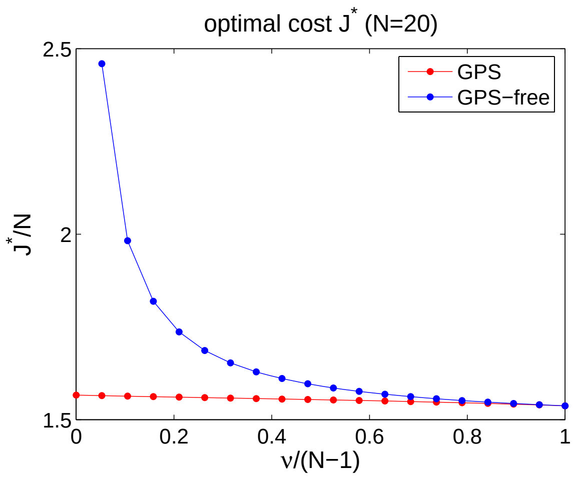

Fig. 1. Minimum cost for $r = q = 1$ for two scenarios: a GPS-based control for a Gaussian distribution of agents' initial positions (see Case 4), and a GPS-free control where only relative agents' distances is known. The dots correspond to the actual instantiation of $\nu = 1, \ldots, N-1$. The choice $\nu/(N-1) = 0$ corresponds to no communication between agents and $\nu/(N-1) = 1$ corresponds to a fully connected communication graph.

when $r = 0$. In this case the previous equation simplifies to:

$$\begin{aligned} s_\infty &= q + \left(\frac{N\nu^2 + \nu(\nu+1)(N-1)^2}{(N-1)^2} h^2 - \frac{2\nu N}{N-1} h + 1 \right) s_\infty \\ &= q + a(h) s_\infty \end{aligned}$$

The minimum cost is equivalent to chose the h that minimizes the convergence rate $a(h)$, i.e. $\frac{da}{dh} = 0$. When the number of agents is large, i.e. $N >> 1$, then the optimal gain and convergence rate converges to:

$$h^* = \frac{1}{\nu+1}, \quad a(h^*) = \frac{1}{\nu+1}$$

Differently from Case 1 where all agents converges to a common location in one step, in the GPS-free scenario the rate of convergence depends approximately as the inverse of the number of communication links per time step ν. The same results ware found in [2].

However, except for the three special scenarios described above, the agents' initial position distribution is necessary to compute the optimal gains k^* and h^*. A common choice for initial distribution of agents' position is the Gaussian distribution which is analyzed in the following special case.

D. Case 4: Gaussian distribution of initial positions

If the initial position distribution of agents x_0 is known, then it can be used to explicitly compute the expectations present in the cost J_∞. Suppose for example that it is known that initial position of agents are i.i.d. random variable with a gaussian distribution, i.e.

$$x_i(0) \sim \mathcal{N}(\bar{x}_0, \sigma^2), \quad i = 1, \ldots, N$$

where $\bar{x}_0, \sigma \in \mathbb{R}, \sigma > 0$, then we can compute the distribution on $x_0 = (x_1(0), \ldots, x_N(0))$. In particular it is possible to explicitly calculate the required statistics defined earlier in this section:

$$\begin{aligned} b_1 &= \bar{x}_0 \\ b_2 &= (N-1)\sigma^2 \\ b_3 &= \sigma^2 \end{aligned}$$

Therefore the optimal choice for the variable α is given by:

$$\alpha^* = \bar{x}_0$$

and the optimization problem reduces to:

$$\begin{aligned} \min_{k,h} \quad & J_\infty(k,h) = \left((N-1)s_\infty + s_\infty^\perp\right)\sigma^2 \\ s.t. \quad & s_\infty = a_1(k,h) + a_2(h)s_\infty^\perp + a_3(k,h)s_\infty \\ & s_\infty^\perp = a_4(k) + a_4(k)s_\infty^\perp \\ & |a_3(k,h)| < 1, \quad |a_4(k)| < 1 \end{aligned}$$

The optimal gains k^*, h^* resulting from the previous optimization problem can be obtained numerically either by explicitly removing the variables $s_\infty^\perp, s_\infty$ using the constraints and the finding the minimum over k and h, or by using Lagrange multipliers and then applying an iterative algorithm that converges to the unique solution. In the interest of space these steps are not reported here and only the numerical results are presented.

The optimal gains h^* and k^* are a function only of the total number of agents N and the number of communication links ν. Fig. 2 shows the optimal gains as a function of ratio of communication links for $N = 20$. As expected when the number of communication links is small, more weight is placed on the gain k^* related to the feedback about the a-priori expected position of the agents center of mass. When more information is exchanged among agents then progressively more weight is placed on the gain h^*.

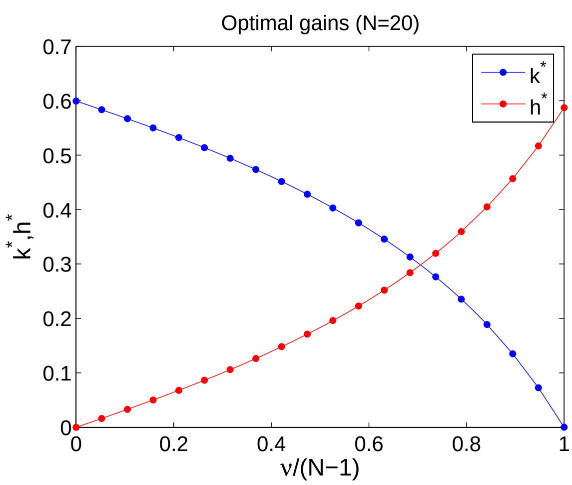

Fig. 2. Optimal gains h^* and k^* for $N = 20$ and cost weights $r = q = 1$ as a function of the ratio between the messages received by each agent ν and the maximum number of messages that one agent can receive, i.e. $N-1$. The dots correspond to the actual instantiation of $\nu = 1, \ldots, N-1$. The choice $\nu/(N-1) = 0$ corresponds to no communication between agents and $\nu/(N-1) = 1$ corresponds to a fully connected communication graph.

Although the optimal gains k^*, h^* are function of the number of agents N, this dependence becomes smaller as N increases and they converge to a constant value as shown in Fig. 3.

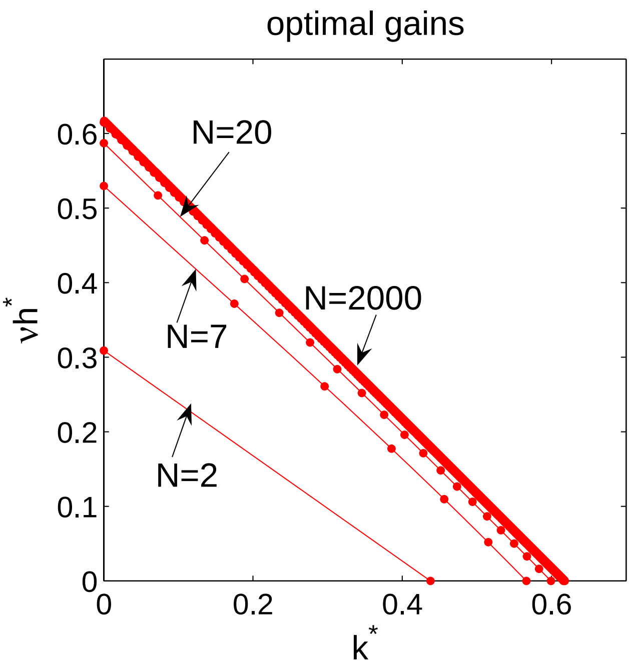

Fig. 3. Optimal gains locus for $r = q = 1$ and different total agents number N. The optimal gains h^* is multiplied by the number of exchanged messages ν to compensate for the fact that the gain h^* must decrease as the number of exchanged messages decreases. As the number of agents increases the locus of optimal gains h^*, k^* converges to a the line $\nu h^* + k^* = \frac{s_{are}}{s_{are}+r} = 0.618$, where s_{are} is defined in Case 2 in the text.

Also the average total cost per vehicles J^*/N for the optimal gains is converges as the number of agents increases as shown in Fig. 4. This is because the difference between the position of the a-priori knowledge of the agents center of mass and the actual position center of mass for a specific realization of agents' initial positions becomes smaller and smaller as the number of agents increases.

IV. Conclusions

In this paper we studied the trade off with respect to rate of convergence, energy consumption and number of information exchanged for a simple model of rendezvous of mobile vehicles. We approached the problem considering a randomized communication scheme where the number of messages exchanged by each vehicle per unit time was fixed and we formulated the control problem as a stochastic linear quadratic optimization problem. We showed how increase of information exchange can improve performance, as one would expect. However we also showed that if some a-priori knowledge about the initial distribution of agents's positions can be effectively used to limit performance degradation when little information exchange among vehicles is available.

Another important observation is that randomized communication schemes result in a fast information distribution

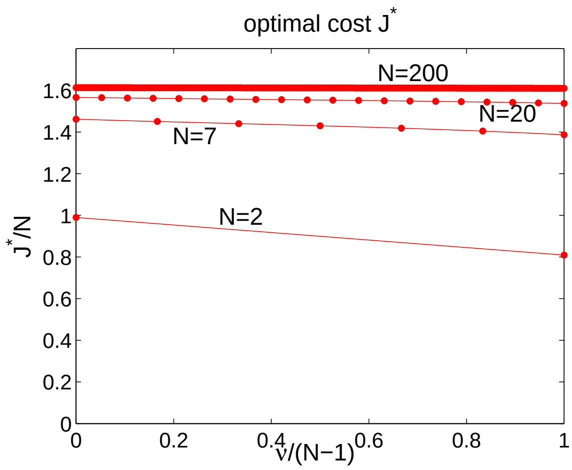

Fig. 4. Optimal average cost per agent J^*/N as a function of the ratio between the messages received by each agent ν and the maximum number of messages that one agent can receive for different N, for $r = q = 1$. The cost is a decreasing function of the number of communication links ν but it converges to the constant value $s_{are} = 1.618$, where s_{are} is defined in Case 2 in the text.

among all agents and thus increasing performance. In other words, it seems that fixed communication schemes might hamper performance, unless they are properly designed based on vehicles topology [2], while randomized communication schemes are easy to implement and have almost optimal performance. To be fair, however, it is important to remark that purely randomized schemes might not be possible in practice. In fact, communication among agents can be established only if they are sufficiently close to each other. For example, in [4] authors considered disk-communication, i.e. agents could communicate only if within a certain distance. A more realist model would be to consider a probabilistic communication model where the probability of successfully exchange a packet depends on the distance. This approach is currently under investigation.

Other possible research directions are the extension of the results presented in this work to more realistic agents dynamics, to synchronization of unstable or oscillatory systems, and to dynamics with process and measurements noise.

References

[1] J. Fax and R. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1465–1476, September 2004.

[2] R. Carli, F. Fagnani, A. Speranzon, and S. Zampieri, "Communication constraints in the state agreement problem," *Submitted for publication*, 2005.

[3] H. Tanner, A. Jadbabaie, and G. Pappas, "Flocking in fixed and switching networks," *submitted to IEEE Transaction on Automatic Control*.

[4] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6.