

The value of communication in the voltage regulation problem

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Abstract—We consider the problem of controlling the reactive power injection of microgenerators in order to regulate the voltage profile in a power distribution network. We formulate a large class of purely local controllers which includes most of the solutions proposed in the literature and in the latest grid code drafts, and we show that these strategies do not guarantee the desired regulation; namely, that for each of them there are equilibria that are not feasible with respect to the desired voltage constraints. We then show that, by adding short range communication between microgenerators, it is possible to design control strategies that provably converge to the feasible set, and we propose one possible strategy. This fundamental performance gap between local and networked strategies is finally illustrated via simulations.

I. INTRODUCTION

Traditionally, the main task of the power distribution grid used to be to deliver power from the transmission grid to the consumers, in a mono-directional fashion. Proper operation of the distribution grid has therefore been mostly a planning/design problem (*fit-and-forget*) for the distribution network operator. Such a planning has been done typically based on a worst-case analysis of the power demand (peak load), resulting in a conservative but acceptable overdimensioning of the infrastructure.

Today's power distribution grid, however, is witnessing some unprecedented challenges [1]–[4], including a large penetration of distributed microgenerators from renewable power sources and a larger diffusion of electric mobility.

Because of these new challenges, a fit-and-forget approach to the management of the grid will not suffice any more. In particular, the voltage profile of low and medium voltage networks is affected by these bidirectional active power flows, and both overvoltage and undervoltage conditions are expected to happen increasingly often. An avenue that is currently being explored by both researchers and practitioners, consists in providing microgenerators with some sensing and computation capabilities, and to exploit the flexibility of their power electronic interface to inject (or withdraw) reactive power from the grid. If properly controlled, these devices can act as a finely distributed network of reactive power compensators, providing a valuable ancillary service to the distribution grid and, ultimately, being an enabler for larger

generation from renewable sources, widespread electric mobility, better grid efficiency, and postponed reinforcement.

Because of the lack of full state monitoring of the distribution grid, most of the efforts towards reactive power control for voltage regulation have focused on purely local feedback strategies (see Figure 1). According to these strategies, the reactive power injection of the power inverter is adjusted based on real time measurements that can be performed at the point of connection of the power inverter to the grid [5]. Different variations have been proposed. In most cases, the reference for reactive power injection is computed as a static function of the measured voltage amplitude, often with a deadband and/or saturation [6]. Since the former strategies could lead to oscillatory behaviors, smoother incremental algorithms have been also proposed, in which the power injection is adjusted based on both the voltage amplitude and the current reactive power setpoint [7], [8].

In some strategies, the static feedback is complemented by a feedforward term, function of the local active and reactive power demand [9], [10]. An offline optimization of the static feedback of these strategies (namely of the slope factor and of the thresholds) based on the analysis of the voltage sensitivity matrix, has been suggested in [11]. In other works, the authors build a separable cost function and then perform a gradient projected descend, until they reach the equilibrium [12], [13]. Finally, a local incremental controller (where the voltage violation is accumulated) has been proposed in [14].

Based on the promising results provided in these works, purely local reactive power control strategies for inverter-connected microgenerators have also been considered for inclusion in the latest revisions of some distribution grid codes [15]–[17]. On the other hand, it has been empirically observed that these strategies might underperform when compared to “benchmark” solutions, where a centralized controller has access to the entire network state and can optimally dispatch reactive power compensators [18], [19]. Furthermore, it has been observed how agent-to-agent communication can be beneficial for this application [20], [21].

In this paper, we investigate whether there is a *fundamental gap* between the performance of purely local voltage regulation strategies, and distributed strategies in which a minimal amount of communication between agents is allowed.

We start by formulating a general class of purely local controllers, which contains the aforementioned examples. The proposed class of controllers includes the smaller class defined in [22], which only models static maps from the measured voltages to the reactive power control. We show via a counterexample how it is possible to construct scenarios where all these controllers are not effective in regulating the

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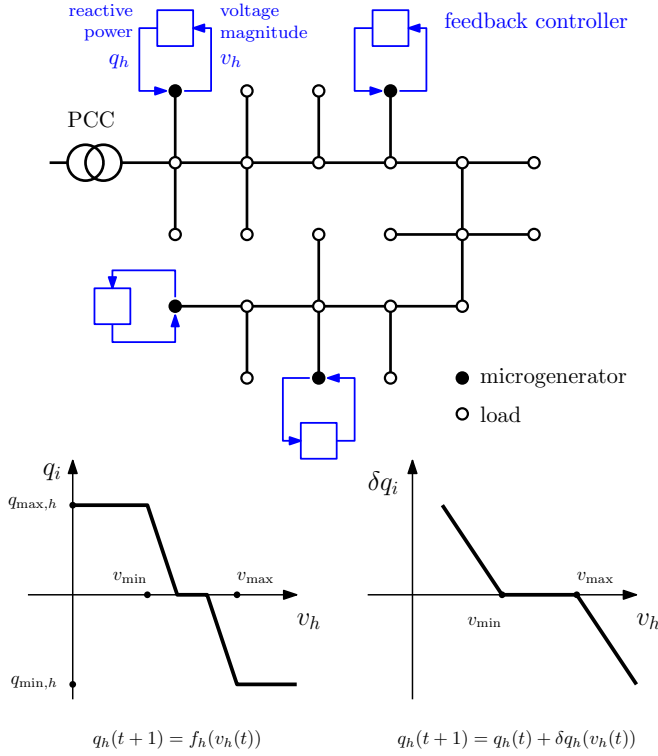


Fig. 1. Example of static (left panel) and incremental (right panel) purely local feedback laws for voltage regulation via reactive power injection.

voltage between the desired limits, even if the available reactive power resources would allow it, if properly dispatched. In other words, local feedback control laws fail to drive the system to a feasible operating point, despite this being inside the reactive power capability region of the inverters.

Inspired by the recent finding by the authors [23], we then present a fully distributed (leader-less) reactive power control strategy in which each compensator only needs to communicate with a limited set of neighboring compensators, leaving all other buses (including loads) unmonitored. This strategy provably converges to an feasible operating point, that guarantees satisfaction of the grid voltage limits.

In order to illustrate this result, we simulate both the local strategies and the proposed distributed strategy on the IEEE 123-bus test feeder.

A. Mathematical notation

We define by $\mathbf{1}$ the column vector of all ones, while by e_v we the vector whose value is 1 in position v , and 0 everywhere else.

Given $u, v, w \in \mathbb{R}^\ell$, with $v_h \leq w_h, h = 1, \dots, \ell$ we define the operator $[u]_v^w$ as the component wise projection of u in the set $\{x \in \mathbb{R}^\ell : v_h \leq x_h \leq w_h, h = 1, \dots, \ell\}$, that is,

$$([u]_v^w)_h = \begin{cases} u_h & \text{if } v_h \leq u_h \leq w_h \\ v_h & \text{if } u_h < v_h \\ w_h & \text{if } u_h > w_h \end{cases} \quad (1)$$

Finally, we denote by \bar{u} , $\text{Re } u$, and $\text{Im } u$, the complex conjugate, the real part, and the imaginary part of u , respectively.

II. POWER DISTRIBUTION GRID MODEL

We consider a grid connected, balanced, radial power distribution network, on which we define the following steady state quantities for each bus $h \in \mathcal{V} := \{1, \dots, n\}$:

- v_h voltage magnitudes
- θ_h voltage angles
- p_h active power injections
- q_h reactive power injections

We then define v (and similarly θ, p, q) as the vectors containing all the scalar quantities v_h (respectively θ_h, p_h, q_h).

All power flows that are compatible with the physics of the grid (namely with Kirchhoff's and Ohm's law) must satisfy the nonlinear complex-valued equation

$$\text{diag}(u) \bar{Y} u = s \quad (2)$$

where $u_h = v_h e^{j\theta_h}$ and $s_h = p_h + jq_h$ denote the complex bus voltages and complex bus power injections, respectively, and where Y is the bus admittance matrix of the grid. We neglect shunt admittances and therefore assume $Y\mathbf{1} = 0$.

We label the PCC as node 1 and consider it as an ideal sinusoidal voltage generator (*slack bus*) at the grid nominal voltage $v_1 = 1$, with arbitrary, but fixed, angle θ_1 . We model all nodes except the PCC as *constant power buses*. These include both loads and microgenerators.

We adopt a linearized model to express the relation between voltages and nodal powers in the grid. It is shown in [24] that by linearizing the power flow equations around a flat voltage profile (corresponding to a no-load condition of the grid) one gets the implicit relation

$$\begin{bmatrix} \text{Re } Y & -\text{Im } Y \\ -\text{Im } Y & -\text{Re } Y \end{bmatrix} \begin{bmatrix} v \\ \theta \end{bmatrix} \approx \begin{bmatrix} p \\ q \end{bmatrix}. \quad (3)$$

It can be shown [20, Lemma 1] that there exists a unique symmetric, positive semidefinite matrix $X \in \mathbb{C}^{n \times n}$ such that

$$\begin{cases} YX = I - e_1 \mathbf{1}^\top \\ Xe_1 = 0, \end{cases} \quad (4)$$

called the Green matrix, which depends only on the topology of the grid power lines and on their impedances, and whose elements are all non-negative. This matrix allows to derive the following convenient explicit expression for the voltage magnitudes.

Lemma 1. *Let Y be a bus admittance matrix satisfying $Y\mathbf{1} = 0$, and let X be defined as in (4). Then the expression*

$$v = \mathbf{1} + \text{Re } Xp + \text{Im } Xq \quad (5)$$

satisfies the linearized power flow model (3).

Proof. The statement can be proved by inspection, plugging (5) into (3), together with

$$\theta = \text{Im } Xp - \text{Re } Xq,$$

and using the first of the properties (4) of the Green matrix expressed in rectangular coordinates, i.e.

$$\begin{cases} \text{Re } Y \text{Re } X - \text{Im } Y \text{Im } X = I - e_1 \mathbf{1}^\top \\ \text{Im } Y \text{Re } X + \text{Re } Y \text{Im } X = 0. \end{cases}$$

□

Equation (5) models the well known fact that the injection or the absorption of reactive power increase or decrease, respectively, the voltage magnitude also in the case of not purely inductive lines. Notice in fact that we have made no assumption on the X/R ratio of the lines.

The quality of this linearization can be studied following the analysis in [25], and relies on having large nominal voltage of the grid and relatively small nodal currents. This assumption is verified in practice, and corresponds to correct design and operation of distribution networks, where indeed the nominal voltage is chosen sufficiently large in order to deliver power to the loads with relatively small power losses.

Finally, we assume that microgenerators are connected to a subset of the grid buses, namely $\mathcal{C} \subseteq \mathcal{V}$ (with $|\mathcal{C}| = m$). As in Figure 1, each microgenerator is provided with some

- *sensing capabilities*, so that it can take local voltage magnitude measurements;
- *computational capabilities* that will be exploited to implement the control algorithms;
- *actuation capabilities*, being able to regulate the amount of reactive power injected into the grid.

In order to underline the difference among microgenerators and passive loads in the notation, we introduce the following block decomposition of the voltage magnitude vector v

$$v = [v_1 \quad v_G^T \quad v_L^T], \quad (6)$$

where v_1 is the voltage magnitude at the PCC, $v_G \in \mathbb{R}^m$ are the voltage magnitudes at the microgenerators, and $v_L \in \mathbb{R}^{n-m-1}$ are the voltage magnitudes at the loads. Similarly, we also define $s_G = p_G + jq_G$ and $s_L = p_L + jq_L$. Accordingly with the same partitioning, we can block-partition the matrix X as

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & X_{GG} & X_{GL} \\ 0 & X_{LG} & X_{LL} \end{bmatrix}.$$

The structure of X descends from the property of the Green matrix, and it can be shown that under reasonable values of the grid admittance parameters, the block X_{GG} is invertible. With this decomposition, we then have

$$v_G = \mathbf{1} + X_{GG}^R p_G + X_{GL}^R p_L + X_{GG}^I q_G + X_{GL}^I q_L, \quad (7)$$

where the superscripts R and I indicate the real and imaginary part of the block, respectively.

III. VOLTAGE CONTROL VIA REACTIVE POWER REGULATION

In this Section, we formulate the problem of controlling the reactive power injected (or withdrawn) by the microgenerators in order to regulate the voltage profile of the distribution feeder. Since we assume that only microgenerators can measure their bus voltage, we consider the constraints

$$v_{\min} \leq v_h \leq v_{\max}, \quad \forall h \in \mathcal{C}, \quad (8)$$

where v_{\min} and v_{\max} are, respectively, the minimum and maximum admissible values for the voltage magnitudes.

Typical scenarios include both symmetric bounds around the nominal voltage (e.g. $\pm 10\%$) and asymmetric bounds (e.g. $v_{\min} = 0.87, v_{\max} = 1.06$).

In addition, since the generators deployed in the distribution network are, typically, of small size, we need to take into account also constraints on their generation capabilities. Precisely, we assume that

$$q_{\min,h} \leq q_h \leq q_{\max,h}, \quad \forall h \in \mathcal{C}, \quad (9)$$

where $q_{\min,h}, q_{\max,h}$ denote, respectively, the minimum and the maximum amount of reactive power that can be injected by agent h . In most cases $q_{\min,h} \leq 0$ and $q_{\max,h} \geq 0$.

Based on the constraints in (8) and in (9), we introduce a proper definition of the set of the *feasible reactive power injections*. Observe that, in the setup we consider, the quantities p_G, p_L and q_L are assumed to be constant and that only q_G is actuated in order to regulate v_G ; in other words, v_G can be described as a function of q_G . In particular a given q_G is said to be feasible if it satisfies (9) and if the induced v_G satisfies (8). More formally, for a given triple (p_G, p_L, q_L) , we define

$$\mathcal{F}(p_G, p_L, q_L) = \{q_G \text{ such that } \forall h \in \mathcal{C} \text{ it holds } q_{\min,h} \leq q_h \leq q_{\max,h}, v_{\min} \leq v_h \leq v_{\max}\}.$$

Since there is no risk of confusion, for the sake of notational convenience, we omit the dependence of \mathcal{F} on (p_G, p_L, q_L) .

The goal of a reactive power control strategy is to drive the reactive power injection of the microgenerators to a point that belongs to the set \mathcal{F} , for any initial condition.

In all the strategies that we consider in this paper, microgenerators measure periodically and synchronously the magnitudes of their voltages; namely, all the agents take their measurements at time-instants τ_0, τ_1, \dots , where $\tau_t = tT$, for a given sampling time T . Based on those measurements, they synchronously update their reactive power injection, and hold the same value until the next measurement. In the following, since there is no risk of confusion, we will denote the t -th iteration of the various algorithms just by the index t .

IV. A CLASS OF PURELY LOCAL CONTROL STRATEGIES

In this section, we define a family of *purely local* strategies, in which each agent h updates q_h based only on its current reactive power injection and on the measurements of the magnitude of its own voltage, i.e., v_h ; in these strategies, agents do not communicate with each other. The family we introduce includes most of the purely local strategies that have been recently proposed in the literature and in the latest grid code drafts, as reviewed in the Introduction.

In these strategies, the reactive power output of each microgenerator $h \in \mathcal{C}$ can be expressed as

$$q_h(t+1) = g_h(q_h(t), v_h(t)). \quad (10)$$

We will say that a controller

$$g_h : [q_{\min}, q_{\max}] \times \mathbb{R}_{\geq 0} \rightarrow [q_{\min}, q_{\max}]$$

belongs to the class \mathcal{G} if it meets the following properties.

- 1) for a given $q \in [q_{\min,h}, q_{\max,h}]$, the function $g_h(q, \cdot)$ is continuous;
- 2) for every $q, q' \in [q_{\min}, q_{\max}]$, with $q > q'$, it must hold that

$$g_h(q, v) - g_h(q', v) < q - q'; \quad (11)$$

- 3) $g_h(q, v)$ is a weakly decreasing function of v , i.e. for every $0 \leq v \leq v'$,

$$g_h(q, v) \geq g_h(q, v'); \quad (12)$$

- 4) $g_h(q, v)$ must satisfies the following condition

$$g_h(0, 1) = 0. \quad (13)$$

Remark. \mathcal{G} is a wide class, containing several algorithms proposed in the literature. For instance, all the strategies in which the power injection is computed as a static function of the measured voltage amplitude, i.e. those that can be expressed as

$$g_h(q, v) = f_h(v) \quad (14)$$

where $f_h(v)$ is a function of the type depicted in the left panel of Figure 1, fit in \mathcal{G} . As reviewed in the Introduction, the formulation (14) describes a large number of proposed control strategies.

Furthermore, also the incremental versions of (14), whose update rules are either

$$g_h(q, v) = [q + \gamma(f_h(v) - q)]_{q_{\min,h}}^{q_{\max,h}} \quad (15)$$

or

$$g_h(q, v) = [q + \gamma(f_h^{-1}(q) - v)]_{q_{\min,h}}^{q_{\max,h}} \quad (16)$$

(presented respectively in [7] and [8]), belong to \mathcal{G} .

A configuration (q_h^*, v_h^*) is an equilibrium for the algorithms g_h if it satisfies the equation

$$q_h^* = g_h(q_h^*, v_h^*) \quad (17)$$

The equilibria of the algorithms in \mathcal{G} have a notable feature: the reactive power output of each agent can be exactly inferred with the knowledge of its equilibrium voltage.

Proposition 2. Let $g_h(q, v)$ belongs to \mathcal{G} . Given $v^* \in \mathbb{R}_{\geq 0}$, there exists only one q^* such that equation (17) holds.

Proof. Let us define the function

$$h(q, v) := g_h(q, v) - q.$$

It is a continuous function, such that for every v

$$\begin{aligned} h(q_{\min}, v) &= g_h(q_{\min}, v) - q_{\min} \geq 0 \\ h(q_{\max}, v) &= g_h(q_{\max}, v) - q_{\max} \leq 0 \end{aligned}$$

Fix $v \in \mathbb{R}_{\geq 0}$. Furthermore, $h(\cdot, v)$ is a strictly decreasing function. In fact, if we consider $q_1 > q_2$, we have that

$$h(q_1, v) - h(q_2, v) = g_h(q_1, v) - g_h(q_2, v) - q_1 + q_2 < 0$$

Thus, there exists a unique configuration q such that $h(q, v) = 0$, i.e. $g_h(q, v) = q$. \square

Thanks to Proposition 2, we can define, for each agent h , the equilibria function

$$\begin{aligned} F_h : \mathbb{R}_{\geq 0} &\rightarrow [q_{\min,h}, q_{\max,h}] \\ v &\mapsto q : q = g_h(q, v) \end{aligned}$$

which, given the equilibrium voltage of agent h , returns its reactive power output.

The next Proposition studies some properties of the equilibria function.

Proposition 3. Let $F_h(v)$ be the equilibria function associated with the controller $g_h(q, v) \in \mathcal{G}$.

- 1) $F_h(v)$ is weakly decreasing.
- 2) $F_h(v)$ is continuous.
- 3) $F_h(1) = 0$

Proof. Let $v, v' \in \mathbb{R}_{\geq 0}$, $v > v'$, and let $q = F_h(v)$, $q' = F_h(v')$. Let us assume $q > q'$. Then, being $g_h(q, \cdot)$ a non-increasing function,

$$\begin{aligned} q' &< g_h(q, v') + q' - q \\ &\leq g_h(q, v) + q' - q = q' \end{aligned}$$

which is absurd.

Standard analysis results state that a non-increasing function whose image is a connected set is continuous. Thus, in order to prove the continuity, we just need to prove that the image of $F(v)$ is a connected set. To this aim, consider $v, v' \in \mathbb{R}_{\geq 0}$, $v > v'$ and $q = F_h(v)$, $q' = F_h(v')$, $q \leq q'$. For every $q'' < q' < q$, from equation (11), we have that

$$g_h(q'', v) < g_h(q, v) + q'' - q = q'' \quad (18)$$

$$g_h(q'', v') > g_h(q', v') + q'' - q' = q'' \quad (19)$$

Since $g_h(q'', \cdot)$ is a continuous function, there exists $v'' \leq v' \leq v$ such that $g_h(q'', v'') = q''$, and thus q'' belongs to the image of $F(v)$.

Equation $F(1) = 0$ follows trivially from equation (13). \square

Remark. For the particular local control strategies described by the update laws (14), (15), (16), it can be easily shown that the equilibria function is

$$F_h(v) = f_h(v).$$

In addition, it is worth mentioning that there are strategies that do not fit in \mathcal{G} , for instance the local algorithm proposed in [14], which can be expressed in the form

$$g_h(q, v) = [q + f_h(v)]_{q_{\min,h}}^{q_{\max,h}} \quad (20)$$

The former algorithm would fit instead in a more general class composed by the controllers that, instead of (11), satisfy

$$g_h(q, v) - g_h(q', v) \leq q - q'.$$

The difficulty in this case is that the equilibria function is a set valued function i.e., given a value of v , there is set of reactive power output for which equation (17) holds. The

characterization of this more general class of algorithms represents a future extension of this paper.

So far, we characterized the equilibria of a single controller. There remains the open question of what happens when every agent in a smart distribution grid is commanded by a local controller belonging to \mathcal{G} . Let us define the function

$$F(\cdot) : \mathbb{R}_{\geq 0}^m \rightarrow [q_{\min,1}, q_{\max,1}] \times \cdots \times [q_{\min,m}, q_{\max,m}]$$

$$[F(v)]_h = F_h(v).$$

$F(\cdot)$ is a diagonal map with entries that are weakly increasing. If there exists a global equilibrium (q_G, v_G) , it must solve the following system

$$q_G = F(v_G)$$

$$v_G = X_{GG}^I q_G + b$$

or, equivalently, the equation

$$v_G = X_{GG}^I F(v_G) + b \quad (21)$$

where

$$b = \mathbf{1} + X_{GG}^R p_G + X_{GL}^R p_L + X_{GL}^I q_L.$$

In principle there could exist one, many or even zero solutions of (21). Since X_{GG}^I is positive definite we can write

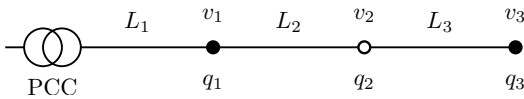
$$(X_{GG}^I)^{-1} v_G - F(v_G) = (X_{GG}^I)^{-1} b,$$

apply the result in [26] and prove this way that, for any b , equation (21) has a unique solution in v_G .

Now that we have introduced a well defined function F that maps voltage profiles v_G into equilibrium reactive power injections q_G , we can discuss the effectiveness of this class of local strategies for the regulation of the voltage.

In the remainder of this section we provide a simple counterexample in which, for any controller in \mathcal{G} , there exist an equilibrium of the algorithm that does not belong to \mathcal{F} (i.e., it is not feasible), even if $\mathcal{F} \neq \emptyset$ (i.e., a feasible reactive power injection exists).

A. A simple example of the ineffectiveness of local strategies



Consider a network composed of four nodes (PCC, a microgenerator, a load, and another microgenerator) connected forming a line, as in figure, via inductances L_1 , L_2 , and L_3 . We assume, with minimal loss of generality, that $v_{\min} = 1 - \delta$ and $v_{\max} = 1 + \delta$, for a given $\delta > 0$. We aim at showing that, for any value of L_1 , L_2 , and L_3 , there exist values of $q_{\max,1}$, $q_{\max,3}$, and q_2 , such that

- i) all local algorithms of the class \mathcal{G} have a unfeasible equilibrium $(q_1^*, q_3^*) \notin \mathcal{F}$;
- ii) there exists a feasible reactive power injection ($\mathcal{F} \neq \emptyset$).

As all lines are purely inductive, the linearized model gives

$$v_1 = 1 + L_1 q_1 + L_1 q_2 + L_1 q_3$$

$$v_2 = 1 + L_1 q_1 + (L_1 + L_2) q_2 + (L_1 + L_2) q_3$$

$$v_3 = 1 + L_1 q_1 + (L_1 + L_2) q_2 + (L_1 + L_2 + L_3) q_3.$$

We know that for any q_2 a equilibrium $(q_1^*, v_1^*), (q_3^*, v_3^*)$ exists. We first show that, if

$$q_2 = Q := -\frac{1}{L_2} \delta - \frac{L_2 + L_3}{L_2} q_{\max,3}, \quad (22)$$

then $v_3^* < 1 - \delta$. Observe that

$$v_1^* - v_3^* = -L_2 q_2 - (L_2 + L_3) q_3^*$$

$$\geq -L_2 Q - (L_2 + L_3) q_{\max,3} = \delta.$$

Observe moreover that, if we define $\phi_1(v_1) := v_1 - 1 - L_1 F_1(v_1)$, then

$$\phi_1(v_1^*) = L_1 q_2 + L_1 q_3^*$$

$$\leq L_1 Q + L_1 q_{\max,3} = -\frac{L_1}{L_2} \delta - \frac{L_1 L_3}{L_2} q_{\max,3} < 0.$$

Since $\phi_1(v_1)$ is strictly increasing and since $\phi_1(1) = 0$, we can argue that $v_1^* < 1$ which, together with the fact that $v_1^* - v_3^* < \delta$ yields $v_3^* < 1 - \delta$. This proves that if $q_2 = Q$ (and, in fact, if $q_2 \leq Q$), then the equilibrium is not feasible.

In order to complete the example we show under which conditions there exist reactive powers $q_1 \in [q_{\min,1}, q_{\max,1}]$, $q_3 \in [q_{\min,3}, q_{\max,3}]$ such that $v_1, v_2 \in [1 - \delta, 1 + \delta]$, when $q_2 = Q$. Let us take $q_3 = q_{\max,3}$. Then

$$v_1 = 1 + L_1 q_1 - \frac{L_1 L_3}{L_2} q_{\max,3} - \frac{L_1}{L_2} \delta$$

$$v_3 = 1 + L_1 q_1 - \frac{L_1 L_3}{L_2} q_{\max,3} - \frac{L_1 + L_2}{L_2} \delta = v_1 - \delta$$

From this we can argue that the state is feasible if and only if q_1 is such that $v_1 \in [1, 1 + \delta]$. This condition is equivalent to the fact that

$$\frac{L_3}{L_2} q_{\max,3} + \frac{1}{L_2} \delta \leq q_1 \leq \frac{L_3}{L_2} q_{\max,3} + \frac{1}{L_2} \delta + \frac{1}{L_1} \delta.$$

We can conclude that there exists $q_1 \in [q_{\min,1}, q_{\max,1}]$ such that the previous inequality holds if and only if

$$q_{\max,1} \geq \frac{L_3}{L_2} q_{\max,3} + \frac{1}{L_2} \delta. \quad (23)$$

It is therefore enough to choose q_2 , $q_{\max,1}$, and $q_{\max,3}$, according to (22) and (23) in order to obtain the desired counterexample that is valid for *all* strategies belonging to the class of local strategies \mathcal{G} .

V. A NETWORKED CONTROL STRATEGY

In this section, after having assessed the limitations of purely local strategies in Section IV, we investigate distributed voltage control strategies (denoted hereafter as DVS) in which communication between agents is allowed.

In particular, we assume that every agent $h \in \mathcal{C}$ can communicate with its neighbors, defined as follows.

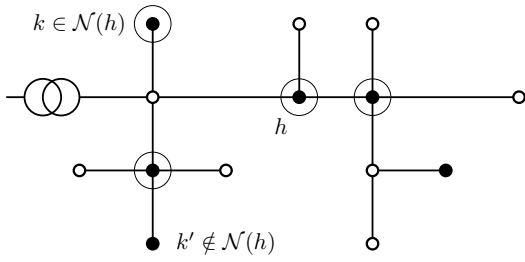


Fig. 2. An example of neighbor microgenerators. Black nodes are microgenerators ($h \in \mathcal{C}$). White nodes are loads. The circled microgenerators belong to the set $\mathcal{N}(h)$ of neighbors of h . For each agent $k \in \mathcal{N}(h)$, the path that connects h to k does not include any other microgenerator.

Definition 4 (Neighbor microgenerators). *Let $h \in \mathcal{C}$ be a microgenerator. The set of neighbors of h , denoted as $\mathcal{N}(h)$, is the subset of \mathcal{C} defined as*

$$\mathcal{N}(h) = \{k \in \mathcal{C} \mid \exists \mathcal{P}_{hk}, \mathcal{P}_{hk} \cap \mathcal{C} = \{h, k\}\},$$

where \mathcal{P}_{hk} is the set of buses that lie on the path going from bus h to bus k (see Figure 2).

The DVS we propose is a dual-ascent like algorithm amenable of distributed implementation which aims at solving the following constrained optimization problem

$$\min_{q_G} \frac{1}{2} q_G^T X_{GG}^I q_G \quad (24a)$$

$$\text{subject to} \quad \begin{aligned} v_{\min} \leq v_h \leq v_{\max} \\ q_{\min,h} \leq q_h \leq q_{\max,h} \end{aligned} \quad \forall h \in \mathcal{C} \quad (24b)$$

To apply the dual decomposition tool, for each $h \in \mathcal{C}$, we introduce the Lagrange multipliers $\lambda_{\min,h}$, $\lambda_{\max,h}$, $\mu_{\min,h}$, $\mu_{\max,h}$ corresponding, respectively, to the constraints $v_h \geq v_{\min}$, $v_h \leq v_{\max}$, $q_h \geq q_{\min,h}$, $q_h \leq q_{\max,h}$. Accordingly let the Lagrangian be defined as

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} q_G^T X_{GG}^I q_G + \\ & \sum_{h \in \mathcal{C}} \lambda_{\min,h} (v_{\min} - v_h) + \sum_{h \in \mathcal{C}} \lambda_{\max,h} (v_h - v_{\max}) + \\ & \sum_{h \in \mathcal{C}} \mu_{\min,h} (q_{\min,h} - q_h) + \sum_{h \in \mathcal{C}} \mu_{\max,h} (q_h - q_{\max,h}) \end{aligned}$$

In the proposed strategy, agent h keeps alternating minimizations on the primal variable q_h with dual-ascent steps on the dual variables $\lambda_{\min,h}$, $\lambda_{\max,h}$, $\mu_{\min,h}$, $\mu_{\max,h}$. Precisely agent h iteratively executes the following actions in order:

- 1) it measures its voltage magnitude $v_h(t)$ and it gathers from its neighbors the values of the multipliers

$$\{\mu_{\min,k}(t), \mu_{\max,k}(t), k \in \mathcal{N}(h)\};$$

- 2) it computes the reactive power set-point

$$\begin{aligned} \hat{q}_h = & \lambda_{\min,h}(t) - \lambda_{\max,h}(t) + \\ & + \sum_{k \in \mathcal{N}(h)} G_{hk} (\mu_{\min,k}(t) - \mu_{\max,k}(t)) \quad (25) \end{aligned}$$

where G_{hk} are the element of the inverse of the matrix X_{GG}^I , therefore a function of the grid topology and parameters, and are assumed to be known by agent h ;

- 3) it updates its power Lagrange multipliers as

$$\begin{aligned} \mu_{\min,h}(t+1) &= [\mu_{\min,h}(t) + \gamma (q_{\min} - \hat{q}_h)]_0^\infty \\ \mu_{\max,h}(t+1) &= [\mu_{\max,h}(t) + \gamma (\hat{q}_h - q_{\max})]_0^\infty \end{aligned}$$

where γ is a positive constant;

- 4) it updates its voltage Lagrange multipliers as

$$\begin{aligned} \lambda_{\min,h}(t+1) &= [\lambda_{\min,h}(t) + \gamma (v_{\min} - v_h(t))]_0^\infty \\ \lambda_{\max,h}(t+1) &= [\lambda_{\max,h}(t) + \gamma (v_h(t) - v_{\max})]_0^\infty \end{aligned}$$

- 5) it adjusts the amount of injected reactive power to the value $q_h(t+1)$ obtained by projecting \hat{q}_h into the feasible set defined by (9), i.e.,

$$q_h(t+1) = [\hat{q}_h]_{q_{\min,h}}^{q_{\max,h}}. \quad (26)$$

Observe that, in order to perform an iteration of DVS, each agent needs information coming only from its neighbors; in this sense the algorithm is distributed. Further explanations are in order and are provided in the next lemma.

Lemma 5. *The inverse G of X_{GG}^I has the sparsity pattern induced by the Definition 4 of neighbor microgenerators, i.e.*

$$G_{hk} \neq 0 \quad \Leftrightarrow \quad k \in \mathcal{N}(h).$$

Proof. The statement can be proved following the steps in [20, Appendix A]. \square

Lemma 5 is the reason why $\arg \min_{q_G} \mathcal{L}$ can be computed in a distributed way. Indeed from $\partial \mathcal{L} / \partial q_G = 0$ we get

$$X_{GG}^I q_G + \frac{\partial v_G}{\partial q_G} (\lambda_{\max,G} - \lambda_{\min,G}) + \mu_{\max,G} - \mu_{\min,G} = 0.$$

By using the linearized model (7), and by left-multiplying by $\hat{G} = (X_{GG}^I)^{-1}$, we have

$$q_G + \lambda_{\max,G} - \lambda_{\min,G} + \hat{G} (\mu_{\max,G} - \mu_{\min,G}) = 0.$$

Therefore, \hat{q}_h in (25) corresponds element-wise to the $\arg \min_{q_G} \mathcal{L}$, at least in the linearized model.

Remark. *The proposed DVS algorithm deals with both constraints on v_G and on q_G . The voltage constraints are treated as soft constraint, namely, they might be violated during the iterations of the algorithm. Instead the reactive power constraints are treated as hard constraints; indeed thanks to the projection step in (26), they are guaranteed to be satisfied at any time t . The fact that $q_h(t+1)$ is in general different from \hat{q}_h makes DVS algorithm slightly different from the standard dual-ascent algorithm.*

Remark. *In the proposed DVS strategy, we have adopted a particular quadratic cost of the reactive power injections, described by the matrix X_{GG}^I . A discussion on the different options in terms of cost function goes beyond the scope of this paper, as the analysis is focused on whether the different strategies are capable of driving the systems to a feasible reactive power injection (i.e., to the set \mathcal{F}) or not. However,*

it is worth noticing that, in the linearized model, $X_{GG}q_G$ is the voltage drop caused by the reactive power injected by the microgenerators, which we call δv_G . From this point of view, the cost function (24a) can be rewritten as

$$\delta v_G^T G \delta v_G.$$

Given that G is a Laplacian, this cost function promotes, among feasible reactive power injections, those that cause uniform voltage drops. Adopting a different cost function can, in general, require the measurement and exchange of other quantities. In [7], for example, a distributed strategy has been proposed to minimize power distribution losses, assuming that microgenerators can also measure voltage angles θ_h .

A. Some insights on the convergence properties

The combined presence of soft and hard constraints makes difficult to characterize the convergence properties of DVS. In this section we consider a slightly modified version of DVS, denoted as soft-DVS, where each agent h performs steps 1 through 4 as in DVS, but replaces (26) with $q_h(t+1) = \hat{q}_h$. In other words, the projection step is not performed, and soft-DVS can be seen as a distributed implementation of the standard dual ascent algorithm.

Following (7), the set \mathcal{F} can be approximated as

$$\tilde{\mathcal{F}} = \{q_G : \tilde{v}_{\min} \leq X_{GG}^I q_G \leq \tilde{v}_{\max}, q_{\min,G} \leq q_G \leq q_{\max,G}\}$$

where

$$\begin{aligned} \tilde{v}_{\min} &= - (X_{GG}^R p_G + X_{GL}^R p_L + X_{GL}^R q_L) + \mathbf{1}(v_{\min} - 1) \\ \tilde{v}_{\max} &= - (X_{GG}^R p_G + X_{GL}^R p_L + X_{GL}^R q_L) + \mathbf{1}(v_{\max} - 1). \end{aligned}$$

Under the approximation introduced by the linearization, it therefore follows that the problem

$$\begin{aligned} \min_{q_G} \quad & \frac{1}{2} q_G^T X_{GG}^I q_G \\ \text{subject to} \quad & q_G \in \tilde{\mathcal{F}} \end{aligned} \quad (27)$$

is a strictly convex problem. Then, if $\tilde{\mathcal{F}} \neq \emptyset$, there exists a unique solution to (27), which we denote as q_G^* .

The following proposition characterizes the convergence properties of the soft-DVS algorithm.

Proposition 6. *Consider the optimization problem in (27) and assume that $\tilde{\mathcal{F}} \neq \emptyset$. Then the trajectory $t \rightarrow q_G(t)$ generated by the soft-DVS algorithm converges to the optimal solution q_G^* if*

$$\gamma < \frac{2}{\rho(\Phi G \Phi^T)} \quad (28)$$

where $\Phi = [X_{GG}^I \quad -X_{GG}^I \quad I \quad -I]^T$.

VI. SIMULATIONS

In order to illustrate the practical relevance of the proposed analysis, we considered a realistic scenario based on the IEEE 123-bus test feeder [27], with the following modifications:



Fig. 3. The IEEE 123-bus test feeder. In red, the three-phase backbone that has been considered in the simulations, as detailed in [28]. The distribution feeder is connected to the MV network at the PCC. Two nodes (circled in red) host a microgenerator each, both capable also of injecting reactive power in order to support the feeder voltage profile.

- we only considered the three-phase backbone of the feeder, lumping all the single-phase loads into balanced three-phase constant-power loads (see [28] for the resulting 56-bus model);
- in order to observe an overvoltage condition, grid loading has been reduced to 1/4 of the testbed values;
- two microgenerators have been added (buses 10 and 32).

The resulting overvoltage contingency is illustrated in the top panel of Figure 4. In this case, generators at bus 10 and 32 inject active power with unity power factor, i.e. with zero reactive power injection, although they have a reactive power capability of 400 and 200 KVAR, respectively. In multiple buses, the voltage magnitude exceeds the limit of 1.05 p.u.

In the second panel, we simulated the effect of a purely local feedback law like the one proposed in [5], [6], [18] and schematically represented in the left panel of Figure 1. We assumed a deadzone for the voltages $[0.99, 1.01]$ p.u. This family of local static feedback strategies do not achieve the desired voltage regulation, resulting in voltage steady state violation at some buses, including bus 32.

In the third panel, we simulated the effect of a purely local incremental feedback like the one proposed in [8], and schematically depicted in the right panel of Figure 1. Notice that, because of the deadzone, this approach does not strictly

belong to the class of local strategies defined in Section V. However, also in this case, the reactive power control strategy fails to regulate the voltage below the overvoltage limit, and therefore this simulation serves as a counterexample.

Finally, in the bottom panel of Figure 4, we simulated the networked controller DVS proposed in Section V, which makes use of communication between the two microgenerators. Via this coordination strategy, now both the agents participate in the regulation of the voltage, and a feasible voltage profile is achieved. Notice how the availability of a communication channel allows to obtain a form of reactive power sharing between microgenerators, so that also the one at bus 10 (which measures a feasible voltage magnitude at its point of connection) participates to the voltage regulation scheme.

VII. CONCLUSIONS

The analysis proposed in this paper shows how agent-to-agent communication plays a fundamental role in the problem of distributed voltage support. In fact, there is a sharp gap between the performance of the large class of purely local strategies that we considered (which fails to drive the power system to a configuration of feasible voltages) and networked strategies (which provably do so, even with just short range communication).

This result strongly suggests that the larger class of networked feedback laws should be considered for the design of voltage regulation strategies in distribution networks, and sheds light on the value of communication for this challenging and timely power system application.

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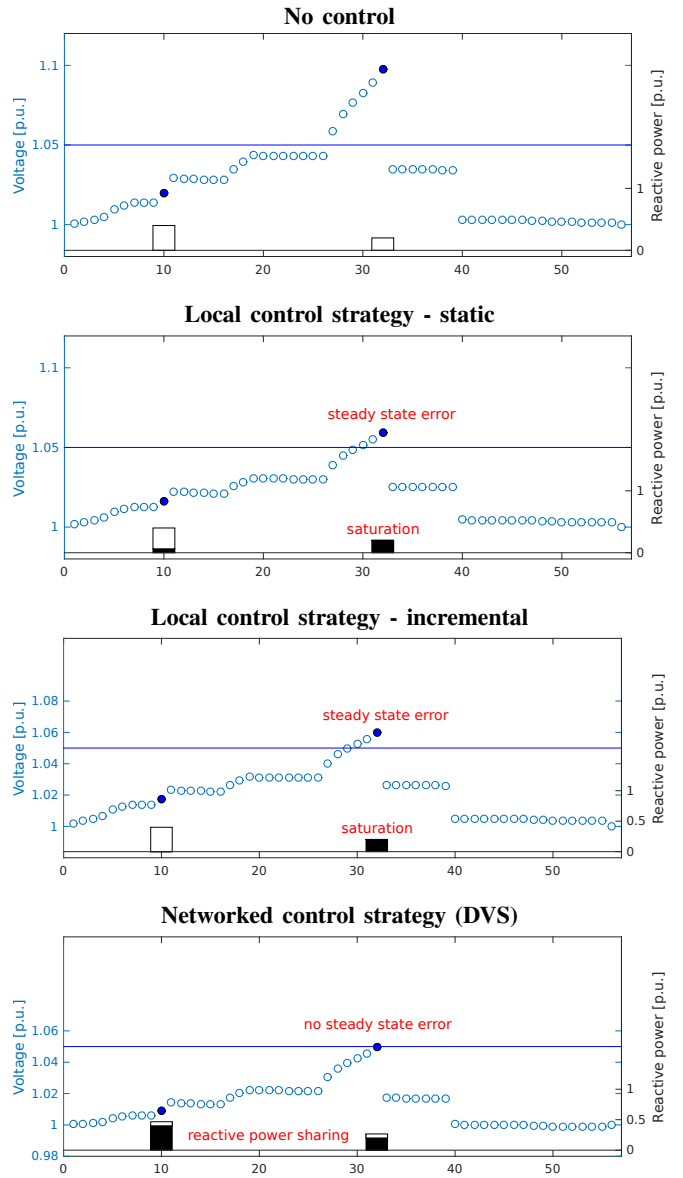


Fig. 4. Simulated voltage profile of the three-phase backbone of the IEEE 123-bus test feeder, when four different reactive power control strategies are employed: zero reactive power, local static feedback, local incremental feedback, and networked control. The empty bars represent the reactive power capability of the microgenerators. The black bars represent the amount of reactive power that is withdrawn, at steady state.

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