A symbolic approach to performance analysis of quantized feedback systems: the scalar case

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Abstract

When dealing with the control of a large number of interacting systems, the fact that the flow of information has to be limited becomes an essential feature of the control design. The first consequence of the limited information flow constraint is that the signals that the controllers and the systems exchange have to be quantized. Though quantization has already been extensively considered in the control literature, its analysis from the point of view of the information flow demand has been considered only recently.

Limiting the information flow between a plant and a controller will necessary lead to a performance degradation of the feedback loop, and we expect a trade-off between the achievable performance and the amount of information exchange allowed in the loop.

Most of the success of modern digital communication theory in the last fifty years is due to the contributions of information theory, which proposed a symbolic based analysis of the communication channel performance. The same goal is much more difficult to be reached in digital control theory.

The present paper proposes an attempt towards this direction. The main contribution of this paper is to provide a complete analysis of the trade-off between performance and information flow in the simple case of the stabilization of a scalar linear system by means of a memoryless quantized feedback map.

Keywords: Stability, stabilization, communication constraint, quantized feedback, chaotic control, symbolic dynamics, Markov chains, entropy.

AMS subject classification: 93D15, 37B10, 37E05

1 Introduction

The stabilization by quantized feedback controllers has been widely investigated in the last few years (see [6, 26, 3, 25, 20, 7, 2, 9, 23] and the reference therein). There are two different situations in which quantization reveals to be a central feature in the control design. The first is related to control systems in which either the sensors or the actuators limitations impose that their measures or their commands can take a limited number of different values. In this case the number of quantization levels provides a measure of the sensor or of the actuator complexity. Another situation in which quantization plays an important role is when plants and controllers exchange information through digital communication channels with limited capacity. In this last case, the measures and the commands need to be quantized before being communicated and the number of quantization levels is strictly related with information flow between the components of the control system and so with the capacity required to transmit the control information.

Two different approaches have been proposed in the literature to solve the control problem with quantized feedback. The first approach considers memoryless feedback quantizers. In particular in [6] there is a first mathematical analysis of control systems with uniform quantized feedback, while in [26, 2] a first bound of the number of quantization intervals needed to stabilize a linear system is proposed. In [7] logarithmic quantizers are shown to yield Lyapunov stability. In [9] a chaos based quantized controller has been proposed and a first comparison between uniform, logarithmic and chaotic quantized feedback controllers has been presented in the scalar case. In [23] performance of uniform quantized feedback controllers are analyzed for general linear systems.

The second approach considers quantized feedback controllers with an internal state. In particular [3] proposes a stabilization technique in which the quantizer is scaled according to the state growth. In [25] this technique is used to show the relation between the degree of instability of the system to be controlled and the number of quantization levels of the feedback quantizer. The same relation has be found independently in [20] in a different context.

In general the analysis of memoryless quantized feedback controllers is hard, while the results become quite neat for quantized feedback controllers with infinite memory. Notice that, while it is reasonable to allow a memory structure on sensors and actuator when designing control systems under communication constraint [25, 20], in situations in which quantization is due to sensors or to actuators poorness, only the memoryless quantized feedback controller becomes a reasonable model.

The present paper considers memoryless quantized controllers for which, as we mentioned, a mathematical analysis is more complicated. The relation between controller complexity and controller performance is investigated by using information theoretic and combinatorial techniques. One of the main contributions of this paper is showing that the controller performance has to be described by two conflicting parameters, one evaluating the steady state, and the other evaluating the transient of the controlled system. Roughly speaking we proved that, for a fixed controller complexity, good steady state implies bad transient and vice versa.

More precisely in this paper we consider the stabilization problem for discrete time linear systems with one-dimensional state, namely a system described by the equation

$$x_{t+1} = ax_t + u_t \, .$$

While in the classical control setting this stabilization problem is completely trivial and there is very little to be said, in the memoryless quantized feedback setting non trivial issues already come up in this simple situation. In this set up a memoryless quantized feedback is a control law $u_t = k(x_t)$, where $k(\cdot)$ is a quantized (i.e. piecewise constant) map. Let N be the number of distinct values which $k(\cdot)$ is allowed to take. The number N will provide a measure of the information flow in the feedback loop. In the literature referenced above several different quantized stabilizing strategies have been proposed in this context. Moreover, in [2, 26] it has been found the minimum value of N (as a function of $|a|$) ensuring the existence of a memoryless quantized controller yielding stability (but not convergence) of the system.

The aim of this paper is to compare the different quantized control strategies proposed in the literature in terms of complexity and performance and to establish a number of results showing fundamental limitations of quantized control. To be more precise about performance, notice first that, if the original system is unstable, a state feedback with finitely many quantization intervals can only yield the so called practical stabilization, namely the convergence of any initial state belonging to a bigger bounded region I into another smaller target region of the state space J. The ratio C between the measure of the starting region and the target region is called contraction and it provides a description of the steady state properties of the closed loop system. Beyond C, the expected time T needed to shrink the state of the plant from the starting set to the target set will measure the transient controller performance. Notice that these two parameters represent a particular way of evaluating the steady state and the transient performance of the controller. There are other possible choices. For instance it is possible to evaluate the transient by means of a quadratic like index. Some preliminary investigations show that the techniques proposed in this paper can be applied also in this set up and yield similar trade-off results.

We will evaluate the relations between the parameters N, C, T and a in a series of different stabilization strategies. In all cases we will see that, for fixed a, as C grows, either N has to grow or T has to grow. However different strategies exhibit different growth rates of the two parameters N and T. In all cases an increasing value of $|a|$ either requires to increase N or yields a degradation of C and T. These results extend the relations between N and $|a|$ proposed in [2, 26] and complete the analysis started in [9], where however the parameter T was interpreted as the sup norm of the entrance time and where a stronger notion of stability was considered. The relations between the parameters N, C, T pointed out in the examples are in accordance with some fundamental bounds which are proved in the second part of the paper, proving in this way the optimality of the proposed quantized controller synthesis techniques.

Now we present an outline of the contents of this paper and of our main results. In Section 2 we present all basic definitions and notations. In particular we introduce the concepts of stability and almost stability and we state precisely the problems we want to solve. Moreover we introduce some basic tools from the ergodic theory of piecewise affine maps. Using these we show that the expected entrance time T is always finite if we have almost stability.

Section 3 is devoted to the introduction and the discussion of a general stabilization strategy based on nesting an initial given quantized stabilizer.

Section 4 is devoted to the analysis of some examples. We show that, by nesting the quantized dead-beat controller in a suitable way, we can obtain a variety of different quantized stabilizers, which can be analyzed in terms of the parameters N and T as functions of a and C. There are three particularly significant cases. The first is the quantized dead beat control which is obtained by using uniform quantized feedback. In this case N grows linearly in C and $|a|$ and T tends to the constant 1. The second is the logarithmic quantized feedback strategy. In this case instead both N and T grow logarithmically in C. The last one is the chaotic quantized feedback strategy. In this last case only almost stability can be achieved and N tends to the constant $\lceil |a| \rceil$ while T grows linearly in C. Notice that the first and the last strategies present dual characteristics of N and T as functions of C. It is interesting to observe that, if we take any linear feedback $u_t = kx_t$, with $k \in \mathbb{R}$, such that the linear closed loop system $x_{t+1} = (a+k)x_t$ is asymptotically stable, then the expected entrance time T of this controlled system is such that $T/\log C$ tends to a constant which is a decreasing function of $|a+k|$. Hence the logarithmic regime corresponds to the performance which can be obtained through the allocation of the eigenvalue inside the unit circle and the absolute value of this eigenvalue determines the logarithmic rate.

In Section 5 we obtain universal bounds relating T, N, and C for fixed $|a|$. The main results are presented in Theorems 3, 4, and Corollaries 3, and 2. All these results, except Theorem 4 needs the assumption $|a| > 2$. Corollary 3 says two things: first, in order to obtain expected entrance time T growing at most logarithmically with respect to C, we need a number of quantization intervals N growing at least logarithmically with respect to C. Second, if we use a number of quantization intervals N growing at most logarithmically with respect to C, we obtain expected entrance times T growing at least logarithmically with respect to C. Moreover the corollary furnishes a quantitative trade-off between the two ratios $T/\log C$ and $N/\log C$ which turns out to be interesting if related to the previous comment on the logarithmic regime which can be obtained in the linear feedback case. Another consequence of the results presented in this section is that the chaos based stabilization strategy is somehow optimal since its performance can not be improved without paying this with a greater information flow. Finally, Theorem 5 shows that any stabilization strategy yielding stability has the ratio $N/\log C$ bounded from below.

For proving the results in Section 5 we need to use the tools of combinatorial analysis of the symbolic dynamics associated to piecewise affine maps. This is developed in Section 6 which contains the deeper mathematical result of this paper which is Theorem 6. This theorem provides a new bound on the number of the paths on a graph with possibly infinite uncountable edges, when this graph has some specific properties. This theorem is very general and has potential applications also in other situations such as in the analysis of quantized feedback systems when the state is multi-dimensional [10].

We conclude this introduction with few remarks to better stress the reasons why we limited our analysis to scalar state space systems. These, from an application viewpoint, may be seen as a relatively uninteresting family of systems to be considered. However, this simple case already contains all the interesting issues of the coupling between control and information and mathematically leads to non-trivial problems. The completeness of the results obtained in this paper, because of the simplified set up we chose, will provide the guidelines for the future investigations on more general situations (see [10]). Observe finally that first order systems can be considered as simplified models of more general systems and that one important case in which control under communication constraint is relevant is just when many simple systems have to be controlled by a unique centralized controller.

Notation We present here some notations which will be used in the paper. If A, B are two sets, then $A \setminus B := \{a \in A : a \notin B\}$. Given a map $f : A \to B$ and $B_1 \subseteq B$ we define

$$f^{-1}(B_1) := \{a \in A : f(a) \in B_1\} .$$

The symbol $A^{\mathbb{N}}$ denotes the set of all sequences taking values on the set A, while symbol A^* denotes the set of all finite words over the alphabet A. The symbol $\#A$ denotes the cardinality of A.

The symbol $\mathbb{R}_+$ denotes the set of all positive real numbers. If $a \in \mathbb{R}_+$, then $\lceil a \rceil$ means the minimum integer greater than or equal to a and $\log a$ is the natural logarithm of a. Given $a, b \in \mathbb{R}$, $a \wedge b$ and $a \vee b$ denote the minimum and the maximum between a and b, respectively. Given $K \subseteq \mathbb{R}$, $\overline{K}$ denotes the closure of K, while ∂K denotes the boundary set of K.

Let I be an interval in $\mathbb{R}$. Given any function $f : I \to \mathbb{R}$ we define

$$\text{supp }(f) := \{x \in I : f(x) \neq 0\}.$$

For any $J \subseteq I$ we denote $\mathbf{1}_J$ the function defined on I which is 1 in J and 0 on $I \setminus J$ and it is called the indicator function of J. With the symbol $L^1(I)$ we mean the set of the absolutely integrable functions which is a normed space with norm

$$||f||_1 := \int_I |f(x)| dx \qquad \forall f \in L^1(I).$$

If $\mathcal{P} : L^1(I) \to L^1(I)$ is a linear continuous operator, then the symbol $||\mathcal{P}||_1$ denotes the induced norm of $\mathcal{P}$. The symbol $L^\infty(I)$ means the set of the bounded functions on I which is a normed space as well. A function $f \in L^1(I)$ such that $f(x) \geq 0$ for all $x \in I$ and such that $||f||_1 = 1$ is called a density function on I. It induces a probability measure on I which will be denoted by $\mathbb{P}_f$, while the symbol $\mathbb{E}_f$ will denote the expected value with respect to $\mathbb{P}_f$. The probability measure and the expected value with respect to uniform Lebesgue measure on I will be simply denoted by the symbols $\mathbb{P}$ and $\mathbb{E}$, respectively.

2 Problem statement

Consider the following discrete-time, one-dimensional linear model

$$x_{t+1} = ax_t + u_t, \tag{1}$$

where $a \in \mathbb{R}$. Most of the paper is devoted to the stabilization problem and so it is assumed that $|a| > 1$. Some results however holds true also for stable systems and so for $|a| \leq 1$.

Let $k : \mathbb{R} \to \mathbb{R}$ be a piecewise constant function with only finitely many discontinuities. If we use k as a static feedback in the system (1), namely we let $u_t = k(x_t)$, we obtain the closed loop system

$$x_{t+1} = \Gamma(x_t), \tag{2}$$

where $\Gamma(x) := ax + k(x)$ is a piecewise affine map with a fixed slope a. Autonomous systems like (2) in which Γ is piecewise affine can exhibit a very wild behavior. Their dynamical properties have been extensively studied in the past [15, 18, 5].

Remark: In fact, the definition we gave is not precise if we do not define what happens at the boundary points of the intervals. We assume there is a finite family of disjoint open intervals I_h such that

$D := \cup_h I_h$ is dense in $\mathbb{R}$ and such that $k(x) = u_h$ for every $x \in I_h$. In this way the associated closed loop map is defined as a map

$$\begin{aligned} &\Gamma : D \to \mathbb{R} \\ &\Gamma(x) = ax + u_h \quad \text{if } x \in I_h\,. \end{aligned} \tag{3}$$

In order to consider iterations of Γ we need to restrict the domain by considering

$$\Omega = \bigcap_{n=0}^{\infty} \Gamma^{-n}(D)\ . \tag{4}$$

It is clear that $\Gamma(\Omega) \subseteq \Omega$. Notice that $\mathbb{R} \setminus \Omega$ is a countable subset of $\mathbb{R}$ and since most of the questions considered in this paper are related to mean properties, it will be sufficient to consider Γ as a map defined on Ω, disregarding all the orbits which will eventually get to a discontinuity point.

However, in those situations in which it is necessary to understand how the dynamics is defined at the boundaries, it is necessary to define the dynamics of Γ on all $\mathbb{R}$. This is done by considering, for any $x_0 \in \mathbb{R}$, the left and right limit of $\Gamma(x)$ for $x \to x_0$ denoted by $\Gamma(x_0-)$ and $\Gamma(x_0+)$ and by defining the enlarged set of orbits as

$$X_\Gamma = \{(x_t) \in \mathbb{R}^{\mathbb{N}} \mid x_{t+1} = \Gamma(x_t+) \text{ or } x_{t+1} = \Gamma(x_t+)\ \forall t \in \mathbb{N}\}\,. \tag{5}$$

The subset $X_\Gamma \cap \Omega^{\mathbb{N}}$ consists in the orbits of Γ on Ω and it is in bijection with Ω through the initial condition.

It is obvious that, by using quantized feedback controllers only a "practical stability" can be obtained as detailed in the following definitions.

Definition: Invariance and almost invariance. Given a closed interval I, we say that I is Γ-invariant if every orbit (x_t) of Γ with $x_0 \in I$ is such that $x_t \in I$ for every t. It is almost Γ-invariant if the assertion above is true for almost every initial condition x_0 with respect to the Lebesgue measure. When an interval I is invariant or almost invariant we will use in any case the notation $\Gamma : I \to I$.

Definition: Stability and almost stability. Given two closed intervals $J \subseteq I$, we say that Γ is (I,J)-*stable* if I and J are invariant by Γ and if for every orbit (x_t) of Γ with $x_0 \in I$, there exists an integer $t \geq 0$ such that $x_t \in J$. We say that Γ is *almost* (I,J)-*stable* if I and J are almost invariant and the convergence to J as defined above occurs for almost all initial condition in the orbit $x_0 \in I$, with respect to the Lebesgue measure. A quantized feedback map $k : \mathbb{R} \to \mathbb{R}$ is said to be *(almost)* (I,J)-*stabilizing* if the corresponding closed loop map Γ is (almost) (I,J)-stable.

Remark: For what concerns almost invariance and almost stability it is sufficient to work with Γ on the set Ω as defined in (4). The concepts of invariance and of stability depend also on the dynamics on boundary points and so the orbits have to be considered as defined in (5).

Assume that Γ is almost (I,J)-stable. The first entrance time function

$$T_{(I,J)} : I \cap \Omega \to \mathbb{N} \cup \{+\infty\}$$

is defined by

$$T_{(I,J)}(x) = \inf\{n \in \mathbb{N} \mid \Gamma^n x \in J\} = \sum_{n=1}^{\infty} \mathbf{1}_{I \setminus J}(\Gamma^n x)\ . \tag{6}$$

We put $T_{(I,J)}(x) := +\infty$ if $\Gamma^t x \notin J$ for all t. Notice that the map $T_{(I,J)}$ is always finite exactly when we have stability, while it is almost surely finite when we have almost stability.

Remark: Notice that, if we want to extend the function $T_{(I,J)}$ to the all I, we can not use definition (6). Indeed, there is a possible ambiguity for orbits touching discontinuity points since, given $x \in I$, there can be infinitely many orbits having x as initial condition and therefore $\Gamma^n x$ is not uniquely defined. In this case definition (6) should be replaced as follows: we say that $T_{(I,J)}(x) = n$ if every orbit $(x_t) \in X_\Gamma$ such that $x_0 = x$ is such that $x_t \in J$ for any $t \geq n$ and if there exists an orbit $(x_t) \in X_\Gamma$ such that $x_0 = x$ and such that $x_{n-1} \notin J$.

The expected value of the entrance time with respect to a density function f on I is given by

$$\mathbb{E}_f(T_{(I,J)}) = \int_I T_{(I,J)}(x)f(x)dx.$$

It is clear that

$$\mathbb{E}_f(T_{(I,J)}) = \int_I \left[\sum_{n=1}^{\infty} \mathbf{1}_{I\setminus J}(\Gamma^n x)f(x)\right] dx = \sum_{n=1}^{\infty} n\mathbb{P}_f[T_{(I,J)} = n] = \sum_{n=0}^{\infty} \mathbb{P}_f[T_{(I,J)} > n]\,.$$

In the sequel, for any given (almost) (I,J)-stabilizing quantized feedback k yielding an (almost) (I,J)-stable piecewise affine closed loop map Γ, we will use the symbol $\mathbf{T}(k)$ or $\mathbf{T}(\Gamma)$ to denote the relative expected entrance time $\mathbf{E}(T_{(I,J)})$ with respect to the uniform density function on I. Notice that this quantity depends only on the restriction of Γ to $I\setminus J$ and so we can assume that Γ is defined only on $I\setminus J$. For this reason the right parameter measuring the information flow will be the number of quantization intervals in $I\setminus J$ which will be denoted by symbols $\mathbf{N}(k)$ or $\mathbf{N}(\Gamma)$. Finally the ratio between the length of I and the length of J will be called contraction rate and will be denoted by $C(k)$ or $C(\Gamma)$.

The performance analysis of the quantized stabilization consists in determining, for a given $C > 1$, $N \in \mathbb{N}$ and $T > 0$, whether there exists or not a (almost) stabilizing quantized feedback k such that $C(k) = C$, $\mathbf{N}(k) = N$ and $\mathbf{T}(k) = T$, or, in other words, in estimating the set

$$\mathcal{A} := \{(C,N,T) \ : \ \text{there exists } k \text{ such that } C(k) = C, \mathbf{N}(k) = N, \mathbf{T}(k) = T\}$$

Remark: The analysis proposed in this paper can be extended to a family of more general performance measures. Let

$$V : I \to \mathbb{R}$$

be such that $0 \le V(x) \le 1$ for every $x \in I$ and $V(x) = 0$ for every $x \in J$. Another measure of the transient properties of the closed loop system is the following number

$$\mathbb{E}\left(\sum_{n=0}^{\infty} V(\Gamma^n x)\right).$$

It is clear that, if $V(x) = \mathbf{1}_{I\setminus J}(x)$, then the previous cost coincides with the expected entrance time in J. If $V(x)$ is a general continuous function, then, for any $\alpha \in [0,1]$ we have that

$$\alpha\mathbf{1}_{I\setminus J(\alpha)}(x) \le V(x) \le \mathbf{1}_{I\setminus J}(x),$$

where $J(\alpha) := \{x \in I \ : \ V(x) \le \alpha\}$. This fact implies that

$$\alpha\mathbb{E}(T_{J(\alpha)}) \le \mathbb{E}\left(\sum_{n=0}^{\infty} V(\Gamma^n x)\right) \le \mathbb{E}(T_{(I,J)}).$$

This shows that the dependence of this generalized performance index and of the expected entrance time on the parameters $C(\Gamma)$ and $\mathbf{N}(\Gamma)$ will be similar.

2.1 The Perron-Frobenius operator for piecewise affine maps

In this subsection we recall some standard results on the ergodic theory of piecewise affine maps and we will present a first preliminary result asserting that the expected entrance time is always finite for almost (I,J)-stable piecewise affine maps.

Let $\Gamma : I \to I$ be a piecewise affine map with fixed slope a and assume here that $|a| > 1$. It is a standard fact that Γ induces a linear transformation

$$\mathcal{P}_\Gamma : L^1(I) \to L^1(I)$$

called the Perron-Frobenius operator associated with Γ which is uniquely defined by the following duality relation

$$\int_I (g\circ\Gamma)(x)f(x)dx = \int_I g(x)(\mathcal{P}_\Gamma f)(x)dx \tag{7}$$

for all $g \in L^\infty(I), f \in L^1(I)$. It can be shown that the operator $\mathcal{P}_\Gamma$ is bounded with $||\mathcal{P}_\Gamma||_1 \leq 1$ and it maps probability densities into probability densities. An important interpretation of $\mathcal{P}_\Gamma$ is as follows. If we have a continuous random variable X defined on I whose density is f, then the density of the transformed random variable $X\circ\Gamma$ is $\mathcal{P}_\Gamma f$. A final important property of the Perron-Frobenius operator $\mathcal{P}_\Gamma$ is that $\mathcal{P}_{\Gamma^n} = \mathcal{P}_\Gamma^n$.

The relevance of the Perron-Frobenius operator in our investigations is due to the fact that

$$\mathbb{P}_f[T_{(I,J)} > n] = \int_{I\setminus J} \mathcal{P}_\Gamma^n f(x)dx$$

which follows by iterating (7) and by taking $g(x) = \mathbf{1}_{I\setminus J}(x)$. This shows that the asymptotics of this operator and so its spectral properties will be relevant for our purposes.

We have the following result.

Lemma 1 *Let Γ be almost (I,J)-stable. If $h(x) \in L^1(I)$ is an invariant density of $\mathcal{P}_\Gamma$, then*

$$\text{supp } h \subseteq J.$$

Proof First we show that, since J is invariant by Γ, the fact that supp $f \subseteq J$ implies that supp $\mathcal{P}_\Gamma^k f \subseteq J$. Indeed, if $K \subseteq I\setminus J$, then $\Gamma^{-1}(K) \subseteq I\setminus J$ and so

$$\int_K (\mathcal{P}_\Gamma f)(x)dx = \int_{\Gamma^{-1}(K)} f(x)dx = 0.$$

We show now that, if h is invariant by $\mathcal{P}_\Gamma$, then also $h\mathbf{1}_J$ and $h\mathbf{1}_{I\setminus J}$ are invariant by $\mathcal{P}_\Gamma$. Indeed, for any $g \in L^\infty(I), f\in L^1(I)$ we have that

$$\begin{aligned}\int_I g(x)(\mathcal{P}_\Gamma h\mathbf{1}_J)(x)dx &= \int_J g(x)(\mathcal{P}_\Gamma h\mathbf{1}_J)(x)dx = \int_J g(x)(\mathcal{P}_\Gamma h)(x)dx = \\ &= \int_I g(x)\mathbf{1}_J(x)(\mathcal{P}_\Gamma h)(x)dx = \int_I g(x)(h\mathbf{1}_J)(x)dx\end{aligned}$$

where in the first equality we used the fact that supp $h\mathbf{1}_J \subseteq J$. This shows that $h\mathbf{1}_J$ is invariant. Since both h and $h\mathbf{1}_J$ are invariant, so is $h\mathbf{1}_{I\setminus J}$, as well.

Finally, if we assume by contradiction that there exists a non-zero invariant density of $\mathcal{P}_\Gamma$ not supported inside J, then for the above considerations, there also exists a non-zero invariant density supported inside $I\setminus J$. Let us call it h_0. We can find $\delta > 0$ and a subset $K \subseteq I\setminus J$ of nonzero Lebesgue measure such that $h_0(x) > \delta$ for every $x \in K$. Consequently, $h_0 - \delta\mathbf{1}_K$ is a non-negative function. Therefore also $\mathcal{P}_\Gamma^n(h_0 - \delta\mathbf{1}_K) = h_0 - \mathcal{P}_\Gamma^n(\delta\mathbf{1}_K)$ is non-negative for all $n \geq 0$. Since h_0 is 0 on J, it follows that $\mathcal{P}_\Gamma^n\mathbf{1}_K$ is 0 on J for every n. This implies that

$$\int_{\Gamma^{-n}(J)} \mathbf{1}_K(x)dx = \int_J (\mathcal{P}_\Gamma^n\mathbf{1}_K)(x)dx = 0$$

which implies that $K\cap\Gamma^{-n}(J)$ has zero Lebesgue measure for all $n \geq 0$, which contradicts the almost (I,J)-stability of Γ. ■

To obtain a good characterization of the spectral properties of $\mathcal{P}_\Gamma$ we need to restrict the type of densities to be considered. Let $\mathrm{BV}(I) \subseteq L^1(I)$ be the subspace of $L^1(I)$ constituted by the bounded variation functions on the interval I. More precisely, if we define the variation of a function f as

$$\bigvee f := \sup\left\{\sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)|\ x_i \in I,\ x_1 < x_2 < \cdots < x_n\right\},$$

then

$$\mathrm{BV}(I) := \{f : I \to \mathbb{R} : \bigvee f < \infty\}.$$

Equip now the space $\mathrm{BV}(I)$ with the new norm

$$|||f||| := \bigvee f + ||f||_1 \ .$$

It is a classical fact that $\mathcal{P}_\Gamma(\mathrm{BV}(I)) \subseteq \mathrm{BV}(I)$ and that $\mathcal{P}_\Gamma|_{\mathrm{BV}(I)}$ is bounded with respect to the norm $||| \cdot |||$. Using now the Lasota-Yorke inequality [15] and the spectral theorem of Ionescu-Tulcea and Marinescu [12] the following facts can be shown to hold true

(i) Let σ_1 be the set of eigenvalues of modulus 1 of $\mathcal{P}_\Gamma$ seen as an operator on $L^1(I)$. Then this set is a finite multiplicative group. Moreover each of these eigenvalues has a finite dimensional eigenspace contained in $\mathrm{BV}(I)$.

(ii) The Perron-Frobenius operator $\mathcal{P}_\Gamma$ on $\mathrm{BV}(I)$ admits the following decomposition

$$\mathcal{P}_\Gamma = \sum_{\lambda \in \sigma_1} \lambda Q_\lambda + R \tag{8}$$

where Q_λ are finite rank operators on $\mathrm{BV}(I)$ and R is a bounded operator on $\mathrm{BV}(I)$ such that

(a) $Q_\lambda \circ R = R \circ Q_\lambda = 0$ for all $\lambda \in \sigma_1$;

(b) $Q_\lambda \circ Q_{\lambda'} = 0$ for all $\lambda, \lambda' \in \sigma_1$ such that $\lambda \neq \lambda'$;

(c) $Q_\lambda \circ Q_\lambda = Q_\lambda$ for all $\lambda \in \sigma_1$;

(d) $|||R^n||| \leq c\gamma^n$ for all $n \in \mathbb{N}$, where c is a positive constant and $0 < \gamma < 1$.

An important consequence of the above results is that the spectrum of $\mathcal{P}_\Gamma$ in $\mathrm{BV}(I)$ is composed of a finite set of eigenvalues on the unit circle (with finite dimensional eigenspaces) and of another part contained in a disk of radius strictly smaller than 1.

We now state and prove the main result of this section.

Proposition 1 *Let Γ be an almost (I, J)-stable piecewise affine map. Then, there exists a constant $K > 0$ such that*

$$\mathbb{E}_f(T_{(I,J)}) \leq K|||f|||$$

for every probability density $f \in \mathrm{BV}(I)$.

Proof Notice preliminarily that there exists $\nu \in \mathbb{N}$ such that $\lambda^\nu = 1$ for every $\lambda \in \sigma_1$. This implies that

$$\mathcal{P}_\Gamma^\nu = \sum_{\lambda \in \sigma_1} Q_\lambda + R^\nu$$

This implies that for any density $f \in \mathrm{BV}(I)$ we have that $Q_\lambda f$ is invariant by $\mathcal{P}_\Gamma^\nu$. Since $\mathcal{P}_\Gamma^\nu$ is the Perron-Frobenius operator for the map Γ^ν which is almost (I, J)-stable, then, by Lemma 1, we have that $\mathrm{supp}\, Q_\lambda f \subseteq J$. Using this fact and formula (8) we obtain

$$\mathbb{P}_f[T_{(I,J)} > n] = \int_{I \setminus J} (\mathcal{P}_\Gamma^n f)(x) dx = \int_{I \setminus J} (R^n f)(x) dx \leq c\gamma^n |||f|||$$

and hence

$$\mathbb{E}_f(T_{(I,J)}) = \sum_{n=0}^{+\infty} \mathbb{P}_f[T_{(I,J)} > n] \leq \frac{c}{1-\gamma} |||f||| \, .$$

∎

3 Nested quantized feedback strategies

Consider the linear discrete time system (1), where $|a| > 1$, and consider two intervals $J \subseteq I$. We want to stabilize it through a quantized state feedback, i.e. we want to find a quantized feedback map k such that the closed loop system (2) drives (almost) any initial state $x_0 \in I$ into a state evolution which, after a transient, enters the interval J. Several solutions to this problem can be proposed. In fact we will show that, starting from a base quantized feedback, it is possible to construct a family of quantized feedbacks by iterating the base one.

More precisely, suppose that we have found a (I, J)-stabilizing quantized feedback $k_1(x)$ and a (J, K)-stabilizing quantized feedback $k_2(x)$. Then it is clear that the quantized feedback

$$k(x) = \begin{cases} k_1(x) & \text{if } x \in I \setminus J \\ k_2(x) & \text{if } x \in J \setminus K \end{cases} \tag{9}$$

will be (I, K)-stabilizing. The analogous conclusion is less straightforward in case we we start from almost stabilizing quantized feedbacks. In the sequel we will show that this is indeed the case, namely, if $k_1(x)$ is almost (I, J)-stabilizing and $k_2(x)$ is almost (J, K)-stabilizing, then $k(x)$ is almost (I, K)-stabilizing.

Let $\Gamma : I \to I$ be a almost (I, J)-stable piecewise affine map with fixed slope a such that $|a| > 1$ and let $\mathcal{P}_\Gamma$ be the Perron-Frobenius operator associated with Γ. From any density function $f \in L^1(I)$ it is possible to define a probability measure μ on J as the image of the measure $\mathbb{P}_f$ through the map

$$\psi(x) := \Gamma^{T_{(I,J)}(x)}(x),$$

where $T_{(I,J)}(x)$ is the first entrance time function of Γ. More precisely, if $A \subseteq J$ is a measurable set, then

$$\mu(A) := \mathbb{P}_f[\psi^{-1}(A)]. \tag{10}$$

The following result gives important information on the measure μ.

Proposition 2 *For any density $f \in L^1(I)$, the measure μ defined in (10) is absolutely continuous with respect to the Lebesgue measure and its corresponding density h is given by*

$$h = \mathbf{1}_J f + \sum_{j=1}^{+\infty} \mathbf{1}_J \mathcal{P}_\Gamma(\mathbf{1}_{I \setminus J} \mathcal{P}_\Gamma^{j-1} f)\,. \tag{11}$$

Moreover, there exists a constant $H > 0$ only depending on Γ such that

$$|||h||| \leq H|||f|||\,, \quad \forall f \in \mathrm{BV}(I).$$

Proof Let $A \subseteq J$ be a measurable set. Then,

$$\begin{aligned} \mu(A) &= \mathbb{P}_f[\psi^{-1}(A)] = \sum_{j=0}^{+\infty} \mathbb{P}_f[\psi^{-1}(A) \cap \{T_{(I,J)}(x) = j\}] \\ &= \sum_{j=0}^{+\infty} \mathbb{P}_f[\Gamma^{-j}(A) \cap \{T_{(I,J)}(x) = j\}] = \mathbb{P}_f[A] + \sum_{j=1}^{+\infty} \mathbb{P}_f[\Gamma^{-j}(A) \cap \Gamma^{-j+1}(I \setminus J)]. \end{aligned} \tag{12}$$

Notice that

$$\begin{aligned} \mathbb{P}_f[\Gamma^{-j}(A) \cap \Gamma^{-j+1}(I \setminus J)] &= \int_I \mathbf{1}_{\Gamma^{-j}(A)}(x) \mathbf{1}_{\Gamma^{-j+1}(I \setminus J)}(x) f(x) dx \\ &= \int_I \mathbf{1}_{\Gamma^{-1}(A)}(x) \left[\mathbf{1}_{I \setminus J}(x) \mathcal{P}_\Gamma^{j-1} f(x)\right] dx \\ &= \int_A \mathcal{P}_\Gamma(\mathbf{1}_{I \setminus J} \mathcal{P}_\Gamma^{j-1} f)(x) dx = \int_A h_j(x) dx, \end{aligned}$$

where $h_j(x) := \mathbf{1}_J(x)\mathcal{P}_\Gamma(\mathbf{1}_{I\setminus J}\mathcal{P}_\Gamma^{j-1}f)(x)$. Using this relation in (12) and the fact that $h_j(x)$ are non negative we obtain, by Fatou's lemma, that

$$\mu(A) = \mathbb{P}_f[A] + \sum_{j=1}^{+\infty}\int_A h_j(x)dx = \int_A \left[\mathbf{1}_J(x)f(x) + \sum_{j=1}^{+\infty} h_j(x)\right] dx$$

which shows that the series $\sum_{j=1}^{+\infty} h_j(x)$ converges in L^1 sense. Hence, the function h, defined in (11), is in L^1 and μ is absolutely continuous with respect to the Lebesgue measure with density h.

We now show that there is also convergence in the norm $|||\cdot|||$ if $f \in \mathrm{BV}(I)$. First notice that, by Yorke inequality [15, formula (6.1.12)], for all $g \in \mathrm{BV}(I)$ we have

$$\bigvee(g\mathbf{1}_J) \leq 2\bigvee g + \frac{2}{|I|}||g||_1$$

which implies that

$$|||g\mathbf{1}_J||| \leq 2\bigvee g + \left(1 + \frac{2}{|I|}\right)||g||_1 \leq \left(2 + \frac{2}{|I|}\right)|||g|||.$$

Using the previous inequality we can argue that

$$|||\mathbf{1}_J\mathcal{P}_\Gamma(\mathbf{1}_{I\setminus J}\mathcal{P}_\Gamma^{j-1}f)||| \leq \left(2 + \frac{2}{|I|}\right)|||\mathcal{P}_\Gamma|||\cdot|||\mathbf{1}_{I\setminus J}\mathcal{P}_\Gamma^{j-1}f|||. \tag{13}$$

Using now the spectral decomposition for $\mathcal{P}_\Gamma$ we can estimate this last term as

$$|||\mathbf{1}_{I\setminus J}\mathcal{P}_\Gamma^{j-1}f\,||| = |||R^{j-1}f||| \leq c\gamma^{j-1}|||f|||, \tag{14}$$

where we used the same arguments used in Proposition 1. Putting together estimates (13), and (14), we finally obtain that the sum (11) indeed converges in the norm $|||\cdot|||$ and, moreover, we have that

$$\left|\left|\left|\sum_{j=1}^{+\infty}\mathbf{1}_J\mathcal{P}_\Gamma(\mathbf{1}_{I\setminus J}\mathcal{P}_\Gamma^{j-1}f)\right|\right|\right| \leq \left(2 + \frac{2}{|I|}\right)\frac{|||\mathcal{P}_\Gamma|||c}{1-\gamma}|||f|||$$

which yields the thesis. ■

From the previous proposition and from Propositions 1 we can argue that the composed quantized feedback $k(x)$ defined in (9) is always almost (I, K)-stabilizing. The previous result can be used also to obtain an estimate of the expected entrance time $\mathbf{T}(k)$. Let $T_{(I,J)}(x)$ for k_1 and $T_{(J,K)}(x)$ be the first entrance time function for k_2. It is clear that the first entrance time function $T_{(I,K)}(x)$ of the quantized feedback k is given by

$$T_{(I,K)}(x) = T_{(I,J)}(x) + T_{(J,K)}\left(\Gamma_1^{T_{(I,J)}(x)}(x)\right) .$$

This implies that

$$\mathbb{E}_f(T) = \int_J T_{(I,J)}(x)f(x)dx + \int_J T_{(J,K)}\left(\Gamma_1^{T_{(I,J)}(x)}(x)\right)f(x)dx = \mathbb{E}_f(T_{(I,J)}) + \mathbb{E}_h(T_{(J,K)}) ,$$

where h is the probability density on J obtained form f as shown in the previous proposition.

This shows a way to estimate the expected entrance time of $k(x)$. As far as the number of quantization intervals, it is clear that we have simply that $\mathbf{N}(k) = \mathbf{N}(k_1) + \mathbf{N}(k_2)$. Finally, the contraction rate of the overall quantized feedback is the product of the contraction rates of the component quantized feedbacks, i.e., $C(k) = C(k_1)C(k_2)$.

The previous considerations can be used to obtain a class of (almost) stabilizing quantized feedbacks starting from one. Indeed, assume that $k(x)$ is a (almost) (I, J)-stabilizing quantized feedback with contraction rate $C(k) = C$, $\mathbf{N}(k)$ quantization intervals and expected entrance time $\mathbf{T}(k)$. Let

$$F(x) := \frac{x}{C} + \beta$$

be a affine map such that $J = F(I)$. It is clear that the quantized feedback

$$F \circ k \circ F^{-1} \ : \ F(I) \to F(I)$$

is (almost) $(F(I), F^2(I))$-stabilizing. Observe that the corresponding closed loop map is $F \circ \Gamma \circ F^{-1}$. The same construction can be iterated, obtaining for every $i = 0, 1, \ldots, \tau - 1$ the quantized feedback $F^i \circ k \circ F^{-i}$ which is (almost) $(F^i(I), F^{i+1}(I))$-stabilizing. The quantized feedback defined as follows

$$k^{(\tau)}(x) := F^i \circ k \circ F^{-i}(x) \quad \text{if} \quad x \in F^i(I) \setminus F^{i+1}(I)$$

will be (almost) $(I, F^\tau(I))$-stabilizing with contraction rate $C(k^{(\tau)}) = C(k)^\tau$ and $\mathbf{N}(k^{(\tau)}) = \tau\mathbf{N}(k)$ quantization intervals. As far as the expected entrance time $\mathbf{T}(k^{(\tau)})$ is concerned, it is difficult in general to estimate its dependence on the number τ of iterations.

Consider the map

$$\Psi : I \to I : x \mapsto F^{-1} \circ \Gamma^{T_{(I,J)}(x)}(x), \tag{15}$$

where $T_{(I,J)}(x)$ is the first entrance time function for k. It follows from Proposition 2 that Ψ transforms absolutely continuous measures into themselves so that also in this case we can consider the associated Perron-Frobenius operator

$$\mathcal{P}_\Psi : L^1(I) \to L^1(I).$$

It is easy to see that

$$\mathcal{P}_\Psi f = C^{-1}(h \circ F),$$

where h is the density which is obtained from f as shown in (11).

It is clear from the previous considerations that

$$\mathbf{T}(k^{(\tau)}) = \sum_{i=0}^{\tau-1} \mathbb{E}_{\mathcal{P}_\Psi^i f}(T_{(I,J)})\,, \tag{16}$$

where f is the uniform probability density on I. From Propositions 2 and 1 we obtain

$$\mathbf{T}(k^{(\tau)}) \leq \sum_{i=0}^{\tau-1} K|||\mathcal{P}_\Psi^i f||| \leq \sum_{i=0}^{\tau-1} KH^i|||f||| \leq \frac{K}{H-1}H^\tau|||f|||\,. \tag{17}$$

This is not a very good estimate, since we expect that in many situations the growth should be linear in τ. For instance, if the uniform density on I is invariant, then we have that $\mathbf{T}(k^{(\tau)}) = \tau\mathbf{T}(k)$. In this case from a triple $(C, N, T) \in \mathcal{A}$ we can obtain a sequence of triples $(C^\tau, \tau N, \tau T)) \in \mathcal{A}$, for all $\tau \in \mathbb{N}$. This method will be used in the following subsections to obtain three specific quantized feedback strategies.

In general we can not guarantee that Ψ will possess invariant probability densities and it seems to be very difficult to obtain estimates which are better than (17). Notice indeed that Ψ is also a piecewise affine map but in general with an infinite number of continuity intervals. For this type of maps the theory is quite weak: invariant probability densities are not guaranteed to exist and we may lose the spectral structure of the corresponding Perron-Frobenius operator we had in the finite case (see [4] for more details). There is however a case in which things go smooth namely when $T(x)$ is bounded. In this case Ψ is an expanding piecewise affine map with only a finite number of continuity intervals and in this case invariant densities do exist and the Perron-Frobenius operator $\mathcal{P}_\Psi$ admits the usual spectral decomposition (8). In this case we have the following result.

Proposition 3 *Assume that $T(x)$ is bounded. Then, there exists a probability density $\overline{f}$ and a bounded sequence $\{a_\tau\}$ such that*

$$\mathbf{T}(k^{(\tau)}) = \tau\mathbb{E}_{\overline{f}}(T) + a_\tau. \tag{18}$$

Proof Let f be the uniform probability density on I. Let moreover $\nu \in \mathbb{N}$ be such that $\lambda^\nu = 1$ for every $\lambda \in \sigma_1$ and let $Q = \sum_\lambda Q_\lambda$. Observe that for all $j \in \mathbb{N}$ we have $Q\mathcal{P}_\Psi^j = \mathcal{P}_\Psi^j - R^j$ and that $Q\mathcal{P}_\Psi^\nu = Q$. This implies that, if we decompose $j = l\nu + r$, with $l \in \mathbb{N}$ and $r \in \{0, \ldots, \nu - 1\}$, we have that

$$\mathcal{P}_\Psi^j = Q\mathcal{P}_\Psi^r + R^j\,.$$

Define

$$\overline{f} = Q\left(\frac{1}{\nu}\sum_{j=0}^{\nu-1}\mathcal{P}_\Psi^j f\right).$$

Then, if $\tau - 1 = l\nu + r$, we have that

$$\begin{aligned}\sum_{j=0}^{\tau-1}\mathcal{P}_\Psi^j f - \tau\overline{f} &= lQ\left(\sum_{j=0}^{\nu-1}\mathcal{P}_\Psi^j f\right) + Q\left(\sum_{j=0}^{r}\mathcal{P}_\Psi^j f\right) + \sum_{j=0}^{\tau-1}R^j f - \tau\overline{f}\\ &= (l\nu-\tau)\overline{f} + Q\left(\sum_{j=0}^{r}\mathcal{P}_\Psi^j f\right) + \sum_{j=0}^{\tau-1}R^j f\end{aligned}$$

Notice that

$$\left|\left|\left|(l\nu-\tau)\overline{f} + Q\left(\sum_{j=0}^{r}\mathcal{P}_\Psi^j f\right) + \sum_{j=0}^{\tau-1}R^j f\right|\right|\right| \le \nu|||\overline{f}||| + |||Q|||\left(\sum_{j=0}^{\nu-1}|||\mathcal{P}_\Psi^j|||\right)|||f||| + \frac{C}{1-\gamma}|||f|||$$

is bounded in τ. Observe finally that

$$|\mathbf{T}(k^{(\tau)}) - \tau\mathbb{E}_{\overline{f}}(T)| = \left|\int_I T(x)\left(\sum_{j=0}^{\tau-1}\mathcal{P}_\Psi^j f(x) - \tau\overline{f}(x)\right)dx\right| \le \int_I T(x)\left|\sum_{j=0}^{\tau-1}\mathcal{P}_\Psi^j f(x) - \tau\overline{f}(x)\right|dx\,.$$

The result now follows by applying Proposition 1. ■

This has the following consequence. If the triple (C, N, T) is in $\mathcal{A}$ and corresponds to a situation in which the entrance time function is bounded, then we can obtain a sequence of triples $(C^\tau, \tau N, \tau\overline{T}+a_\tau) \in \mathcal{A}$, for all $\tau \in \mathbb{N}$, where $\overline{T}$ is the expected entrance time with respect to a suitable probability density and $\{a_\tau\}$ is a bounded sequence.

4 Three stabilizing quantized feedback strategies

The method presented in the previous section will be used in the following subsections to obtain three specific quantized feedback strategies. In the sequel we assume for simplicity that $I = [-1, 1]$ and $J = [\epsilon, \epsilon]$, with $\epsilon \le 1$ and so we have that $C = 1/\epsilon$. In this section we will simply write $C, \mathbf{N}, \mathbf{T}$ dropping the explicit dependence from k. All probabilistic considerations in this section will be carried on with respect to the uniform probability on I.

4.1 Deadbeat quantized feedback strategy

The first strategy, which has been analyzed in a certain detail by Delchamps in [6], consists in approximating the 1-step deadbeat controller $k(x) := -ax$ by its quantized version, i.e., by a uniform quantized function $k(x)$ such that $-ax - \epsilon \le k(x) \le -ax + \epsilon$. One possibility is to take

$$k(x) := -(2h+1)\epsilon \quad \text{for} \quad h\frac{2\epsilon}{a} < x \le (h+1)\frac{2\epsilon}{a} \tag{19}$$

which yields the closed loop map $\Gamma(x)$ illustrated in Figure 1.

This controller drives any state belonging to I into J in one step. In this case we have that

$$\mathbf{N} = 2\left\lceil |a|\frac{C-1}{2}\right\rceil \sim |a|C$$

and that

$$\mathbf{T} = \sum_{n=1}^{\infty}\mathbb{P}[T_J \ge n] = \mathbb{P}[T_J \ge 1] = 1 - \mathbb{P}[J] = 1 - 1/C\ ,$$

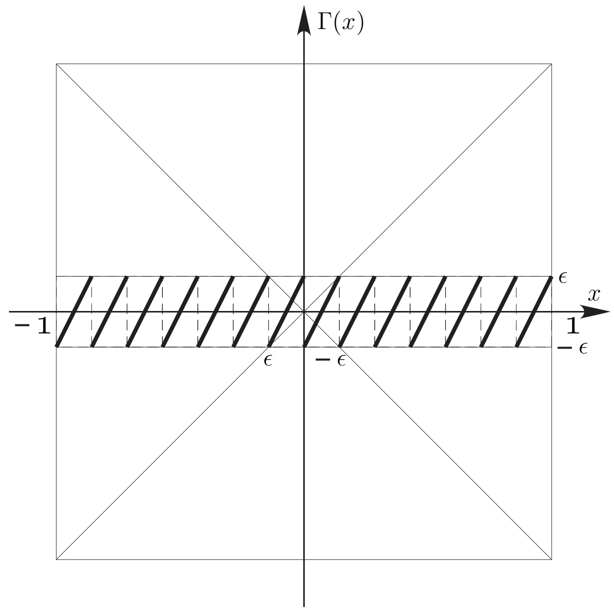

Figure 1: The map Γ associated with the quantized feedback defined in (19).

where $f(C) \sim g(C)$ means that $f(C)/g(C)$ tends to 1 as $C \to \infty$.

Using the nesting strategy presented above we can construct a τ steps deadbeat quantized feedback simply iterating the 1 step deadbeat quantized feedback. We only need to pay attention to the fact that the uniform density in I is invariant with respect to the map Ψ defined in (15). This happens if $|a|(C-1)/2$ is an integer. Assume that this is the case and denote it by n. We obtain a triple contraction rate, quantization intervals, expected entrance time equal to

$$\left(\frac{2n+|a|}{|a|}, 2n, \frac{2n}{2n+|a|}\right) \in \mathcal{A}\,.$$

Using the strategy presented above, we can iterate the construction τ times, obtaining in this way a sequence of triples

$$\left(\left(\frac{2n+|a|}{|a|}\right)^{\tau}, 2\tau n, \tau\frac{2n}{2n+|a|}\right) \in \mathcal{A}, \qquad n, \tau \in \mathbb{N}.$$

which provides a family of quantized feedbacks parameterized by the two integers τ, n. We are mainly interested in understanding what asymptotic behavior can be obtained of $\mathbf{N}$ and $\mathbf{T}$ as $C \to \infty$. To this aim observe that

$$\frac{\mathbf{N}/|a|}{\mathbf{T}C^{1/\mathbf{T}}} = \left(\frac{2n+|a|}{|a|}\right)^{-\frac{|a|}{2n}} \in [1/e, 1].$$

Making the change of variable

$$C = \left(\frac{2n+|a|}{|a|}\right)^{\tau}, \qquad n = \frac{|a|}{2}(C^{\frac{1}{\tau}} - 1) \tag{20}$$

we obtain

$$\begin{aligned} \mathbf{N}/|a| &= \tau(C^{\frac{1}{\tau}} - 1) \\ \mathbf{T} &= \tau(1 - C^{-\frac{1}{\tau}}) \end{aligned}$$

where τ is any function of C that, by (20), can be chosen arbitrarily subject to the fact that $\tau(C)/\log C$ is bounded from above. If in particular τ is fixed, we obtain

$$\begin{aligned} \mathbf{N}/|a| &\sim \tau C^{\frac{1}{\tau}} \\ \mathbf{T} &\sim \tau\,. \end{aligned}$$

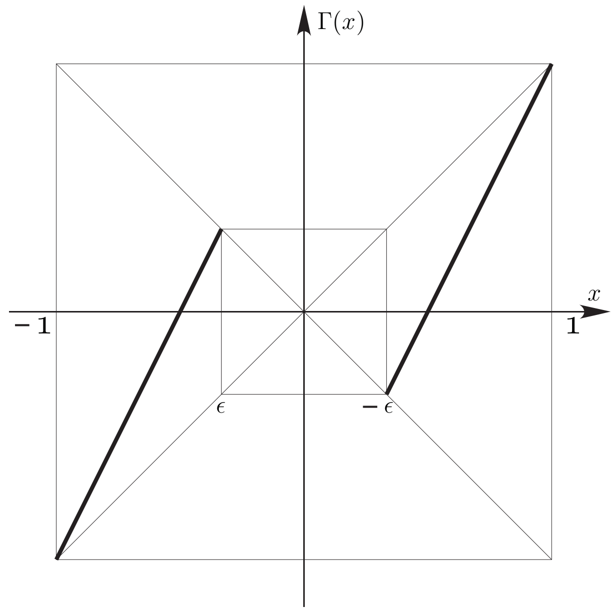

Figure 2: The map Γ associated with the quantized feedback defined in (23).

If instead we think of τ as a possible function of C, we can distinguish two different behaviors: the case when $\tau(C)/\log C \to 0$ and the case when $\tau(C) \sim K \log C$. In the first case we have that

$$\mathbf{N}/|a| \sim \mathbf{T} C^{1/\mathbf{T}}\,. \tag{21}$$

and moreover $\mathbf{N}/\log C \to \infty$, namely we have a superlogarithmic growth of the number of quantization intervals, while the expected entrance time have a sublogarithmic growth $\mathbf{T}/\log C \to 0$. In the second situation when $\tau(C) \sim K \log C$ we have that both $\mathbf{N}$ and $\mathbf{T}$ grow logarithmically in C. More precisely, we have that

$$\begin{array}{rcl} \mathbf{N}/|a| & \sim & K(e^{1/K}-1)\log C \\ \mathbf{T} & \sim & K(1-e^{-1/K})\log C \end{array} \tag{22}$$

4.2 Logarithmic quantized feedback strategy

The second strategy is based on the quantized feedback (we assume $a > 0$, the case $a < 0$ being completely analogous)

$$k(x) = \begin{cases} -a+1 & \text{if } \epsilon \le x \le 1 \\ +a-1 & \text{if } -1 \le x \le -\epsilon \end{cases} \tag{23}$$

where

$$\epsilon = \frac{a-1}{a+1}.$$

In this way we obtain an almost (I, J)-stabilizing quantized feedback where $I = [-1, 1]$ and $J = [-\epsilon, \epsilon]$. The graph of closed loop map $\Gamma)x)$ is illustrated in Figure 2.

In this case we have a contraction rate $1/\epsilon$ and 2 quantization intervals. The expected entrance time can be found by noticing that

$$\Gamma^{-n}(I \setminus J) = [-1, -\epsilon_n] \cup [\epsilon_n, 1],$$

where $\epsilon_n = 1 - 2/(a+1)a^n$, which implies that the expected entrance time is

$$\sum_{n=0}^{\infty} \mathbb{P}[T_{(I,J)} > n] = \sum_{n=0}^{\infty} \mathbb{P}[\Gamma^{-n}(I \setminus J)] = \frac{2}{a+1} \sum_{n=0}^{\infty} a^{-n} = \frac{2a}{a^2-1}$$

In general, when we do not restrict to positive a, we obtain a triple contraction rate, quantization intervals, expected entrance time equal to

$$\left(\frac{|a|-1}{|a|+1}, 2, \frac{2|a|}{|a|^2-1}\right) \in \mathcal{A}.$$

Using the strategy presented above, we can iterate the construction τ times. In this case it is less obvious to show that the Lebesgue measure is invariant with respect to the map Ψ defined from Γ as in (15). To show this observe preliminarily that, if we assume that $\Gamma(x) = x$ for all $x \in J$, then

$$\lim_{n\to\infty} \Gamma^n(x) = \Gamma^{T_{(I,J)}(x)}(x), \qquad \text{for almost all } x \in I$$

which implies that $\Gamma^n(x)$ converges to $\Gamma^{T_{(I,J)}(x)}(x)$ in distribution. Observe moreover that, if the density function f_n of the random variable $\Gamma^n(x)$ is of the form

$$f_n(a) = \begin{cases} \alpha_n & \text{if } a \in J \\ \beta_n & \text{if } a \in I \setminus J, \end{cases}$$

then also f_{n+1} has the same structure with $\alpha_{n+1} = 2\beta_n/|a| + \alpha_n$ and $\beta_{n+1} = \beta_n/|a|$. This implies that

$$\lim_{n\to\infty} f_n(a) = \begin{cases} 1/\epsilon & \text{if } a \in I_1 \\ 0 & \text{if } a \in I_0 \setminus I_1 \end{cases}$$

from which we can argue that the Lebesgue measure is invariant with respect to the map Ψ.

These facts allow us to obtain a sequence of triples

$$\left(\left(\frac{|a|+1}{|a|-1}\right)^{\tau}, 2\tau, \frac{2|a|}{|a|^2-1}\tau\right) \in \mathcal{A}, \qquad \tau \in \mathbb{N}.$$

Making the change of variable

$$C = \left(\frac{|a|+1}{|a|-1}\right)^{\tau}, \qquad \tau = \frac{\log C}{\log(|a|+1) - \log(|a|-1)}$$

we obtain

$$\begin{aligned} \mathbf{N}/|a| &= \frac{2}{|a|} \frac{\log C}{\log(|a|+1) - \log(|a|-1)} \\ \mathbf{T} &= \frac{2|a|}{|a|^2-1} \frac{\log C}{\log(|a|+1) - \log(|a|-1)} \end{aligned}$$

These expressions motivate the fact that this this quantized feedback is called logarithmic quantizer. The strategy obtained in this way coincides with the one proposed in [7, 9] which yields a Lyapunov stability.

4.3 Chaotic quantized feedback strategy

In [9] another possible quantized feedback yielding almost stability has been proposed. This control strategy exploits the chaotic behavior of the state evolution inside $I = [-1, 1]$ produced by the feedback map

$$k_0(x) := -(2h+1) \quad \text{for} \quad \frac{2}{a}h < x \leq \frac{2}{a}(h+1), \tag{24}$$

when we have that $|a| \geq 2$. In this way we have that, for almost every initial condition x_0, the state evolution x_t is maintained inside the interval I and is dense in this interval. For this reason x_t will visit the interval $J = [-\epsilon, \epsilon]$. Therefore, if we modify this feedback map in J as follows

$$k(x) = \begin{cases} k_0(x) & \text{if } x \in I \setminus J \\ k_1(x) & \text{if } x \in J \end{cases} \tag{25}$$

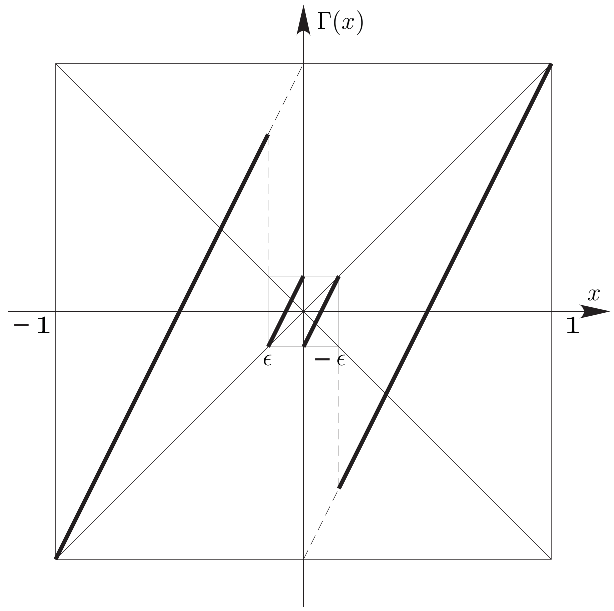

Figure 3: The map Γ associated with the quantized feedback defined in (25).

where $k_1(x)$ is any quantized feedback making J invariant (take for instance $k_1(x) = \epsilon k_0(x/\epsilon)$) we obtain that the state will move chaotically inside I till it will enter the interval J and there it will be entrapped. In this way we obtain a feedback map requiring

$$\mathbf{N} = \lceil |a| \rceil$$

quantization intervals. The closed loop map $\Gamma(x)$ is shown in Figure 3 in case $a = 2$.

In this case the evaluation of the expected entrance time can be done using Markov chain techniques. Assume that $\epsilon = 2^{-n}$. It is clear that, for evaluating the expected entrance time, we can refer to the system with feedback $k_0(x)$. Define the sets $I_i := [-i2^{-n}, -(i-1)2^{-n}] \cup [(i-1)2^{-n}, i2^{-n}]$, $i = 1, \ldots, 2^n$. In this way we have that $J = I_1$. Assuming that that initial state x_0 is uniformly distributed in I, we can argue that

$$\mathbb{P}[x_0 \in I_i] = 2^{-n}.$$

The initial distribution is described by the row vector

$$\pi := 2^{-n} \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \in \mathbb{R}^{1 \times 2^n} .$$

Assuming that the iterated state x_t is uniformly distributed in each quantization interval I_i, then the structure of the closed loop map $\Gamma_0(x) = ax + k_0(x)$ ensures that also the updated state $x_{t+1} = \Gamma_0(x_t)$ will be uniformly distributed in each quantization interval. Moreover we have that

$$\mathbb{P}[x_{t+1} \in I_j | x_t \in I_i] = \Pi_{ij} \ ,$$

where Π_{ij} is the i, j-element of the matrix

$$\Pi = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{2^n \times 2^n} \ .$$

Then (see [13]) the expected first entrance time in the state 1 is given by the formula

$$\mathbf{T} = \mathbb{E}(T_{(I,J)}) = \frac{d}{dz} w(z)_{|z=1} \ ,$$

where

$$w(z) := \frac{\pi \Pi(z) e_1}{e_1^T \Pi(z) e_1}$$

and where $\Pi(z) := \sum_{n \geq 0} \Pi^n z^n$ and $e_1 := [1 \ 0 \ \cdots \ 0]^T$. Since $\pi\Pi = \pi$, then

$$\pi \Pi(z) := \frac{1}{1-z} \pi.$$

It can be seen that

$$e_1^T \Pi(z) e_1 = 1 + 2^{-n} \frac{z^n}{1-z},$$

obtaining in this way

$$w(z) = \frac{1}{z^n + (1-z)2^n}$$

and

$$\mathbf{T} = \frac{d}{dz} w(z)_{|z=1} = 2^n - n \ .$$

In this way we obtained the triple

$$(2^n, 2, 2^n - n) \in \mathcal{A} \ .$$

Using the strategy presented above we can iterate this construction τ times. It can be shown that also in this case the Lebesgue measure is invariant with respect to the closed map Ψ defined from Γ as in (15). To show this we use the same kind of reasoning used in the previous subsection. Again, by defining Γ in such a way that $\Gamma(x) = x$ for all $x \in J$, we have that the random variable $\Gamma^n(x)$ converges to $\Gamma^{T_{(I,J)}(x)}(x)$ in distribution. Observe moreover that, if the density function f_n of the random variable $\Gamma^n(x)$ is constant in each quantization interval I_i, then it can be shown that also f_{n+1} has the same property. This implies that also the limit density will be a function which is constant in each set I_i and in particular in J. From this we can argue that the Lebesgue measure is invariant with respect to the map Ψ. These facts allow us to obtain a sequence of triples

$$(2^{\tau n}, \tau 2, \tau 2^n - \tau n) \in \mathcal{A}, \qquad n, \tau \in \mathbb{N}.$$

The previous reasoning can be extended to any situation in which $|a|$ is an integer. In this case it can be obtained sequence of triples

$$(|a|^{\tau n}, \tau |a|, \tau |a|^n - \tau n) \in \mathcal{A}, \qquad n, \tau \in \mathbb{N}.$$

which provides a family of quantized feedbacks parameterized by the two integers τ, n. We are mainly interested in understanding what asymptotic behavior can be obtained for $\mathbf{N}$ and $\mathbf{T}$ as $C \to \infty$. To this aim observe that

$$\frac{\mathbf{T}}{\frac{\mathbf{N}}{|a|} C^{\frac{|a|}{\mathbf{N}}}} = 1 - \frac{n}{|a|^n} \in [1 - \frac{1}{e \log |a|}, 1] \ .$$

Making the change of variable

$$C = |a|^{\tau n}, \qquad n = \frac{\log C}{\tau \log |a|} \tag{26}$$

we obtain that

$$\begin{aligned} \mathbf{N}/|a| &= \tau \\ \mathbf{T} &= \tau C^{\frac{1}{\tau}} - \frac{\log C}{\log |a|} \end{aligned}$$

where τ is any function of C that, by (26), can be chosen arbitrarily subject to the fact that $\tau(C)/\log C$ is bounded from above. If in particular τ is fixed, we obtain

$$\begin{aligned} \mathbf{N}/|a| &= \tau \\ \mathbf{T} &\sim \tau C^{\frac{1}{\tau}}. \end{aligned}$$

If instead we think of τ as a possible function of C, we can distinguish the case when $\tau(C)/\log C \to 0$ and the case when $\tau(C) \sim K \log C$. In the first case we have that

$$\mathbf{T} \sim \frac{\mathbf{N}}{|a|} C^{\frac{|a|}{\mathbf{N}}} \tag{27}$$

and moreover $\mathbf{N}/\log C \to 0$, namely a sublogarithmic growth of the number of quantization intervals, while the expected entrance time have a superlogarithmic growth $\mathbf{T}/\log C \to \infty$. In the second situation when $\tau(C) \sim K \log C$ we have that both $\mathbf{N}$ and $\mathbf{T}$ grow logarithmically in C. More precisely, we have that

$$\begin{aligned} \mathbf{N}/|a| &= K \log C \\ \mathbf{T} &= \left(K e^{1/K} - \tfrac{1}{\log |a|}\right) \log C \end{aligned} \tag{28}$$

Chaotic stabilizers can also be considered for non integers slopes a. Some preliminary results on this case have been obtained in [9]. In [8] the following more refined result is proved.

Theorem 1 *Let a be such that $|a| > 2$, $I = [-1, 1]$ and $J = [-\epsilon, \epsilon]$, where $0 < \epsilon < 1$. There exists an almost (I, J)-stabilizing quantized feedback $k : I \to \mathbb{R}$ such that*

$$\begin{aligned} \mathbf{N} &= \lceil |a| \rceil + 1 \\ \mathbf{T} &\leq KC, \end{aligned}$$

where K is a positive constant only depending on a.

Remark: The following table summarizes the properties of the different quantized feedback strategies.

	$\mathbf{N}/\|a\|$	$\mathbf{T}$
τ steps deadbeat quantizer	$\tau C^{\frac{1}{\tau}}$	τ
Logarithmic quantizer	$\frac{2}{\|a\|} \frac{\log C}{\log(\|a\|-1) - \log(\|a\|+1)}$	$\frac{2\|a\|}{\|a\|^2 - 1} \frac{\log C}{\log(\|a\|-1) - \log(\|a\|+1)}$
τ steps chaotic quantizer	τ	$\tau C^{\frac{1}{\tau}}$

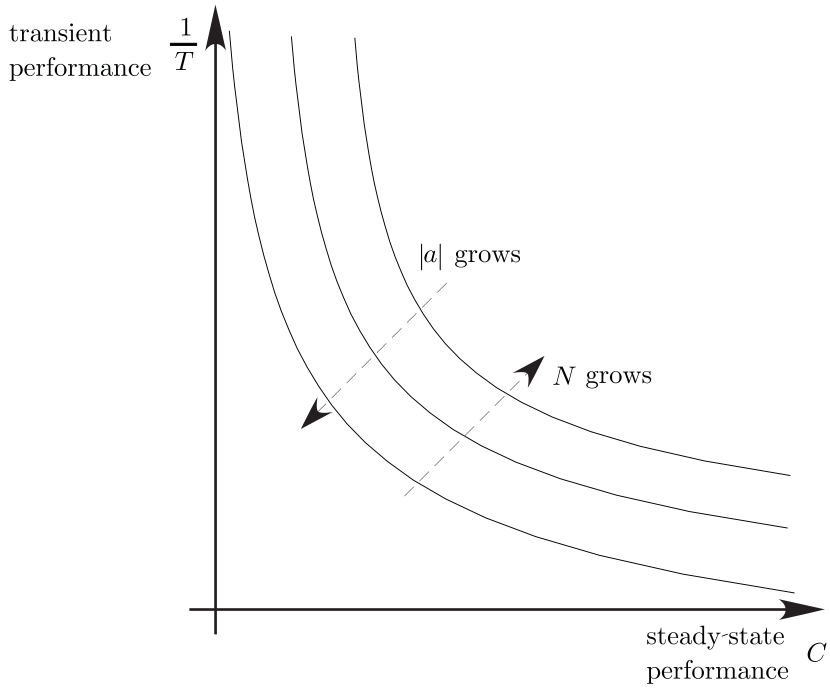

Figure 4: The qualitative relations between the parameters $|a|$, N, C and T. The parameter C describes the steady state performance, the parameter $1/T$ describes the transient performance, the curves describes the trade-off between these two parameters for fixed N and $|a|$. Different curves refer to different values of N and $|a|$.

This table highlights the relations between the parameters $|a|$, N, C and T. In all cases it is possible to see that the steady state performance parameter C and transient performance parameter T are conflicting, namely, for fixed $|a|$ and N, an increasing value of C implies an increasing value of T and vice versa. Moreover, both the performance parameters are improved when increasing N and are worsened when increasing $|a|$. A qualitative description of the relations between the parameters $|a|$, N, C and T is given in Figure 4.

This suggests that looking for the stabilizing quantized feedback with minimal quantization intervals is a rather naive approach to the quantized control problem. Indeed, in case we don't consider the transient performance described by the parameter T, the optimal strategy would be clearly the chaos based one. However, this provides only a partial view of the problem, since in fact the different strategies provide closed loop systems with different trade-off relations between the performance parameters T and C.

5 Bounds of the performance of a quantized feedback system

In this section we will present some general bounds involving the parameters $C(\Gamma), \mathbf{N}(\Gamma), \mathbf{T}(\Gamma)$. These will be obtained by means of a symbolic representation of the dynamical system and using basic combinatorial arguments.

5.1 Symbolic descriptions of the dynamical system

Let $\Gamma : I \to I$ be a piecewise affine map with fixed slope a. Let $J \subseteq I$ be another almost invariant interval. We can write

$$J = \overline{J_1 \cup J_2 \cup \cdots \cup J_M}, \qquad I = \overline{I_1 \cup I_2 \cup \cdots \cup I_N} \cup J \ ,$$

where the I_h's and the J_l's are disjoint open intervals on which Γ is affine. In the sequel we will use the shorthand notation $C = C(\Gamma)$, $\mathbf{N} = \mathbf{N}(\Gamma)$ and $\mathbf{T} = \mathbf{T}(\Gamma)$. In this section we will always consider Γ defined on the set Ω as defined in (4). Define the finite sets

$$\mathcal{I} = \{I_1, I_2, \dots I_{\mathbf{N}}\}, \quad \mathcal{J} = \{J_1, J_2, \dots J_M\}$$

and define a map $\psi : \Omega \to (\mathcal{I} \cup \mathcal{J})^{\mathbb{N}}$ by

$$\psi(x)_n = \omega_n \ \text{ if } \ \Gamma^n(x) \in \omega_n \, . \tag{29}$$

Notice that the above map is well defined by the way in which Ω has been defined. Consider the language $\Sigma_*(\Gamma)$ over the alphabet $\mathcal{I} \cup \mathcal{J}$ which is the subset of $(\mathcal{I} \cup \mathcal{J})^*$ consisting of all the finite words appearing in the infinite sequences in the range of ψ. If $|a| > 1$, then Γ is locally expanding and, as a consequence, the map ψ is injective. Indeed, it follows from (29) that $x \in \omega_0 \cap \cdots \cap \Gamma^{-n}\omega_n$ for every n. On the other hand it follows from the simple bound (33) presented below that the length of this interval goes to 0 for $n \to +\infty$, this yields injectivity. This implies that all the dynamical and statistical properties of the map Γ can be read out from the language $\Sigma_*(\Gamma)$. Notice, for further use, the following properties of simple verification:

1. $\omega_0\omega_1 \cdots \omega_n \in \Sigma_*(\Gamma)$ if and only if $\omega_0 \cap \Gamma^{-1}\omega_1 \cap \ldots \cap \Gamma^{-n}\omega_n \neq \emptyset$.
2. For all $\omega_0\omega_1 \cdots \omega_n \in \Sigma_*(\Gamma)$ the map Γ^{n+1} is affine on the interval $\omega_0 \cap \Gamma^{-1}\omega_1 \cap \ldots \cap \Gamma^{-n}\omega_n$.
3. If $\omega_0\omega_1 \cdots \omega_n$ and $\nu_0\nu_1 \cdots \nu_m$ are in $\Sigma_*(\Gamma)$ and none of the two happens to be the initial subword of the other, then the two intervals $\omega_0 \cap \Gamma^{-1}\omega_1 \cap \ldots \cap \Gamma^{-n}\omega_n$ and $\nu_0 \cap \Gamma^{-1}\nu_1 \cap \ldots \cap \Gamma^{-m}\nu_m$ are disjoint.

As we mentioned above, language $\Sigma_*(\Gamma)$ contains all the dynamical and statistical properties of the map Γ. In particular this is true for the expected entrance time. Indeed, as the following lemma shows, the expected entrance time can be estimated by knowing how the number of words in $\Sigma_*(\Gamma)$ grows with respect to their length. More precisely, denote by γ_n the number of distinct words in sublanguage $\Sigma_*(\Gamma) \cap \mathcal{I}^*$ of length n, i.e.,

$$\gamma_n := \#\{\omega_0\omega_1 \cdots \omega_{n-2}\omega_{n-1} \in \Sigma_*(\Gamma) \cap \mathcal{I}^*\} \, . \tag{30}$$

Then we have the following result.

Lemma 2 *Given any $n \in \mathbb{N}$ we have that*

$$\mathbb{P}[T_{(I,J)} = n] \quad \leq \quad \mathbb{P}[J]\frac{\gamma_n}{|a|^n} \tag{31}$$

$$\mathbb{P}[T_{(I,J)} \geq n] \quad \geq \quad \mathbb{P}[I \setminus J] - \mathbb{P}[J]\sum_{k=1}^{n-1} \frac{\gamma_k}{|a|^k} \, . \tag{32}$$

Proof As mentioned above, the family of intervals of the form

$$\omega_0 \cap \Gamma^{-1}(\omega_1) \cap \cdots \cap \Gamma^{-(n-1)}(\omega_{n-1}) \cap \Gamma^{-n}(\omega_n) \qquad \omega_0, \dots, \omega_{n-1} \in \mathcal{I}, \omega_n \in \mathcal{J}$$

constitute a partition of the set of points of I which end inside J in exactly n steps. Moreover, since Γ^n is affine on each of these intervals, it follows that

$$\mathbb{P}[\omega_0 \cap \Gamma^{-1}(\omega_1) \cap \cdots \cap \Gamma^{-(n-1)}(\omega_{n-1}) \cap \Gamma^{-n}(\omega_n)] \leq \frac{\mathbb{P}[J]}{|a|^n} \, . \tag{33}$$

Therefore, if we let

$$\tilde{\gamma}_n := \#\{\omega_0\omega_1 \cdots \omega_{n-2}\omega_{n-1} \in \Sigma_*(\Gamma) \mid \omega_0\omega_1 \cdots \omega_{n-2}\omega_{n-1} \in \mathcal{I}^* \text{ and } \omega_{n-1} \in \mathcal{J}\} \, ,$$

we can argue that

$$\mathbb{P}[T_{(I,J)} = n] \leq \mathbb{P}[J]\frac{\tilde{\gamma}_{n+1}}{|a|^n} \leq \mathbb{P}[J]\frac{\gamma_n}{|a|^n} \ ,$$

where we used the fact that for all $n \geq 1$ have that $\tilde{\gamma}_{n+1} \leq \gamma_n$.

We prove now the second assertion by induction on n. The assertion is trivial for $n = 1$. Assume by induction that the assertion holds for n and let us prove it for $n+1$. We can now write

$$\begin{aligned}\mathbb{P}[T_{(I,J)} \geq n+1] &= \mathbb{P}[T_{(I,J)} \geq n] - \mathbb{P}[T_{(I,J)} = n] \geq \mathbb{P}[T_{(I,J)} \geq n] - \mathbb{P}[J]\frac{\gamma_n}{|a|^n} \\ &\geq \mathbb{P}[I \setminus J] - \mathbb{P}[J]\sum_{k=1}^{n-1}\frac{\gamma_k}{|a|^k} - \mathbb{P}[J]\frac{\gamma_n}{|a|^n} = \mathbb{P}[I \setminus J] - \mathbb{P}[J]\sum_{k=1}^{n}\frac{\gamma_k}{|a|^k} \ .\end{aligned}$$

■

Notice that $\mathbb{P}[J] = C^{-1}$. This implies that formula (32) can be rewritten as

$$\mathbb{P}[T_{(I,J)} \geq n] \geq 1 - C^{-1} - C^{-1}\sum_{k=1}^{n-1}\frac{\gamma_k}{|a|^k} \ . \tag{34}$$

from which we can argue that for any arbitrarily fixed $\overline{n} \in \mathbb{N}$ we have that

$$\mathbf{T} = \mathbb{E}(T_{(I,J)}) = \sum_{n=1}^{+\infty}\mathbb{P}[T_{(I,J)} \geq n] \geq \sum_{n=1}^{\overline{n}}\mathbb{P}[T_{(I,J)} \geq n] \geq \overline{n}(1 - C^{-1}) - C^{-1}\sum_{n=1}^{\overline{n}}\sum_{k=1}^{n-1}\frac{\gamma_k}{|a|^k} \ . \tag{35}$$

If we can establish upper bounds on γ_k, through (35) we can thus achieve lower bounds on $\mathbf{T}$. The following theorem provides the most relevant contribution of this paper, since it presents a bound on the growth of γ_k depending on the number of quantization intervals $\mathbf{N}$. The proof of this theorem is very long and it will be presented in the last section.

Theorem 2 *Assume that $|a| > 2$. Then*

$$\frac{\gamma_k}{|a|^k} \leq 2\left[\sum_{s=1}^{r\wedge k}\binom{k-1}{s-1}\binom{r}{s}\left(\frac{s}{r}\right)^s\right]\left(\frac{\mathbf{N}M}{k \wedge \frac{\mathbf{N}M}{e}}\right)^{k\wedge\frac{\mathbf{N}M}{e}} \qquad \forall k \geq 1 \ , \tag{36}$$

where $r \in \{1, \ldots, \mathbf{N}\}$ is independent of k, but may depend on the specific system, while M depends only on $|a|$.

Remark: In symbolic dynamics [17] the set $\overline{\Psi(\Omega)}$ (where the closure is to be intended in the product topology of $(\mathcal{I} \cup \mathcal{J})^{\mathbb{N}}$) is called shift. It can be shown that its topological entropy is $\log|a|$. As a consequence, for every $\epsilon > 0$, there exists $M_\epsilon > 0$ such that

$$\gamma_k \leq M_\epsilon(|a| + \epsilon)^k \tag{37}$$

This type of estimate is of no use for our purposes for two reasons: first because the geometric growth rate $|a| + \epsilon$ causes a too quick growth in the double summation in (35), making impossible any useful estimate. Second, because it is not clear how explicitly M_ϵ depends on the map Γ. In fact, the estimate (36) is uniform with respect to all the possible piecewise affine maps having slope a and $\mathbf{N}$ quantization intervals. Notice moreover that for large k ($k \geq \max\{\mathbf{N}, \mathbf{N}/Me\}$), (36) can be written as

$$\gamma_k \leq (\overline{M}k)^{\mathbf{N}}|a|^k$$

where $\overline{M}$ is a suitable constant only depending on a. This is clearly a much better estimate than (37).

Using Theorem 2 we obtain a lower bound estimate on $\mathbf{T}$.

Corollary 1 *For any $\overline{n} \in \mathbb{N}$ we have that*

$$\mathbf{T} \geq \overline{n}(1 - C^{-1}) - 2C^{-1} \left[\sum_{s=1}^{r \wedge \overline{n}-1} \binom{\overline{n}}{s+1} \binom{r}{s} \left(\frac{s}{r}\right)^s \right] \left(\frac{\mathbf{N}M}{\overline{n} - 1 \wedge \frac{\mathbf{N}M}{e}} \right)^{\overline{n}-1 \wedge \frac{\mathbf{N}M}{e}} . \tag{38}$$

Proof From Theorem 2 we can argue that

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{\gamma_k}{|a|^k} &\leq 2 \sum_{k=1}^{n-1} \sum_{s=1}^{r \wedge k} \binom{k-1}{s-1} \binom{r}{s} \left(\frac{s}{r}\right)^s \left(\frac{\mathbf{N}M}{k \wedge \frac{\mathbf{N}M}{e}} \right)^{k \wedge \frac{\mathbf{N}M}{e}} \\ &= 2 \sum_{s=1}^{r \wedge n-1} \sum_{k=s}^{n-1} \binom{k-1}{s-1} \binom{r}{s} \left(\frac{s}{r}\right)^s \left(\frac{\mathbf{N}M}{k \wedge \frac{\mathbf{N}M}{e}} \right)^{k \wedge \frac{\mathbf{N}M}{e}} \\ &\leq 2 \sum_{s=1}^{r \wedge n-1} \binom{r}{s} \left(\frac{s}{r}\right)^s \max_{k=s}^{n-1} \left\{ \left(\frac{\mathbf{N}M}{k \wedge \frac{\mathbf{N}M}{e}} \right)^{k \wedge \frac{\mathbf{N}M}{e}} \right\} \sum_{k=s}^{n-1} \binom{k-1}{s-1} \\ &= 2 \left[\sum_{s=1}^{r \wedge n-1} \binom{n-1}{s} \binom{r}{s} \left(\frac{s}{r}\right)^s \right] \left(\frac{\mathbf{N}M}{n - 1 \wedge \frac{\mathbf{N}M}{e}} \right)^{n-1 \wedge \frac{\mathbf{N}M}{e}} , \end{aligned} \tag{39}$$

where we used the identity (88) and the bound (93) of the Appendix.

From (39) we can further obtain

$$\begin{aligned} \sum_{n=1}^{\overline{n}} \sum_{k=1}^{n-1} \frac{\gamma_k}{|a|^k} &\leq 2 \sum_{n=1}^{\overline{n}} \sum_{s=1}^{r \wedge n-1} \binom{n-1}{s} \binom{r}{s} \left(\frac{s}{r}\right)^s \left(\frac{\mathbf{N}M}{n - 1 \wedge \frac{\mathbf{N}M}{e}} \right)^{n-1 \wedge \frac{\mathbf{N}M}{e}} \\ &= 2 \sum_{s=1}^{r \wedge \overline{n}-1} \sum_{n=s+1}^{\overline{n}} \binom{n-1}{s} \binom{r}{s} \left(\frac{s}{r}\right)^s \left(\frac{\mathbf{N}M}{n - 1 \wedge \frac{\mathbf{N}M}{e}} \right)^{n-1 \wedge \frac{\mathbf{N}M}{e}} \\ &\leq 2 \sum_{s=1}^{r \wedge \overline{n}-1} \binom{r}{s} \left(\frac{s}{r}\right)^s \max_{n=s+1}^{\overline{n}} \left\{ \left(\frac{\mathbf{N}M}{n - 1 \wedge \frac{\mathbf{N}M}{e}} \right)^{n-1 \wedge \frac{\mathbf{N}M}{e}} \right\} \sum_{n=s+1}^{\overline{n}} \binom{n-1}{s} \\ &= 2 \left[\sum_{s=1}^{r \wedge \overline{n}-1} \binom{\overline{n}}{s+1} \binom{r}{s} \left(\frac{s}{r}\right)^s \right] \left(\frac{\mathbf{N}M}{\overline{n} - 1 \wedge \frac{\mathbf{N}M}{e}} \right)^{\overline{n}-1 \wedge \frac{\mathbf{N}M}{e}} , \end{aligned}$$

where again we used the identity (88) and the bound (93). From this (38) follows by a simple substitution. ∎

In the following subsections we will exploit the previous result to obtain bounds describing the trade-off between the number of quantization intervals $\mathbf{N}$ and the expected entrance time $\mathbf{T}$ for a given almost (I, J)-stable piecewise affine map Γ with contraction rate C. Three situations will be distinguished. First, we will consider the regime when $\mathbf{N}/\log C$ is sufficiently small. It contains the case when $\mathbf{N}/\log C \to 0$, namely the regime of sublogarithmic growth of $\mathbf{N}$ in C: the corresponding expected entrance time $\mathbf{T}$ will exhibit a superlogarithmic growth in C. The second case considered will be a sort of a dual of the first one, since we will assume that $\mathbf{T}/\log C$ is sufficiently small. It contains the case when $\mathbf{T}/\log C \to 0$, namely the regime of sublogarithmic growth of $\mathbf{T}$ in C: this time the corresponding number of quantization intervals $\mathbf{N}$ will exhibit a superlogarithmic growth in C. From these two cases we will then be able to study in detail a third situation, the logarithmic regime, which is when both $\mathbf{N}$ and $\mathbf{T}$ exhibit a logarithmic growth. In this case we will establish quantitative bounds relating the ratios $\mathbf{N}/\log C$ and $\mathbf{T}/\log C$.

5.2 The regime of sublogarithmic growth of $\mathbf{N}$ in C.

In this subsection we will assume that $\mathbf{N}/\log C$ is small enough. In this case it is convenient to proceed the estimates in (38) as follows

$$\begin{aligned}
\sum_{s=1}^{r\wedge\overline{n}-1}\binom{\overline{n}}{s+1}\binom{r}{s}\left(\frac{s}{r}\right)^s \;&\leq\; \sum_{s=0}^{r\wedge\overline{n}-1}\binom{\overline{n}}{s+1}\binom{r}{s}=\binom{\overline{n}+r}{r+1}=\frac{\overline{n}}{r+1}\binom{\overline{n}+r}{r} \\
&\leq\; \frac{1}{\sqrt{\pi}}\frac{\overline{n}}{r+1}\left(1+\frac{\overline{n}}{r}\right)^r e^r \leq \overline{n}\left(1+\frac{\overline{n}}{\mathbf{N}}\right)^{\mathbf{N}} e^{\mathbf{N}}\ ,
\end{aligned}$$

where we used the bound (91), the fact that $\frac{2}{(r+1)\sqrt{\pi}} \leq 1$ and that $\left(1+\frac{\overline{n}}{r}\right)^r e^r$ is an increasing function in r. We obtain in this way

$$\mathbf{T} \geq \overline{n}\left[1 - C^{-1} - \left(1+\frac{\overline{n}}{\mathbf{N}}\right)^{\mathbf{N}} A^{\mathbf{N}} C^{-1}\right]\ , \tag{40}$$

where $A := e^{\left(\frac{M}{e}+1\right)}$. We are now ready to prove the following result:

Theorem 3 *There exist $K_1 > 0$, $\beta_1 > 0$ and $C_1 > 1$ such that*

$$C \geq C_1 \quad \text{and} \quad \frac{\mathbf{N}}{\log C} \leq \beta_1 \implies \mathbf{T} \geq K_1 \mathbf{N} C^{1/\mathbf{N}}\ . \tag{41}$$

Proof If in (40) we choose $\overline{n} = \lceil D\mathbf{N}C^{1/\mathbf{N}}\rceil$ for some constant $D > 0$ which will be fixed later, we have that

$$\begin{aligned}
\frac{\mathbf{T}}{\mathbf{N}C^{1/\mathbf{N}}} \;&\geq\; \frac{\lceil D\mathbf{N}C^{1/\mathbf{N}}\rceil}{\mathbf{N}C^{1/\mathbf{N}}}\left[1 - C^{-1} - \left(1+\frac{\lceil D\mathbf{N}C^{1/\mathbf{N}}\rceil}{\mathbf{N}}\right)^{\mathbf{N}} A^{\mathbf{N}} C^{-1}\right] \\
&\geq\; D\left[1 - C^{-1} - \left(1+\frac{D\mathbf{N}C^{1/\mathbf{N}}+\mathbf{N}}{\mathbf{N}}\right)^{\mathbf{N}} A^{\mathbf{N}} C^{-1}\right] \\
&=\; D\left[1 - C^{-1} - \left(2C^{-1/\mathbf{N}}+D\right)^{\mathbf{N}} A^{\mathbf{N}}\right]\ .
\end{aligned} \tag{42}$$

Assume now that $\mathbf{N} \leq \beta \log C$ for some β which will be chosen later. This implies that

$$(2C^{-1/\mathbf{N}} + D)A \leq (2e^{-1/\beta} + D)A$$

By choosing β and D small enough, we obtain that $(2e^{-1/\beta} + D)A \leq 1/2$. Let β_1 and D_1 be possible solutions of the this inequality. In this situation we can argue that

$$\frac{\mathbf{T}}{\mathbf{N}C^{1/\mathbf{N}}} \geq D_1[1 - C^{-1} - (1/2)^{\mathbf{N}}] \geq D_1[1/2 - C^{-1}]$$

and so there exist $C_1 > 1$ and $K_1 > 0$ such that (41) holds true. ∎

Theorem 3 will be important for later results on the logarithmic regime. Notice moreover that the bound established in Theorem 3 resembles the relation (27) between the expected entrance time and the number of quantization intervals that can be obtained when using the nested chaotic scheme proposed in Subsection 4.3. However, there is a difference and in fact the bound provided by Theorem 3 is not tight in this case. Consider for simplicity the case in which $\tau = 1$ so that we have a simple chaotic quantized feedback. In this case we have $\mathbf{N} = \lceil |a| \rceil$ quantization intervals and this, by Theorem 3, yields the bound

$$\mathbf{T} \geq K_1 C^{1/\lceil |a| \rceil}.$$

However, this is not a good bound since we expect in this case that $\mathbf{T} \sim C$. In fact, this bound can be improved in this particular case by using Proposition 5 that is a modification of Theorem 2 in which r is fixed equal to 1.

Corollary 2 *There exist $K_1 > 0$, and $C_1 > 1$ such that*

$$C \geq C_1 \quad \text{and} \quad \mathbf{N} = \lceil |a| \rceil \implies \mathbf{T} \geq K_1 C \ .$$

Proof By Proposition 5 we can argue that

$$\frac{\gamma_k}{|a|^k} \leq 2 \left(\frac{\mathbf{N}M}{k \wedge \frac{\mathbf{N}M}{e}} \right)^{k \wedge \frac{\mathbf{N}M}{e}} \leq 2e^{\frac{\mathbf{N}M}{e}} = 2e^{\frac{\lceil |a| \rceil M}{e}} \ .$$

Let $A := e^{\frac{\lceil |a| \rceil M}{e}}$. Then, by (35) this implies that

$$\mathbf{T} \geq \overline{n}(1 - C^{-1}) - C^{-1} 2 \binom{\overline{n}}{2} A = \overline{n}[1 - C^{-1} - C^{-1}(\overline{n} - 1)A] \ .$$

Let $\overline{n} = \lceil DC \rceil$ for some constant $D > 0$ which will be fixed later. We have that

$$\frac{\mathbf{T}}{C} \geq D \left[1 - C^{-1} - (\lceil DC \rceil - 1) AC^{-1} \right] \geq D \left[1 - C^{-1} - DA \right]$$

and this implies the thesis. ■

5.3 The regime of sublogarithmic growth of $\mathbf{T}$ in C.

In this subsection we will assume instead that $\mathbf{T}/\log C$ is small enough. In this case it is convenient to proceed the estimates in (38) as follows

$$\begin{aligned}
\sum_{s=1}^{r \wedge \overline{n} - 1} \binom{\overline{n}}{s+1} \binom{r}{s} \left(\frac{s}{r} \right)^s &\leq \frac{1}{\sqrt{\pi}} \sum_{s=1}^{\overline{n}-1} \binom{\overline{n}}{s+1} \left(1 + \frac{r-s}{s} \right)^s e^s \left(\frac{s}{r} \right)^s \\
&= \frac{1}{\sqrt{\pi}} \sum_{s=1}^{\overline{n}-1} \binom{\overline{n}}{s+1} e^s \leq \frac{1}{\sqrt{\pi}} \sum_{s=0}^{\overline{n}} \binom{\overline{n}}{s} e^{s-1} = \frac{1}{\sqrt{e\pi}} (1+e)^{\overline{n}} \ ,
\end{aligned}$$

where again we used the bound (91). We thus obtain

$$\begin{aligned}
\mathbf{T} &\geq \overline{n}(1 - C^{-1}) - C^{-1} \frac{2}{\sqrt{e\pi}} (1+e)^{\overline{n}} \left(\frac{\mathbf{N}M}{\overline{n} - 1 \wedge \frac{\mathbf{N}M}{e}} \right)^{\overline{n} - 1 \wedge \frac{\mathbf{N}M}{e}} \\
&\geq \overline{n}(1 - C^{-1}) - C^{-1} A^{\overline{n}-1} \left(\frac{\mathbf{N}M}{\overline{n} - 1 \wedge \frac{\mathbf{N}M}{e}} \right)^{\overline{n} - 1 \wedge \frac{\mathbf{N}M}{e}} \ ,
\end{aligned} \tag{43}$$

where $A := \frac{2(1+e)^2}{e\sqrt{\pi}}$ and where the last inequality holds if $\overline{n} \geq 2$. We are now ready to prove the following result.

Theorem 4 *There exist $K_2 > 0$, $\gamma_2 > 0$ and $C_2 > 1$ such that*

$$C \geq C_2 \quad \text{and} \quad \frac{\lceil \mathbf{T} \rceil}{\log C} \leq \gamma_2 \implies \mathbf{N} \geq K_2 \lceil \mathbf{T} \rceil C^{\frac{1}{\lceil \mathbf{T} \rceil}} \ . \tag{44}$$

Proof We first show that we can find $C' > 1$ and $\gamma > 0$ such that

$$C \geq C' \quad \text{and} \quad \frac{\lceil \mathbf{T} \rceil}{\log C} \leq \gamma \implies \lceil \mathbf{T} \rceil \leq \frac{\mathbf{N}M}{e} \ . \tag{45}$$

Assume by contradiction that $\lceil \mathbf{T} \rceil > \mathbf{N}M/e$. Then, choosing $\overline{n} := \lceil \mathbf{T} \rceil + 1$, it follows from (35) and (43) that

$$\mathbf{T} \geq (\lceil \mathbf{T} \rceil + 1)(1 - C^{-1}) - C^{-1} A^{\lceil \mathbf{T} \rceil} e^{\frac{\mathbf{N}M}{e}} \geq (\lceil \mathbf{T} \rceil + 1)(1 - C^{-1}) - C^{-1} (eA)^{\lceil \mathbf{T} \rceil} \tag{46}$$

which implies that

$$\begin{aligned} 0 &\geq C(\lceil \mathbf{T} \rceil - \mathbf{T} + 1) - (eA)^{\lceil \mathbf{T} \rceil} - \lceil \mathbf{T} \rceil - 1 \\ &\geq C - (eA)^{\lceil \mathbf{T} \rceil} - \lceil \mathbf{T} \rceil - 1 \\ &\geq C - (eA)^{\gamma \log C} - \gamma \log C - 1 = C - C^{\gamma \log eA} - \gamma \log C - 1 \ . \end{aligned} \tag{47}$$

If we choose $\gamma < (\log eA)^{-1}$, it is clear that there exists $C' > 1$ such that

$$C - C^{\gamma \log eA} - \gamma \log C - 1 > 0$$

for all $C \geq C'$. For such values of C, (47) can not hold. Hence (45) must hold.

Assume now that (45) holds true and choose again $\overline{n} := \lceil \mathbf{T} \rceil + 1$ in (43). Then we obtain

$$\mathbf{T} \quad \geq \quad (\lceil \mathbf{T} \rceil + 1)(1 - C^{-1}) - \left(\frac{\mathbf{N}MA}{\lceil \mathbf{T} \rceil} \right)^{\lceil \mathbf{T} \rceil} C^{-1} \ . \tag{48}$$

Solving with respect to $\mathbf{N}$, we obtain

$$\mathbf{N} \geq \frac{\lceil \mathbf{T} \rceil}{AM} [C(\lceil \mathbf{T} \rceil - \mathbf{T} + 1) - \lceil \mathbf{T} \rceil - 1]^{1/\lceil \mathbf{T} \rceil} \geq \frac{\lceil \mathbf{T} \rceil}{AM} [C - \lceil \mathbf{T} \rceil - 1]^{1/\lceil \mathbf{T} \rceil} \geq \frac{\lceil \mathbf{T} \rceil}{AM} [C - \gamma \log C - 1]^{1/\lceil \mathbf{T} \rceil} \ . \tag{49}$$

Observe finally that

$$\lim_{C \to \infty} \frac{C - \gamma \log C - 1}{C} = 1$$

which implies that for any $\epsilon > 0$ there exists $C'' > 0$ such that $C - \gamma \log C - 1 > (1 - \epsilon)C$ for all $C > C''$. From this we can argue that

$$\mathbf{N} \geq \frac{\lceil \mathbf{T} \rceil}{AM} [(1 - \epsilon)C]^{1/\lceil \mathbf{T} \rceil} \geq \frac{1 - \epsilon}{AM} \lceil \mathbf{T} \rceil C^{1/\lceil \mathbf{T} \rceil} \ .$$

By letting $K_2 := \frac{1-\epsilon}{AM}$, $C_2 := C' \vee C''$ and $\gamma_2 := \gamma$ we have thus proved the thesis. ■

Also in this case it is interesting to compare the bound provided by the previous theorem with the relation (21) between the number of quantization intervals and the expected entrance time that can be obtained when using the nested strategy proposed in subsection 4.1. In this case this comparison shows that, up to a multiplication by a constant, the bound is tight.

5.4 The logarithmic regime

We have the following direct consequence of previous theorems.

Corollary 3 *There exists $C_0 > 1$ and two functions $F, G : \mathbb{R}_+ \to \mathbb{R}_+$ which are decreasing and converging to 0 at $+\infty$, such that for all $C > C_0$ we have that*

$$\frac{\mathbf{N}}{\log C} \geq F\left(\frac{\lceil \mathbf{T} \rceil}{\log C} \right) \quad \textit{and} \quad \frac{\lceil \mathbf{T} \rceil}{\log C} \geq G\left(\frac{\mathbf{N}}{\log C} \right) \tag{50}$$

Proof Notice first that the function $f :]0, 1] \to \mathbb{R} : x \mapsto x e^{1/x}$ is strictly decreasing and its image is $[e, +\infty)$. Let $C_0 := C_1 \vee C_2$, where C_1, C_2 are the constants introduced, respectively, in Theorems 3 and 4. Define the function

$$F(x) = \begin{cases} 1 \wedge \beta_1 & \text{if } 0 \leq x \leq K_1 f(1 \wedge \beta_1) \\ f^{-1}(x/K_1) & \text{if } x > K_1 f(1 \wedge \beta_1) \ , \end{cases}$$

where K_1 and β_1 are the constants provided by Theorem 3. This function is decreasing and such that $F(+\infty) = 0$. We want to show that, if $C > C_0$, then

$$\frac{\mathbf{N}}{\log C} \geq F\left(\frac{\lceil \mathbf{T} \rceil}{\log C} \right) \ .$$

If $\mathbf{N}/\log C > 1\wedge\beta_1$, then

$$\frac{\mathbf{N}}{\log C} > \max_{x\in\mathbb{R}_+} F(x) \geq F\left(\frac{\lceil\mathbf{T}\rceil}{\log C}\right)$$

If instead $\mathbf{N}/\log C \leq 1\wedge\beta_1$, then by Theorem 3 we can argue that

$$\frac{\mathbf{T}}{\log C} \geq K_1 \frac{\mathbf{N}}{\log C} C^{1/\mathbf{N}} = K_1 f\left(\frac{\lceil\mathbf{N}\rceil}{\log C}\right)$$

which implies that

$$\frac{\mathbf{N}}{\log C} \geq f^{-1}\left(\frac{\mathbf{T}}{K_1\log C}\right) = F\left(\frac{\mathbf{T}}{\log C}\right).$$

In the same way it can be shown that

$$\frac{\lceil\mathbf{T}\rceil}{\log C} \geq G\left(\frac{\mathbf{N}}{\log C}\right)$$

where

$$G(x) = \begin{cases} 1\wedge\gamma_2 & \text{if } 0\leq x\leq K_2 f(1\wedge\beta_1) \\ f^{-1}(x/K_2) & \text{if } x > K_2 f(1\wedge\gamma_2)\ , \end{cases}$$

where K_2 and γ_2 are the constants provided by Theorem 4. ■

Remark: The constraint provided by the previous corollary are illustrated in Figure 5 which shows explicitly the region in which the pairs $(\mathbf{N}/\log C, \mathbf{T}/\log C)$ can not belong to. Observe moreover that the functions $F(x)$ and $G(x)$ in the previous corollary which determines the boundary of this region tend to 0 as the function $f(x) = xe^{1/x}$. This is in agreement with the behavior of the logarithmic regime exhibited in the nesting of both dead-beat quantized feedbacks and of chaotic quantized feedbacks (see (22) and (28)). This implies that, up to multiplicative constants, our bounds appear to be quite tight and that the examples presented in Section 4 can not be improved much.

5.5 The case when $|a|\leq 2$

All previous results have been obtained under the assumption $|a|>2$. In fact, part of the results presented in this subsection can be extended to the case $|a|\leq 2$. Indeed, in this case, using the second part of Theorem 6, we obtain the estimate

$$\frac{\gamma_k}{2^k} \leq \left[\sum_{s=1}^{r\wedge k} \binom{k+s-1}{2s-1}\binom{r}{s}\left(\frac{s}{r}\right)^s\right]\left(\frac{\mathbf{N}M}{k\wedge\frac{\mathbf{N}M}{e}}\right)^{k\wedge\frac{\mathbf{N}M}{e}}$$

By similar arguments used to deal with the case $|a|>2$ we obtain

$$\mathbf{T} \geq \overline{n}(1-C^{-1}) - C^{-1}2\left[\sum_{s=1}^{r\wedge\overline{n}-1}\binom{\overline{n}+s}{2s+1}\binom{r}{s}\left(\frac{s}{r}\right)^s\right]\left(\frac{\mathbf{N}M}{\overline{n}-1\wedge\frac{\mathbf{N}M}{e}}\right)^{\overline{n}-1\wedge\frac{\mathbf{N}M}{e}}\left(\frac{2}{|a|}\right)^{\overline{n}-1}$$

for all $\overline{n}\in\mathbb{N}$. Observing that

$$\sum_{s=1}^{r\wedge\overline{n}-1}\binom{\overline{n}+s}{2s+1}\binom{r}{s}\left(\frac{s}{r}\right)^s \leq \frac{1}{\sqrt{\pi}}\sum_{s=1}^{\overline{n}-1}\binom{2\overline{n}-1}{2s+1}e^s \leq \frac{1}{\sqrt{\pi}}\sum_{s=1}^{\overline{n}-1}\binom{2\overline{n}-1}{2s+1}e^{2s+1} \leq \frac{1}{\sqrt{\pi}}(1+e)^{2\overline{n}-1}\ ,$$

we thus obtain

$$\begin{aligned}\mathbf{T} &\geq \overline{n}(1-C^{-1}) - C^{-1}\frac{2}{\sqrt{\pi}}(1+e)^{2\overline{n}-1}\left(\frac{\mathbf{N}M}{\overline{n}-1\wedge\frac{\mathbf{N}M}{e}}\right)^{\overline{n}-1\wedge\frac{\mathbf{N}M}{e}}\left(\frac{2}{|a|}\right)^{\overline{n}-1} \\ &\geq \overline{n}(1-C^{-1}) - C^{-1}A^{\overline{n}-1}\left(\frac{\mathbf{N}M}{\overline{n}-1\wedge\frac{\mathbf{N}M}{e}}\right)^{\overline{n}-1\wedge\frac{\mathbf{N}M}{e}}\ ,\end{aligned}$$

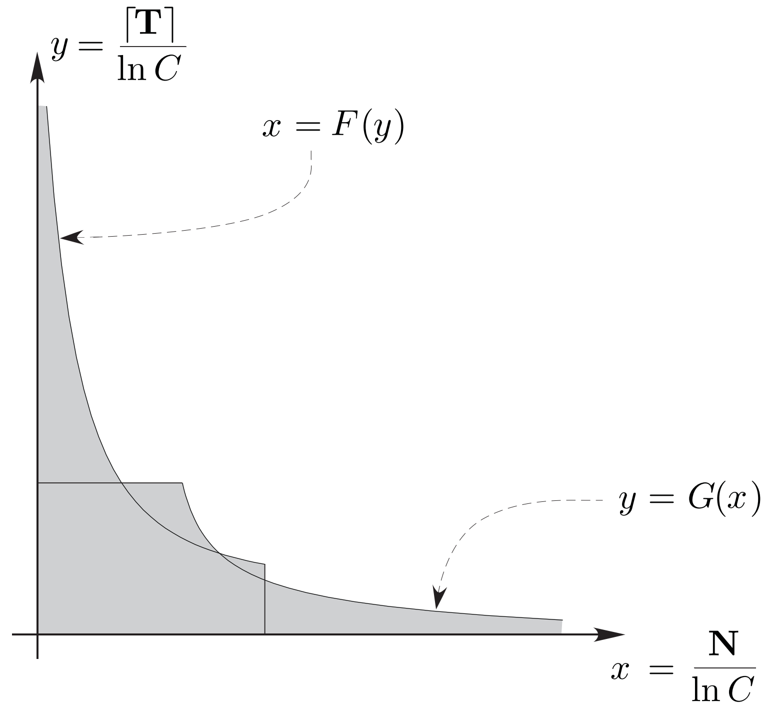

Figure 5: The grey region in this figure represents the set in which the pairs $(\mathbf{N}/\log C, \mathbf{T}/\log C)$ can not belong to.

where $A := \frac{4(1+e)^3}{|a|\sqrt{\pi}}$ and where the last inequality holds if $\overline{n} \geq 2$. The previous inequality looks exactly like (43). This immediately implies that Theorem 4 holds true also for $|a| \leq 2$. We can instead only recover a part of Corollary 3: (50) remains true for small values of γ, as it is easy to see from the proof we gave.

5.6 Stabilizing quantized feedbacks

In this section we will show that quantized control strategies yielding stability or even almost stability, but with only a countable subset of points never entering inside J, require a number of quantization intervals which grows at least logarithmically in C. The result is based on Theorem 7 which in the last section.

Theorem 5 *If Γ is almost (I, J)-stable and if the set of points in I never entering inside J is at most countable, then there exists $\beta > 0$, only depending on a, such that*

$$\mathbf{N}/\log C \geq \beta$$

for all $C > 1$.

Proof Using (87) we can argue that

$$\sum_{k=1}^{n-1} \frac{\gamma_k}{|a|^k} \leq e^{\frac{\mathbf{N}}{e}} \sum_{k=0}^{+\infty} \binom{k+2\mathbf{N}-1}{2\mathbf{N}-1} \left(\frac{2}{|a|}\right)^k = \frac{e^{\frac{\mathbf{N}}{e}}}{\left(1-\frac{2}{|a|}\right)^{2\mathbf{N}}} = \left(\frac{e^{1/e}|a|^2}{(|a|-1)^2}\right)^{\mathbf{N}} .$$

By letting $A := \frac{e^{1/e}|a|^2}{(|a|-1)^2}$, from (34) we can argue that

$$\mathbb{P}[T_{(I,J)} \geq n] \geq 1 - C^{-1} - A^{\mathbf{N}} C^{-1} \geq 1 - C^{-1}(1+A)^{\mathbf{N}} . \tag{51}$$

Since Γ is almost stable, by Proposition 1 we have that $\mathbb{E}(T_{(I,J)}) < +\infty$ which implies that

$$\lim_{n\to\infty} \mathbb{P}[T_{(I,J)} \geq n] = 0\,.$$

From this and from (51) we can argue that $1 - C^{-1}(1+A)^{\mathbf{N}} \leq 0$ which implies that

$$\mathbf{N} \geq \frac{\log C}{\log(1+A)}\,.$$

■

6 Estimation of paths in a class of weighted graphs

For proving our main result, namely Theorem 2, we introduce a class of weighted graphs and we propose a method for bounding the number of paths on this graphs. In the last section we will show how this bound can be used for proving Theorem 2. We use this strategy which consider this graph abstraction because the general result we are going to prove is useful to deal with more general situations, such as quantized controller with memory or the case in which the state of the system is multidimensional (see [10]).

Consider a direct graph $\mathcal{G}$ on a vertex set $\mathcal{X}$ (which is not necessarily finite or countable). For any choice of $\mathcal{X}_1, \dots, \mathcal{X}_k \subset \mathcal{X}$ we define $\mathcal{F}_k[\mathbf{x}_1 \in \mathcal{X}_1, \dots, \mathbf{x}_k \in \mathcal{X}_k]$ to be the set of paths $\mathbf{x}_1 \cdots \mathbf{x}_k \in \mathcal{X}^*$ on the graph $\mathcal{G}$ such that $\mathbf{x}_1 \in \mathcal{X}_1, \dots, \mathbf{x}_k \in \mathcal{X}_k$.

Assume the the graph $\mathcal{G}$ has the following structure. We assume there exist a finite partition

$$\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots \cup \mathcal{X}_{\mathbf{N}}\,,$$

a subset $\mathcal{X}_P \subseteq \mathcal{X}$ and a function $q : \mathcal{X} \to]0,1[$ with the following properties:

(A) There exist numbers $q_1, \dots, q_{\mathbf{N}} \in]0,1[$ such that

$$\begin{array}{ll} q(\mathbf{x}) \leq q_i\,, & \forall \mathbf{x} \in \mathcal{X}_i \\ q(\mathbf{x}) = q_i\,, & \forall \mathbf{x} \in \mathcal{X}_{P,i} := \mathcal{X}_P \cap \mathcal{X}_i\,. \end{array}$$

(B) There exist positive numbers D_1 and α_1 such that, for every $\mathbf{x}' \in \mathcal{X}$, $\mathcal{X}'' \subseteq \mathcal{X}$, and $k \geq 2$,

$$\#\mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}'\,, \mathbf{x}_2\,, \dots, \mathbf{x}_{k-1} \in \mathcal{X}, \mathbf{x}_k \in \mathcal{X}''] \leq D_1 \frac{q(\mathbf{x}')}{\inf_{\mathbf{x}'' \in \mathcal{X}''} q(\mathbf{x}'')} \alpha_1^{k-2}\,.$$

(C) There exist positive numbers D_2 and α_2 such that, for every $\mathbf{x}' \in \mathcal{X}$, $i \in \{1, \dots, \mathbf{N}\}$, and $k \geq 2$,

$$\#\mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}'\,, \mathbf{x}_2\,, \dots, \mathbf{x}_{k-1} \in \mathcal{X} \setminus \mathcal{X}_P, \mathbf{x}_k \in \mathcal{X}_i] \leq D_2 \alpha_2^{k-2}\,.$$

Then, if we define

$$\begin{aligned} \gamma_{k,h} &= \sup_{\mathbf{x}' \in \mathcal{X}_{P,h}} \#\mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2\,, \dots, \mathbf{x}_k \in \mathcal{X}], \qquad h = 1, \dots, \mathbf{N} \\ \gamma_k &= \sum_{h=1}^{\mathbf{N}} \gamma_{k,h}\ , \end{aligned} \tag{52}$$

we have the following result.

Theorem 6 *We have the following bounds:*

(1) If $\alpha_1 > \alpha_2$, then

$$\frac{\gamma_k}{\alpha_1^k} \leq 2 \left[\sum_{s=1}^{r \wedge k} \binom{k-1}{s-1} \binom{r}{s} \left(\frac{s}{r}\right)^s \right] \left(\frac{\mathbf{N}M}{k \wedge \frac{\mathbf{N}M}{e}} \right)^{k \wedge \frac{\mathbf{N}M}{e}} \qquad \forall k \geq 1\ . \tag{53}$$

(2) If $\alpha_1 \le \alpha_2$, then

$$\frac{\gamma_k}{\alpha_2^k} \le \left[\sum_{s=1}^{r\wedge k} \binom{k+s-1}{2s-1} \binom{r}{s} \left(\frac{s}{r}\right)^s \right] \left(\frac{\mathbf{N}M}{k \wedge \frac{\mathbf{N}M}{e}} \right)^{k\wedge \frac{\mathbf{N}M}{e}} \qquad \forall k \ge 1 \ . \tag{54}$$

The constant $r \in \{1,\dots,\mathbf{N}\}$ is independent of k, but may depend on the specific graph, while M depends only on the constants $D_1, D_2, \alpha_1, \alpha_2$.

The proof of the previous theorem is quite involved. For this reason we prefer to divide it into various steps.

Remark As specified in the previous theorem M depends only on the constants $D_1, D_2, \alpha_1, \alpha_2$ and r depends on the the specific graph. These conditions can be exchanged and the same bounds can be shown to hold true in which instead r depends only on the constants $D_1, D_2, \alpha_1, \alpha_2$ and M depends on the the specific graph. However, this exchange makes the bounds useless in general. Only in the specific situation considered in Proposition 5 this point of view yields some advantages.

6.1 The proof of Theorem 6: hierarchies of paths

Assume with no loss of generality that the subsets $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_{\mathbf{N}}$ are ordered in such a way that

$$q_1 \ge q_2 \ge \dots \ge q_{\mathbf{N}} \ .$$

For any choice of integers

$$0 = N_0 < N_1 < \dots < N_{r-1} < N_r = \mathbf{N}$$

we can partition $\mathcal{X}_P$ into the subfamilies

$$\mathcal{X}_P^1 := \{\mathcal{X}_{P,N_0+1}, \dots, \mathcal{X}_{P,N_1}\}\,, \ \ \mathcal{X}_P^2 := \{\mathcal{X}_{P,N_1+1}, \dots, \mathcal{X}_{P,N_2}\}\,, \dots, \ \mathcal{X}_P^r := \{\mathcal{X}_{P,N_{r-1}+1}, \dots, \mathcal{X}_{P,N_r}\} \tag{55}$$

and consider moreover

$$\begin{aligned}
\mathcal{X}_P^{l+} &:= \bigcup_{j=l}^{r} \mathcal{X}_P^j \\
\mathcal{X}^{l} &:= (\mathcal{X} \setminus \mathcal{X}_P) \cup \mathcal{X}_P^l \\
\mathcal{X}^{l+} &:= (\mathcal{X} \setminus \mathcal{X}_P) \cup \mathcal{X}_P^{l+} \ .
\end{aligned}$$

For each $k \in \mathbb{N}$, $h = 1, \dots, \mathbf{N}$, and $l = 1, \dots, r, r+1$, define

$$\gamma_{k,h,l} := \sup_{\mathbf{x}' \in \mathcal{X}_{P,h}} \#\mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathcal{X}^{l+}] \ .$$

From these definitions it follows that $\mathcal{X}_P^{1+} = \mathcal{X}_P$, $\mathcal{X}^{1+} = \mathcal{X}$, and $\mathcal{X}^{(r+1)+} := \mathcal{X} \setminus \mathcal{X}_P$. This implies that $\gamma_{k,h,1} = \gamma_{k,h}$.

We present now two bounds on $\gamma_{k,h,l}$ which will be used in the sequel. The first bound is based on the decomposition of the paths in $\mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathcal{X}^{l+}]$ according to the last exit from $\mathcal{X}_P^l$ among the indices $j = 2, \dots, k$:

$$\begin{aligned}
&\mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathcal{X}^{l+}] = \\
&\left\{ \bigcup_{j=2}^{k} \bigcup_{s=N_{l-1}+1}^{N_l} \mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_{j-1} \in \mathcal{X}^{l+}, \mathbf{x}_j \in \mathcal{X}_{P,s}, \mathbf{x}_{j+1}, \dots, \mathbf{x}_k \in \mathcal{X}^{(l+1)+}] \right\} \bigcup \\
&\qquad\qquad \bigcup \mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathcal{X}^{(l+1)+}] \ .
\end{aligned}$$

Applying Property (B) it follows that for all $l = 1, \dots, r$

$$\begin{aligned}\gamma_{k,h,l} \quad &\le \quad \sum_{j=2}^{k} \sum_{s=N_{l-1}+1}^{N_l} \sup_{\mathbf{x}' \in \mathcal{X}_{P,h}} \#\mathcal{F}_j[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_{j-1} \in \mathcal{X}^{l+}, \mathbf{x}_j \in \mathcal{X}_{P,s}] \cdot \\ &\qquad \cdot \sup_{\mathbf{x}'' \in \mathcal{X}_{P,s}} \#\mathcal{F}_{k-j+1}[\mathbf{x}_j = \mathbf{x}'', \mathbf{x}_{j+1}, \dots, \mathbf{x}_k \in \mathcal{X}^{(l+1)+}] + \\ &\qquad + \sup_{\mathbf{x}' \in \mathcal{X}_{P,h}} \#\mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathcal{X}^{(l+1)+}] \qquad (56) \\ &\le \quad \sum_{j=2}^{k} \sum_{s=N_{l-1}+1}^{N_l} D_1 \frac{q_h}{q_s} \alpha_1^{j-2} \gamma_{k-j+1,s,l+1} + \gamma_{k,h,l+1} \ .\end{aligned}$$

The second bound is based on the decomposition of the paths in $\mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathcal{X}^{l+}]$ according to the first entrance in $\mathcal{X}_P^{l+}$ among the indices $j = 2, \dots, k$:

$$\begin{aligned}&\mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathcal{X}^{l+}] = \\ &\left\{ \bigcup_{j=2}^{k} \bigcup_{s=N_{l-1}+1}^{\mathbf{N}} \mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_{j-1} \in \mathcal{X} \setminus \mathcal{X}_P, \mathbf{x}_j \in \mathcal{X}_{P,s}, \mathbf{x}_{j+1}, \dots, \mathbf{x}_k \in \mathcal{X}^{l+}] \right\} \bigcup \\ &\qquad \bigcup \mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathcal{X} \setminus \mathcal{X}_P] \ .\end{aligned}$$

Applying property (C) it follows that

$$\begin{aligned}\gamma_{k,h,l} \quad &\le \quad \sum_{j=2}^{k} \sum_{s=N_{l-1}+1}^{\mathbf{N}} \sup_{\mathbf{x}' \in \mathcal{X}_{P,h}} \mathcal{F}_j[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_{j-1} \in \mathcal{X} \setminus \mathcal{X}_P, \mathbf{x}_j \in \mathcal{X}_{P,s}] \cdot \\ &\qquad \cdot \sup_{\mathbf{x}'' \in \mathcal{X}_{P,s}} \mathcal{F}_{k-j+1}[\mathbf{x}_j = \mathbf{x}'', \mathbf{x}_{j+1}, \dots, \mathbf{x}_k \in \mathcal{X}^{l+}] + \\ &\qquad + \sup_{\mathbf{x}' \in \mathcal{X}_{P,h}} \mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathcal{X} \setminus \mathcal{X}_P] \qquad (57) \\ &\le \quad \sum_{j=2}^{k} \sum_{s=N_{l-1}+1}^{\mathbf{N}} D_2 \alpha_2^{j-2} \gamma_{k-j+1,s,l} + D_2 \alpha_2^{k-1} \ .\end{aligned}$$

Notice that the previous bound holds true for $l = 1, \dots, r, r+1$.

Define now

$$\delta_{k,l} := \sum_{h=N_{l-1}+1}^{\mathbf{N}} \gamma_{k,h,l}, \quad \tilde{\delta}_{k,l} := \sum_{h=N_{l-2}+1}^{N_{l-1}} \gamma_{k,h,l} \qquad l = 1, \dots, r, r+1$$

which imply that $\delta_{k,r+1} = 0$ for all $k \in \mathbb{N}$. Observe that from (56) we can argue that

$$\begin{aligned}\delta_{k,l} \quad &\le \quad \sum_{h=N_{l-1}+1}^{\mathbf{N}} \left(\sum_{j=2}^{k} \sum_{s=N_{l-1}+1}^{N_l} D_1 \frac{q_h}{q_s} \alpha_1^{j-2} \gamma_{k-j+1,s,l+1} + \gamma_{k,h,l+1} \right) \le \\ &\le \quad \sum_{j=2}^{k} D_1 \frac{\sum_{h=N_{l-1}+1}^{\mathbf{N}} q_h}{q_{N_l}} \alpha_1^{j-2} \sum_{s=N_{l-1}+1}^{N_l} \gamma_{k-j+1,s,l+1} + \sum_{h=N_{l-1}+1}^{N_l} \gamma_{k,h,l+1} + \sum_{h=N_l+1}^{\mathbf{N}} \gamma_{k,h,l+1} \le \\ &\le \quad D_1 \beta_l \sum_{j=2}^{k} \alpha_1^{j-2} \tilde{\delta}_{k-j+1,l+1} + \tilde{\delta}_{k,l+1} + \delta_{k,l+1} = D_1 \beta_l \sum_{j=0}^{k-2} \alpha_1^{j} \tilde{\delta}_{k-j-1,l+1} + \tilde{\delta}_{k,l+1} + \delta_{k,l+1} \ ,\end{aligned}$$

where we define

$$\beta_l := \frac{\displaystyle\sum_{h=N_{l-1}+1}^{\mathbf{N}} q_h}{q_{N_l}} \ . \qquad (58)$$

On the other hand (57) implies that

$$\tilde{\delta}_{k,l} \quad \leq \quad D_2(N_{l-1} - N_{l-2}) \left(\sum_{j=2}^{k} \alpha_2^{j-2} \delta_{k-j+1,l} + \alpha_2^{k-1} \right) ,$$

which, using the convention

$$\delta_{0,l} = 1 \qquad l = 1, \ldots, r, r+1 \ ,$$

is equivalent to

$$\tilde{\delta}_{k,l} \quad \leq \quad D_2 \Delta N_{l-1} \sum_{j=2}^{k+1} \alpha_2^{j-2} \delta_{k-j+1,l} = D_2 \Delta N_{l-1} \sum_{j=0}^{k-1} \alpha_2^{j} \delta_{k-j-1,l} \ ,$$

where we defined $\Delta N_l := N_l - N_{l-1}$.

Summarizing we have the following two inequalities, holding for $k \geq 1$

$$\begin{aligned} \delta_{k,l} \quad &\leq \quad D_1 \beta_l \sum_{j=0}^{k-2} \alpha_1^j \tilde{\delta}_{k-j-1,l+1} + \tilde{\delta}_{k,l+1} + \delta_{k,l+1} \qquad l = 1, \ldots, r \\ \tilde{\delta}_{k,l} \quad &\leq \quad D_2 \Delta N_{l-1} \sum_{j=0}^{k-1} \alpha_2^j \delta_{k-j-1,l} \qquad l = 1, \ldots, r, r+1 \ . \end{aligned} \tag{59}$$

Define now the sequences $\eta_{k,l}, \tilde{\eta}_{k,l}$ for $k = 0, 1, 2, \ldots$ and $l = 1, \ldots, r, r+1$ by letting $\eta_{k,r+1} = \delta_{k,r+1} = 0$ for $k = 0, 1, \ldots$, and satisfying, for every $k \geq 0$, the following recursive relations

$$\begin{aligned} \eta_{k,l} \quad &= \quad D_1 \beta_l \sum_{j=0}^{k-2} \alpha_1^j \tilde{\eta}_{k-j-1,l+1} + \tilde{\eta}_{k,l+1} + \eta_{k,l+1} \\ \tilde{\eta}_{k,l} \quad &= \quad D_2 \Delta N_{l-1} \sum_{j=0}^{k-1} \alpha_2^j \eta_{k-j-1,l} \ . \end{aligned} \tag{60}$$

Notice that, from the above recursive relations it follows that $\eta_{0,l} = 1$ for every l. This implies, in particular, that $\delta_{k,l} \leq \eta_{k,l}$ for every k and l. In the sequel we will estimate $\eta_{k,l}$ by using the zeta transforms formalism.

Let

$$\eta_l(z) := \sum_{k=0}^{+\infty} \eta_{k,l} z^k, \quad \tilde{\eta}_l(z) := \sum_{k=0}^{+\infty} \tilde{\eta}_{k,l} z^k \ .$$

Then by some standard manipulations from (60) we obtain

$$\begin{aligned} \eta_l(z) \quad &= \quad D_1 \beta_l \frac{z}{1 - \alpha_1 z} \tilde{\eta}_{l+1}(z) + \tilde{\eta}_{l+1}(z) + \eta_{l+1}(z) \\ \tilde{\eta}_l(z) \quad &= \quad D_2 \Delta N_{l-1} \frac{z}{1 - \alpha_2 z} \eta_l(z) \end{aligned} \tag{61}$$

which yields

$$\eta_l(z) = \left\{ \left[D_1 \beta_l \frac{z}{1 - \alpha_1 z} + 1 \right] D_2 \Delta N_l \frac{z}{1 - \alpha_2 z} + 1 \right\} \eta_{l+1}(z) \ .$$

By iterating this formula we obtain

$$\eta_1(z) = \prod_{l=1}^{r} \left\{ \left[D_1 \beta_l \frac{z}{1 - \alpha_1 z} + 1 \right] D_2 \Delta N_l \frac{z}{1 - \alpha_2 z} + 1 \right\} \ , \tag{62}$$

where we used the fact that $\eta_{r+1}(z) = 1$.

6.2 The proof of Theorem 6: combinatorial bounds

We now want to estimate the coefficients $\eta_{k,1}$ of $\eta_1(z)$. We recall that $\gamma_k = \delta_{k,1} \leq \eta_{k,1}$. In order to obtain such bounds we will first need to work out some combinatorics.

Bounds on the coefficients of elementary symmetric polynomials

Consider the following polynomial in the indeterminates x and y

$$p(x,y) := \prod_{l=1}^{r} \{[\beta_l x + 1]\,\alpha_l y + 1\} = \sum_{s=0}^{r} \sum_{\sigma=0}^{s} \bar{p}_{\sigma,s} x^\sigma y^s \ . \tag{63}$$

The aim of this part of the section is to determine bounds on the coefficients $\bar{p}_{\sigma,s}$ if we assume that

$$\sum_{l=0}^{r} \alpha_l \leq \alpha, \qquad \sum_{l=0}^{r} \beta_l \leq \beta \ .$$

Consider preliminarily the polynomial

$$\prod_{l=1}^{r} \{\alpha_l y + 1\} = \sum_{s=0}^{r} p_s^r(\alpha_1, \ldots, \alpha_r) y^s \ . \tag{64}$$

The polynomial $p_s^r(\alpha_1, \ldots, \alpha_r)$ are called elementary symmetric polynomials [11] and they can be expressed by the formula

$$p_s^r(\alpha_1, \ldots, \alpha_r) = \sum_{1 \leq l_1 < \cdots l_s \leq r} \prod_{j=1}^{s} \alpha_{l_j} \ .$$

We have the following first elementary result.

Lemma 3 *Assume that* $\sum_{l=0}^{r} \alpha_l \leq \alpha$. *Then*

$$p_s^r(\alpha_1, \ldots, \alpha_r) \leq \binom{r}{s} \left(\frac{\alpha}{r}\right)^s \ . \tag{65}$$

Proof We will actually prove that bound (65) holds true and it is attained when $\alpha_i = \alpha/r$ for all $i = 1, \ldots, r$. For $r = 2$ it can be proven directly. For the general case, it is sufficient to notice that

$$\begin{aligned}
&p_s^r(\alpha_1, \alpha_2, \ldots, \alpha_r) = \\
&= p_s^{r-2}(\alpha_3, \ldots, \alpha_r) + p_1^2(\alpha_1, \alpha_2) p_{s-1}^{r-2}(\alpha_3, \ldots, \alpha_r) + p_2^2(\alpha_1, \alpha_2) p_{s-2}^{r-2}(\alpha_3, \ldots, \alpha_r) \\
&\leq p_s^{r-2}(\alpha_3, \ldots, \alpha_r) + p_1^2\left(\tfrac{\alpha_1+\alpha_2}{2}, \tfrac{\alpha_1+\alpha_2}{2}\right) p_{s-1}^{r-2}(\alpha_3, \ldots, \alpha_r) + p_2^2\left(\tfrac{\alpha_1+\alpha_2}{2}, \tfrac{\alpha_1+\alpha_2}{2}\right) p_{s-2}^{r-2}(\alpha_3, \ldots, \alpha_r) \\
&= p_s^r\left(\tfrac{\alpha_1+\alpha_2}{2}, \tfrac{\alpha_1+\alpha_2}{2}, \alpha_3, \ldots, \alpha_r\right) \ .
\end{aligned}$$

■

We come back to the problem of finding bounds on the coefficients $\bar{p}_{\sigma,s}$ of the polynomial (63).

Lemma 4 *For every* $1 \leq s \leq r$ *and* $0 \leq \sigma \leq s$, *the following bound holds*

$$\bar{p}_{\sigma,s} \leq \binom{s}{\sigma} \left(\frac{\beta}{s}\right)^\sigma \binom{r}{s} \left(\frac{\alpha}{r}\right)^s \ . \tag{66}$$

Proof Observe first that

$$p(x,y) = \prod_{l=1}^{r} \{[\beta_l x + 1]\,\alpha_l y + 1\} = \sum_{s=0}^{r} p_s^r(\alpha_1(1+\beta_1 x), \ldots, \alpha_r(1+\beta_r x)) y^s \ .$$

Moreover we have that

$$\begin{aligned} p_s^r(\alpha_1(1+\beta_1 x),\ldots,\alpha_r(1+\beta_r x)) &= \sum_{1\le l_1<\cdots l_s\le r} \prod_{j=1}^s \alpha_{l_j} \prod_{j=1}^s (1+\beta_{l_j} x) = \\ &= \sum_{1\le l_1<\cdots l_s\le r} \prod_{j=1}^s \alpha_{l_j} \sum_{\sigma=0}^s p_\sigma^s(\beta_{l_1},\ldots,\beta_{l_s}) x^\sigma \end{aligned}$$

from which we can argue that, using Lemma 3,

$$\begin{aligned} \bar p_{\sigma,s} &= \sum_{1\le l_1<\cdots l_s\le r} p_\sigma^s(\beta_{l_1},\ldots,\beta_{l_s}) \prod_{j=1}^s \alpha_{l_j} \le \binom{s}{\sigma}\left(\frac{\beta}{s}\right)^\sigma \sum_{1\le l_1<\cdots l_s\le r} \prod_{j=1}^s \alpha_{l_j} = \\ &= \binom{s}{\sigma}\left(\frac{\beta}{s}\right)^\sigma p_s^r(\alpha_1,\ldots,\alpha_r) \le \binom{s}{\sigma}\left(\frac{\beta}{s}\right)^\sigma \binom{r}{s}\left(\frac{\alpha}{r}\right)^s . \end{aligned}$$

■

To apply the result provided by the previous lemma to our problem, we need to have bounds on $\sum_{l=1}^r \Delta N_l$ and $\sum_{l=1}^r \beta_l$. While it is evident that

$$\sum_{l=1}^r \Delta N_l = \mathbf{N} ,$$

it is less clear how to bound the other sum. This will depend indeed on the way the subfamilies $\mathcal{X}_P^i$ are selected. It follows from (58) that

$$\beta_l = \sum_{k=0}^{r-l} \frac{\sum_{h=N_{l+k-1}+1}^{N_{l+k}} q_h}{q_l} \le \sum_{k=0}^{r-l} \Delta N_{l+k} \frac{q_{N_{l+k-1}+1}}{q_{N_l}} = \frac{q_{N_{l-1}+1}}{q_{N_l}} \sum_{k=0}^{r-l} \Delta N_{l+k} \frac{q_{N_{l+k-1}+1}}{q_{N_{l-1}+1}} . \tag{67}$$

Choose inductively the numbers N_l as follows

$$N_l = \max\left\{k \ge N_{l-1}+1 \mid q_k \ge \frac{1}{2} q_{N_{l-1}+1}\right\} . \tag{68}$$

In this way we have that

$$\frac{q_{N_{l-1}+1}}{q_{N_l}} \le 2 \qquad \frac{q_{N_{l+k-1}+1}}{q_{N_{l-1}+1}} \le 2^{-k} .$$

Inserting in (67) we thus obtain

$$\beta_l \le 2\sum_{k=0}^{r-l} \Delta N_{l+k} 2^{-k} \quad \forall l = 1,\ldots,r \tag{69}$$

which implies that

$$\sum_{l=1}^r \beta_l \le 2\sum_{l=1}^r \sum_{k=0}^{r-l} \Delta N_{l+k} 2^{-k} = 2\sum_{k=0}^{r-1}\left(\sum_{l=1}^{r-k} \Delta N_{l+k}\right) 2^{-k} \le 2\mathbf{N}\sum_{k=0}^{r-1} 2^{-k} \le 4\mathbf{N} .$$

Hence it follows from Lemma 3 that in our case the coefficients $\bar p_{\sigma,s}$ can be bounded as

$$\bar p_{\sigma,s} \le \binom{r}{s}\binom{s}{\sigma}\left(\frac{\mathbf{N}D_2}{r}\right)^s\left(\frac{4\mathbf{N}D_1}{s}\right)^\sigma . \tag{70}$$

Bounds on the coefficients of the power series

Define the coefficients $a_k^{\sigma,s}$ by

$$\left(\frac{1}{1-\alpha_1 z}\right)^{\sigma}\left(\frac{1}{1-\alpha_2 z}\right)^{s}=\sum_{k=0}^{+\infty} a_k^{\sigma,s} z^k \ . \tag{71}$$

The aim of this part of the section is to determine bounds on the coefficients $a_k^{\sigma,s}$. Simple combinatorial manipulation shows that

$$a_k^{\sigma,0}=\binom{k+\sigma-1}{\sigma-1}\alpha_1^k \ \ \forall\sigma\geq 1\ \forall k\geq 0\ . \tag{72}$$

In general we have the bound given by the following lemma.

Lemma 5 *Assume that $\alpha_1>\alpha_2$. Then, for every $s\geq 0$, $\sigma\geq 1$ and $k\geq 0$ we have*

$$0\leq a_k^{\sigma,s}\leq\left(\frac{\alpha_1}{\alpha_1-\alpha_2}\right)^{s} a_k^{\sigma,0}\ .$$

Proof We start by proving that

$$a_k^{\sigma,1}\leq\frac{\alpha_1}{\alpha_1-\alpha_2}a_k^{\sigma,0}$$

by induction on k. It is trivial if $k=0$. Assume it to be true for $k-1$ (with $k\geq 1$) and let us prove it for k. Then

$$\begin{aligned} a_k^{\sigma,1} &= \sum_{h=0}^{k} a_h^{\sigma,0}\alpha_2^{k-h}=\sum_{h=0}^{k}\binom{h+\sigma-1}{\sigma-1}\alpha_1^h\alpha_2^{k-h} \\ &= \sum_{h=0}^{k-1}\binom{h+\sigma-1}{\sigma-1}\alpha_1^h\alpha_2^{k-h}+\binom{k+\sigma-1}{\sigma-1}\alpha_1^k=\alpha_2 a_{k-1}^{\sigma,1}+\binom{k+\sigma-1}{\sigma-1}\alpha_1^k\ . \end{aligned}$$

Using the induction we obtain

$$\begin{aligned} a_k^{\sigma,1} &\leq \frac{\alpha_2\alpha_1}{\alpha_1-\alpha_2}a_{k-1}^{\sigma,0}+\binom{k+\sigma-1}{\sigma-1}\alpha_1^k=\left[\frac{\alpha_2}{\alpha_1-\alpha_2}\binom{k+\sigma-2}{\sigma-1}+\binom{k+\sigma-1}{\sigma-1}\right]\alpha_1^k \\ &= \left[\frac{\alpha_2}{\alpha_1-\alpha_2}\frac{k}{k+\sigma-1}+1\right]\binom{k+\sigma-1}{\sigma-1}\alpha_1^k\leq\left[\frac{\alpha_2}{\alpha_1-\alpha_2}+1\right]\binom{k+\sigma-1}{\sigma-1}\alpha_1^k=\frac{\alpha_1}{\alpha_1-\alpha_2}a_k^{\sigma,0}\ . \end{aligned}$$

Finally assume that the assertion of the lemma holds true for $s-1$. Then

$$a_k^{\sigma,s}=\sum_{h=0}^{k}a_h^{\sigma,s-1}\alpha_2^{k-h}\leq\left(\frac{\alpha_1}{\alpha_1-\alpha_2}\right)^{s-1}\sum_{h=0}^{k}a_h^{\sigma,0}\alpha_2^{k-h}=\left(\frac{\alpha_1}{\alpha_1-\alpha_2}\right)^{s-1}a_h^{\sigma,1}\leq\left(\frac{\alpha_1}{\alpha_1-\alpha_2}\right)^{s}a_h^{\sigma,0}\ .$$

■

6.3 The proof of Theorem 6: the final step

We now want to use the estimates obtained above for bounding the coefficients $\eta_{k,1}$.

From (62) we can argue that

$$\begin{aligned} \eta_1(z) &= \sum_{s=0}^{r}\sum_{\sigma=0}^{s}\bar{p}_{\sigma,s}\left(\frac{1}{1-\alpha_1 z}\right)^{\sigma}\left(\frac{1}{1-\alpha_2 z}\right)^{s}z^{\sigma+s}=\sum_{s=0}^{r}\sum_{\sigma=0}^{s}\bar{p}_{\sigma,s}\sum_{h=0}^{+\infty}a_h^{\sigma,s}z^{h+\sigma+s} \\ &= \sum_{s=0}^{r}\sum_{\sigma=0}^{s}\sum_{k=\sigma+s}^{+\infty}\bar{p}_{\sigma,s}a_{k-\sigma-s}^{\sigma,s}z^k=\sum_{k=0}^{+\infty}\sum_{s=0}^{r\wedge k}\sum_{\sigma=0}^{s\wedge k-s}\bar{p}_{\sigma,s}a_{k-\sigma-s}^{\sigma,s}z^k\ , \end{aligned} \tag{73}$$

where $\bar{p}_{\sigma,s}$ was defined in (63) and $a_k^{\sigma,s}$ in (71). Hence we have

$$\eta_{k,1} = \sum_{s=0}^{r\wedge k} \sum_{\sigma=0}^{s\wedge k-s} \bar{p}_{\sigma,s} a_{k-\sigma-s}^{\sigma,s} \ .$$

Decompose $\eta_{k,1}$ as follows

$$\eta_{k,1} = \eta'_{k,1} + \eta''_{k,1} \ ,$$

where

$$\eta'_{k,1} = \sum_{s=1}^{r\wedge k} \sum_{\sigma=1}^{s\wedge k-s} \bar{p}_{\sigma,s} a_{k-\sigma-s}^{\sigma,s} \qquad \eta''_{k,1} = \sum_{s=0}^{r\wedge k} \bar{p}_{0,s} a_{k-s}^{0,s} \tag{74}$$

Assume now that $\alpha_1 > \alpha_2$ and fix

$$M := \frac{8D_1}{\alpha_1} \vee \frac{2D_2}{\alpha_1 - \alpha_2} \vee \frac{D_2}{\alpha_2} \ . \tag{75}$$

Inserting bounds of Lemmas 4 and 5 we now obtain

$$\begin{aligned}
\frac{\eta'_{k,1}}{\alpha_1^k} &= \sum_{s=1}^{r\wedge k} \sum_{\sigma=1}^{s\wedge k-s} \bar{p}_{\sigma,s} \frac{a_{k-\sigma-s}^{\sigma,s}}{\alpha_1^k} \\
&\leq \sum_{s=1}^{r\wedge k} \sum_{\sigma=1}^{s\wedge k-s} \binom{s}{\sigma}\binom{r}{s}\left(\frac{4\mathbf{N}D_1}{s}\right)^{\sigma}\left(\frac{\mathbf{N}D_2}{r}\right)^{s}\left(\frac{\alpha_1}{\alpha_1-\alpha_2}\right)^{s}\binom{k-s-1}{\sigma-1}\frac{\alpha_1^{k-\sigma-s}}{\alpha_1^k} \\
&\leq \sum_{s=1}^{r\wedge k} \sum_{\sigma=1}^{s\wedge k-s} \binom{r}{s}\binom{s}{\sigma}\left(\frac{s}{r}\right)^{s}\left(\frac{\mathbf{N}M}{2s}\right)^{s+\sigma}\binom{k-s-1}{\sigma-1} \\
&\leq \left[\sum_{s=1}^{r\wedge k}\binom{r}{s}\left(\frac{s}{r}\right)^{s}\sum_{\sigma=1}^{s\wedge k-s}\binom{s}{\sigma}\binom{k-s-1}{\sigma-1}\right]\max_{s=1}^{r\wedge k}\max_{\sigma=0}^{s\wedge k-s}\left\{\left(\frac{\mathbf{N}M}{2s}\right)^{s+\sigma}\right\} \ ,
\end{aligned}$$

Observe that

$$\max_{\sigma=0}^{s\wedge k-s}\left\{\left(\frac{\mathbf{N}M}{2s}\right)^{s+\sigma}\right\} = \left(\frac{\mathbf{N}M}{2s}\right)^{s}\bigvee\left(\frac{\mathbf{N}M}{2s}\right)^{2s\wedge k}$$

and that, by (93),

$$\max_{s=1}^{r\wedge k}\left\{\left(\frac{\mathbf{N}M}{2s}\right)^{s}\right\} \leq \left(\frac{\mathbf{N}M/2}{k\wedge\frac{\mathbf{N}M}{2e}}\right)^{k\wedge\frac{\mathbf{N}M}{2e}} \qquad \max_{s=1}^{r\wedge k}\left\{\left(\frac{\mathbf{N}M}{2s}\right)^{2\left(s\wedge\frac{k}{2}\right)}\right\} \leq \left(\frac{\mathbf{N}M}{k\wedge\frac{\mathbf{N}M}{e}}\right)^{k\wedge\frac{\mathbf{N}M}{e}}$$

which implies

$$\max_{s=1}^{r\wedge k}\max_{\sigma=0}^{s\wedge k-s}\left\{\left(\frac{\mathbf{N}M}{2s}\right)^{s+\sigma}\right\} \leq \left(\frac{\mathbf{N}M}{k\wedge\frac{\mathbf{N}M}{e}}\right)^{k\wedge\frac{\mathbf{N}M}{e}} \ .$$

From this fact and using the combinatorial identity (89) we obtain

$$\frac{\eta'_{k,1}}{\alpha_1^k} \leq \left[\sum_{s=1}^{r\wedge k}\binom{k-1}{s-1}\binom{r}{s}\left(\frac{s}{r}\right)^{s}\right]\left(\frac{\mathbf{N}M}{k\wedge\frac{\mathbf{N}M}{e}}\right)^{k\wedge\frac{\mathbf{N}M}{e}} \ . \tag{76}$$

On the other hand, assuming $k \geq 1$, similar computations show that

$$
\begin{aligned}
\frac{\eta''_{k,1}}{\alpha_1^k} &= \sum_{s=1}^{r\wedge k} \bar{p}_{0,s} \frac{a_{k-s}^{0,s}}{\alpha_1^k} = \left[\sum_{s=1}^{r\wedge k} \binom{r}{s} \left(\frac{\mathbf{N}D_2}{r}\right)^s \binom{k-1}{s-1} \alpha_2^{-s}\right] \frac{\alpha_2^k}{\alpha_1^k} \\
&\leq \left[\sum_{s=1}^{r\wedge k} \binom{r}{s} \left(\frac{\mathbf{N}M}{r}\right)^s \binom{k-1}{s-1}\right] \\
&\leq \left[\sum_{s=1}^{r\wedge k} \binom{k-1}{s-1} \binom{r}{s} \left(\frac{s}{r}\right)^s\right] \max_{1\leq s\leq r\wedge k} \left\{\left(\frac{\mathbf{N}M}{s}\right)^s\right\} ,
\end{aligned}
$$

which yields

$$
\frac{\eta''_{k,1}}{\alpha_1^k} \leq \left[\sum_{s=1}^{r\wedge k} \binom{k-1}{s-1} \binom{r}{s} \left(\frac{s}{r}\right)^s\right] \left(\frac{\mathbf{N}M}{k \wedge \frac{\mathbf{N}M}{e}}\right)^{k\wedge \frac{\mathbf{N}M}{e}} \tag{77}
$$

Putting together (76) and (77) we obtain the final bound

$$
\frac{\gamma_k}{\alpha_1^k} \leq \frac{\eta_{k,1}}{\alpha_1^k} \leq 2 \left[\sum_{s=1}^{r\wedge k} \binom{k-1}{s-1} \binom{r}{s} \left(\frac{s}{r}\right)^s\right] \left(\frac{\mathbf{N}M}{k \wedge \frac{\mathbf{N}M}{e}}\right)^{k\wedge \frac{\mathbf{N}M}{e}} ,
$$

which proves Theorem 6 in the case when $\alpha_1 > \alpha_2$. Observe that M depends only on the parameters α_1, α_2, D_1, and D_2.

In the case when $\alpha_2 \geq \alpha_1$, we replace the estimate in Lemma 5 with

$$
0 \leq a_k^{\sigma,s} \leq \binom{k+s+\sigma-1}{s+\sigma-1} \alpha_2^k \quad \forall s+\sigma \geq 1 \; \forall k \geq 0 \tag{78}
$$

and we fix

$$
M = \frac{8D_1}{\alpha_2} \vee \frac{2D_2}{\alpha_2} .
$$

Similar computations show that, for $k \geq 1$, we can estimate

$$
\begin{aligned}
\frac{\gamma_k}{\alpha_2^k} \leq \frac{\eta_{k,1}}{\alpha_2^k} &\leq \left[\sum_{s=1}^{r\wedge k} \sum_{\sigma=0}^{s\wedge k-s} \binom{r}{s} \binom{s}{\sigma} \left(\frac{\mathbf{N}M}{2r}\right)^s \left(\frac{\mathbf{N}M}{2s}\right)^\sigma \binom{k-1}{s+\sigma-1}\right] \\
&\leq \left[\sum_{s=1}^{r\wedge k} \binom{k+s-1}{2s-1} \binom{r}{s} \left(\frac{s}{r}\right)^s\right] \left(\frac{\mathbf{N}M}{k \wedge \frac{\mathbf{N}M}{e}}\right)^{k\wedge \frac{\mathbf{N}M}{e}} .
\end{aligned} \tag{79}
$$

The proof of Theorem 6 is now complete.

7 Proof of Theorem 2

The aim of this section is to obtain a representation of the language $\Sigma_*(\Gamma)$ by a finite state automaton or equivalently by a graph. This will be called a Markov representation of the language. Then we will show that this representation satisfies conditions (A), (B) and (C) of the previous section and so we will be in a position to apply the estimates proposed there.

7.1 The Markov representation

Assume $\Gamma : I \to I$ is any piecewise affine map. The graph representation of the language $\Sigma_*(\Gamma)$ can be constructed as follows. We define as the set of vertices the set $\mathcal{V} := \Sigma_*(\Gamma)$ and as set of edges $\mathcal{E}$ the set given by

$$
(\omega_0\omega_1 \cdots \omega_{n-1} \to \omega_0\omega_1 \cdots \omega_{n-1}\omega_n) \in \mathcal{E} \quad \Longleftrightarrow \quad \omega_0\omega_1 \cdots \omega_{n-1}\omega_n \in \Sigma_*(\Gamma) . \tag{80}
$$

Moreover we introduce the following labeling $\xi : \mathcal{E} \to \mathcal{I} \cup \mathcal{J}$ on the edges

$$\xi(\omega_0\omega_1 \cdots \omega_{n-1} \to \omega_0\omega_1 \cdots \omega_{n-1}\omega_n) = \omega_n \ .$$

Notice that $\Sigma_*(\Gamma)$ coincides with the set of all the labeled words associated with the finite paths on the graph starting from the empty word ϵ. This representation of $\Sigma_*(\Gamma)$ will be called a *Markov representation*. This can be simplified by considering an equivalence relation on the vertices. With each finite word $\omega_0\omega_1 \cdots \omega_n \in \Sigma_*(\Gamma)$, we associate its *symbolic future*

$$\mathrm{fut}_\Sigma(\omega_0\omega_1 \cdots \omega_n) = \{\overline{\omega}_0\overline{\omega}_1 \cdots \overline{\omega}_k \mid \overline{\omega}_0 = \omega_n \text{ and } \omega_0\omega_1 \cdots \omega_n\overline{\omega}_1 \cdots \overline{\omega}_k \in \Sigma_*(\Gamma)\} \ ,$$

which is a subset of $\Sigma_*(\Gamma)$. More roughly, the symbolic future of a word $\omega_0\omega_1 \cdots \omega_n$ is the set of words whose concatenation with $\omega_0\omega_1 \cdots \omega_n$ is in $\Sigma_*(\Gamma)$.

Consider also the *geometric future* which is

$$\mathrm{fut}(\omega_0\omega_1 \cdots \omega_n) = \Gamma^n(\omega_0 \cap \Gamma^{-1}\omega_1 \cap \ldots \cap \Gamma^{-n}\omega_n) \ .$$

The following result is in [5].

Proposition 4 *Let $\omega_0\omega_1 \cdots \omega_n$ and $\nu_0\nu_1 \cdots \nu_m$ be two words in $\Sigma_*(\Gamma)$. Then*

$$\mathrm{fut}(\omega_0\omega_1 \cdots \omega_n) = \mathrm{fut}(\nu_0\nu_1 \cdots \nu_m) \quad \Longleftrightarrow \quad \mathrm{fut}_\Sigma(\omega_0\omega_1 \cdots \omega_n) = \mathrm{fut}_\Sigma(\nu_0\nu_1 \cdots \nu_m) \tag{81}$$

Now define $\overline{\mathcal{X}}$ to be the quotient of the set $\Sigma_*(\Gamma)$ by the equivalence relation

$$\omega'_0 \cdots \omega'_n \equiv \omega''_0 \cdots \omega''_m \ \Leftrightarrow \ \mathrm{fut}_\Sigma(\omega'_0 \cdots \omega'_n) = \mathrm{fut}_\Sigma(\omega''_0 \cdots \omega''_m) \ . \tag{82}$$

The elements of $\overline{\mathcal{X}}$ will be called *states* and will be denoted by the symbol $\mathbf{x}$. The symbol $\langle\omega_0\omega_1 \cdots \omega_n\rangle$ represents the state consisting of the equivalent class which contains the word $\omega_0\omega_1 \cdots \omega_n$. States representable by words of length 1 will be called *principal states*. The equivalence relation defining $\overline{\mathcal{X}}$ ensures that any state $\mathbf{x} \in \overline{\mathcal{X}}$ has a well defined geometric future $\mathrm{fut}(\mathbf{x})$. In fact, the geometric future $\mathrm{fut}(\mathbf{x})$ uniquely determines the state $\mathbf{x}$. Edges and labels can be naturally redefined on $\overline{\mathcal{X}}$ to obtain a new labeled graph denoted $\overline{\mathcal{G}}$ which is still a Markov representation of $\Sigma_*(\Gamma)$ and so with the property that the labeled sequences associated to the finite paths on $\overline{\mathcal{G}}$, starting from empty word, correspond to all the possible sequences in $\Sigma_*(\Gamma)$.

Notice that there is an edge connecting a state $\mathbf{x}'$ to another state $\mathbf{x}''$ labeled with ω if and only if $\mathrm{fut}(\mathbf{x}'') = \Gamma(\mathrm{fut}(\mathbf{x}')) \cap \omega$. This shows that the Markov representation $\overline{\mathcal{G}}$ has the property that the terminal state of any edge is determined by its initial state and by its label. This means that $\overline{\mathcal{G}}$ is a deterministic automaton. This implies, in particular, that there is a one to one correspondence between paths $\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_k$ on the graph $\overline{\mathcal{G}}$ starting from a principal state and words in $\Sigma_*(\Gamma)$. In order to count the number of words in $\Sigma_*(\Gamma)$ of length k it will thus be equivalent to count the paths in $\overline{\mathcal{G}}$ of the same length k.

Example 1 We provide here a simple example which should clarify the concepts introduced so far. Consider the piecewise affine map $\Gamma : [-1, 1] \to \ [-1, 1]$ defined as follows

$$\Gamma(x) := \begin{cases} ax + 1 & \text{if } -1 < x < 0 \\ ax - 1 & \text{if } 0 < x < 1 \end{cases}$$

where $a = \frac{1+\sqrt{5}}{2}$. The map $\Gamma(x)$ is shown in Figure 6. Let $I_0 :=]-1, 0[$ and $I_1 :=]0, 1[$. For this particular choice of a we have that the set of states is finite

$$\overline{\mathcal{X}} = \{\langle I_0\rangle, \langle I_1\rangle, \langle I_0I_0\rangle, \langle I_1I_1\rangle\} \ .$$

The graph $\overline{\mathcal{G}}$ is shown in Figure 6.

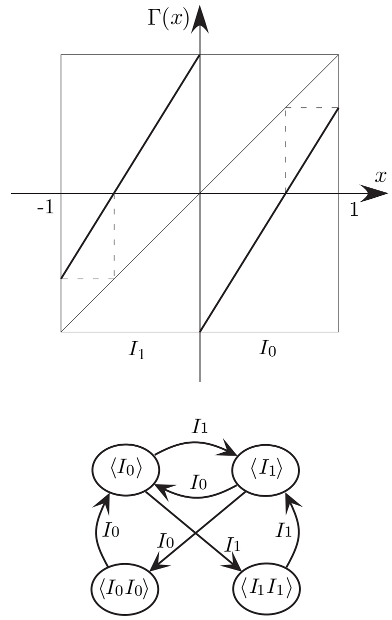

Figure 6: The map Γ of Example 1 and the graph $\overline{\mathcal{G}}$ describing the language associated with its dynamics.

7.2 Properties of the Markov representation

Assume $\Gamma : I \to I$ is a piecewise affine map and that $J \subseteq I$ is another invariant interval as in the setting of Section . We want now to show that the just introduced Markov representation restricted to $\Sigma_*(\Gamma) \cap \mathcal{I}^*$ (we are using the notation established in Section 5.1) satisfies the properties (A), (B) and (C) introduced in the previous section. To this aim we define

$$
\begin{aligned}
&\mathcal{X}_P := \{\langle I_1\rangle, \langle I_2\rangle, \ldots, \langle I_{\mathbf{N}}\rangle\}\\
&\mathcal{X}_i := \{\langle \omega_0\omega_1\cdots\omega_k I_i\rangle \in \overline{\mathcal{X}} \mid \omega_0\omega_1\cdots\omega_k \in \Sigma_*(\Gamma)\cap\mathcal{I}^*\} = \{\mathbf{x}\in\overline{\mathcal{X}} \mid \mathrm{fut}(\mathbf{x}) \subseteq I_i\}\\
&\mathcal{X} := \bigcup_{i=1}^{\mathbf{N}} \mathcal{X}_i = \{\mathbf{x}\in\overline{\mathcal{X}} \mid \mathrm{fut}(\mathbf{x}) \subseteq I \setminus J\}\\
&q \;:\; \mathcal{X} \;\to]0,1[\;:\; \mathbf{x} \mapsto \; q(\mathbf{x}) := \mathbb{P}[\mathrm{fut}(\mathbf{x})]\ .
\end{aligned}
$$

and the graph $\mathcal{G}$ which coincides with the graph $\overline{\mathcal{G}}$ restricted to the set of states $\mathcal{X}$.

By taking $q_i = \mathbb{P}[I_i]$ we have that property (A) holds true. The next two lemmas will show that also properties (B) and (C) hold true with $\alpha_1 = |a|$, $\alpha_2 = 2$, $D_1 = |a|$, and $D_2 = 1$.

Lemma 6 *Let $\mathbf{x}' \in \mathcal{X}$, $\mathcal{X}'' \subseteq \mathcal{X}$, and $k \geq 2$. Then*

$$
\#\mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \ldots, \mathbf{x}_{k-1} \in \mathcal{X}, \mathbf{x}_k \in \mathcal{X}''] \leq \frac{\mathbb{P}[\mathrm{fut}(\mathbf{x}')]}{\inf_{\mathbf{x}''\in\mathcal{X}''}\mathbb{P}[\mathrm{fut}(\mathbf{x}'')]}|a|^{k-1}\ .
$$

Proof Notice that the intervals of the form

$$
\mathrm{fut}(\mathbf{x}') \cap \Gamma^{-1}\mathrm{fut}(\mathbf{x}_2) \cap \cdots \cap \Gamma^{-(k-2)}\mathrm{fut}(\mathbf{x}_{k-1}) \cap \Gamma^{-(k-1)}\mathrm{fut}(\mathbf{x}_k) \qquad \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathcal{X}
$$

constitute a family of disjoint subsets of $\mathrm{fut}(\mathbf{x}')$. This shows that

$$
\mathbb{P}[\mathrm{fut}(\mathbf{x}')] \geq \sum_{\substack{\mathbf{x}_2,\ldots,\mathbf{x}_{k-1}\in\mathcal{X}\\ \mathbf{x}_k\in\mathcal{X}''}} \mathbb{P}\left[\mathrm{fut}(\mathbf{x}') \cap \Gamma^{-1}\mathrm{fut}(\mathbf{x}_2) \cap \cdots \cap \Gamma^{-(k-2)}\mathrm{fut}(\mathbf{x}_{k-1}) \cap \Gamma^{-(k-1)}\mathrm{fut}(\mathbf{x}_k)\right]\ .
$$

Notice moreover that Γ^{k-1} is affine on each of these intervals and that

$$\Gamma^{k-1}(\mathrm{fut}(\mathbf{x}') \cap \Gamma^{-1}\mathrm{fut}(\mathbf{x}_2) \cap \cdots \cap \Gamma^{-(k-2)}\mathrm{fut}(\mathbf{x}_{k-1}) \cap \Gamma^{-(k-1)}\mathrm{fut}(\mathbf{x}_k)) = \mathrm{fut}(\mathbf{x}_k).$$

This implies that that

$$\mathbb{P}\Big[\mathrm{fut}(\mathbf{x}') \cap \Gamma^{-1}\mathrm{fut}(\mathbf{x}_2) \cap \cdots \cap \Gamma^{-(k-2)}\mathrm{fut}(\mathbf{x}_{k-1}) \cap \Gamma^{-(k-1)}\mathrm{fut}(\mathbf{x}_k)\Big] \geq \frac{\inf_{\mathbf{x}''\in\mathcal{X}''} \mathbb{P}[\mathrm{fut}(\mathbf{x}'')]}{|a|^{k-1}}$$

if $\mathbf{x}'\mathbf{x}_2\cdots\mathbf{x}_{k-1}\mathbf{x}_k \in \mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \ldots, \mathbf{x}_{k-1} \in \mathcal{X}, \mathbf{x}_k \in \mathcal{X}'']$ and it is 0 otherwise. This yields the result. ■

Lemma 7 *Let $\mathbf{x}' \in \mathcal{X}$ and let $i = 1, \ldots, \mathbf{N}$. Then*

$$\#\mathcal{F}_k[\mathbf{x}_1 = \mathbf{x}', \mathbf{x}_2, \ldots, \mathbf{x}_{k-1} \in \mathcal{X} \setminus \mathcal{X}_P, \mathbf{x}_k \in \mathcal{X}_i] \leq 2^{k-2}\ .$$

Proof As mentioned above, there is an edge connecting a state $\mathbf{x}'$ to another state $\mathbf{x}''$ with label ω if and only if $\mathrm{fut}(\mathbf{x}'') = \Gamma(\mathrm{fut}(\mathbf{x}')) \cap \omega$. Since the map Γ is affine on $\mathrm{fut}(\mathbf{x}')$, then $\Gamma(\mathrm{fut}(\mathbf{x}'))$ is an interval and so at most two followers of a state can be nonprincipal. The result follows by applying this argument. ■

It follows from Lemmas 6 and 7 that the graph $\mathcal{G}$ satisfies the properties (A), (B) and (C) and hence Theorem 6 holds true in this case. Notice that this yields Theorem 2, since the γ_k defined in (30) coincides with the γ_k defined in (30). Indeed, in this case we have that

$$\gamma_{k,h} = \#\mathcal{F}_k[\mathbf{x}_1 = \langle I_h\rangle, \mathbf{x}_2\,, \ldots, \mathbf{x}_k \in \mathcal{X}]$$

so that $\gamma_k = \sum_h \gamma_{k,h}$ coincides with the number of paths of length k in the graph $\overline{\mathcal{G}}$, starting from a principal state and always remaining in $\mathcal{X}$. This, by the previous discussion, corresponds to the number of distinct subwords in $\Sigma_*(\Gamma) \cap \mathcal{I}^*$ of length k.

7.3 Estimation of the number of paths in the chaotic case

As mentioned in the remark after Theorem 6, in the bound (53) we can fix r instead of the constant M. More precisely, instead of fixing the contraction factor equal to 1/2 in (68), we can choose any $\delta \in]0,1[$. In this case, instead of (70), we obtain

$$\bar{p}_{\sigma,s} \leq \binom{r}{s}\binom{s}{\sigma}\left(\frac{\mathbf{N}D_2}{r}\right)^s \left(\frac{\mathbf{N}D_1}{s\delta(1-\delta)}\right)^\sigma \tag{83}$$

In the case $\alpha_1 > \alpha_2$, the only consequence on the subsequent computations is that the factor $\frac{1}{\delta(1-\delta)}$ will enter in the definition (75) of M. On the other hand, also the number r depends on the contraction factor δ. An important situation in which it is possible to take advantage of this degree of freedom is the following.

If we fix $\delta := q_{\mathbf{N}}/q_1 \wedge 1/2$, then $r = 1$ and in this way we obtain a simplified bound on γ_k in which however the constant M is a decreasing function of δ. In order to obtain an effective bound we need to have bound from above on M and so a bound from below on $q_{\mathbf{N}}/q_1$. In the context of piecewise affine maps this means that we need to have bound from below on $\delta = \mathbb{P}[I_{\mathbf{N}}]/\mathbb{P}[I_1]$. An interesting situation in which this is possible is when $\mathbf{N} = \lceil |a| \rceil$, namely for the chaotic quantized stabilizers.

Proposition 5 *Let $|a| > 2$ and $\mathbf{N} = \lceil |a| \rceil$. There exist constants $C_1 > 1$ and $M > 0$, only depending on $|a|$, such that, if $C > C_1$, then*

$$\frac{\gamma_k}{|a|^k} \leq 2\left(\frac{\mathbf{N}M}{k \wedge \frac{\mathbf{N}M}{e}}\right)^{k\wedge\frac{\mathbf{N}M}{e}} \qquad \forall k \geq 1\ . \tag{84}$$

Proof For the arguments presented above, we need only to prove that there exist constants $\delta_1 > 0$ and $C_1 > 1$, only depending on $|a|$, such that

$$C \geq C_1 \;\Rightarrow\; \frac{\mathbb{P}[I_{\mathbf{N}}]}{\mathbb{P}[I_1]} \geq \delta_1 \ .$$

First notice that $1 \geq \mathbb{P}[\Gamma(I_1)] \geq |a|\,\mathbb{P}[I_1]$, from which we can argue that $\mathbb{P}[I_1] \leq 1/|a|$. Moreover,

$$\mathbb{P}[I_{\mathbf{N}}] = 1 - \mathbb{P}[J] - \sum_{h=1}^{\mathbf{N}-1} \mathbb{P}[I_h] \geq 1 - C^{-1} - (\mathbf{N}-1)\mathbb{P}[I_1] \geq 1 - C^{-1} - \frac{\lceil |a| \rceil - 1}{|a|}$$

and hence

$$\frac{\mathbb{P}[I_{\mathbf{N}}]}{\mathbb{P}[I_1]} \geq \frac{\mathbb{P}[I_{\mathbf{N}}]}{1/|a|} \geq |a| - |a|C^{-1} - \lceil |a| \rceil + 1 \stackrel{C\to\infty}{\longrightarrow} |a| - \lceil |a| \rceil + 1 > 0 \ .$$

This proves the result. ■

7.4 Estimation of the number of paths in the stable case

In this section we will propose a bound on γ_k which holds true when Γ is (I, J)-stable or when Γ is almost (I, J)-stable but with only a countable subset of points in I never entering inside J. For obtaining this bound we need the following lemma.

Lemma 8 *Assume that there exists a state $\mathbf{x} \in \mathcal{X}$ such that there exist two distinct paths in the graph $\mathcal{G}$ both starting and ending in $\mathbf{x}$ and not passing by $\mathbf{x}$ in any intermediate step (simple loops through $\mathbf{x}$). Then there is an uncountable set of points in I never entering inside J.*

Proof The proof is based on a general argument on the symbolic description of a one-dimensional expansive map as Γ which consists in constructing a sort of inverse of the map ψ defined in (29), see [5].

Given any loop $\nu = \mathbf{x}\mathbf{x}_1 \cdots \mathbf{x}_{k-1}\mathbf{x}$ in $\mathcal{G}$, if we consider the open interval

$$K_\nu = \mathrm{fut}(\mathbf{x}) \cap \Gamma^{-1}(\mathrm{fut}(\mathbf{x}_1)) \cap \cdots \cap \Gamma^{-(k-1)}(\mathrm{fut}(\mathbf{x}_{k-1})) \cap \Gamma^{-k}(\mathrm{fut}(\mathbf{x}))\,,$$

we have that Γ^k is affine on K_ν and $\Gamma^k(K_\nu) = \mathrm{fut}(\mathbf{x})$. In particular, it follows that

$$\mathbb{P}[K_\nu] = \mathbb{P}[\mathrm{fut}(\mathbf{x})]|a|^{-k} \tag{85}$$

We now set some notation: if $\nu_1 = \mathbf{x}\mathbf{x}_1^1 \cdots \mathbf{x}_{k_1-1}^1\mathbf{x}$ and $\nu_2 = \mathbf{x}\mathbf{x}_1^2 \cdots \mathbf{x}_{k_2-1}^2\mathbf{x}$ are two loops through $\mathbf{x}$, we define the concatenation of ν_1 and ν_2 as the new loop

$$\nu = \nu_1 \wedge \nu_2 = \mathbf{x}\mathbf{x}_1^1 \cdots \mathbf{x}_{k_1-1}^1\mathbf{x}\mathbf{x}_1^2 \cdots \mathbf{x}_{k_2-1}^2\mathbf{x}\,.$$

Assume that there are two distinct simple loops ν_1 and ν_2 of length, respectively, k_1 and k_2 through $\mathbf{x}$. The corresponding open intervals K_1 and K_2 as defined above are then disjoint. Define now a map $\Upsilon : \{1,2\}^{\mathbb{N}} \to I$ in the following way: given a sequence $(a_n) \in \{1,2\}^{\mathbb{N}}$, consider the set

$$\overline{K}_{a_1} \cap \Gamma^{-k_{a_1}}(\overline{K}_{a_2}) \cap \Gamma^{-k_{a_1}-k_{a_2}}(\overline{K}_{a_3}) \cap \cdots = \bigcap_{n=1}^{+\infty} \Gamma^{-\sum_{j=1}^{n-1} k_{a_j}}(\overline{K}_{a_n})\,. \tag{86}$$

Since

$$\bigcap_{n=1}^{q} \Gamma^{-\sum_{j=1}^{n-1} k_{a_j}}(\overline{K}_{a_n})$$

is simply the closure of the open interval K associated to the loop $\nu_{a_1} \wedge \nu_{a_2} \wedge \cdots \wedge \nu_{a_q}$, it follows that it is non-empty and that, by (85), its size decreases of a factor

$$|a|^{-\sum_{j=1}^{n-1} k_{a_j}}\,.$$

Hence, this implies that the set in (86) consists of exactly one point x. We then put $\Upsilon((a_n)) = x$. Call $\Delta = \Upsilon(\{1,2\}^{\mathbb{N}})$. A standard argument of symbolic dynamics of one dimensional maps now show that there exists $\Delta_1 \subseteq \Delta$, at most countable, such that the counterimage set $\Upsilon^{-1}(x)$ is a singleton for every $x \in \Delta \setminus \Delta_1$. Indeed, it follows by the definition, that the only points x which have more than one counterimage (and in fact exactly two) are those in the union of boundaries of the intervals

$$\bigcap_{n=1}^{q} \Gamma^{-\sum\limits_{j=1}^{n-1} k_{a_j}} (\overline{K}_{a_n}) ,$$

namely those in the subset:

$$\Delta_1 = \bigcup_{q=1}^{+\infty} \bigcup_{a_1,\dots a_q} \partial \left(\bigcap_{n=1}^{q} \Gamma^{-\sum\limits_{j=1}^{n-1} k_{a_j}} (\overline{K}_{a_n}) \right)$$

which is clearly at most countable. Finally, the subset of points in Δ which will never enter inside Δ_1,

$$\Delta_2 = \bigcap_{k=0}^{+\infty} \Gamma^{-k}(\Delta \setminus \Delta_1) ,$$

is clearly uncountable.

We claim that no point in Δ_2 will ever enter inside J. Notice first that, by construction, $\Delta_2 \subseteq \Omega$. Take now $x \in \Delta_2$ and let $(a_n) \in \{1,2\}^{\mathbb{N}}$ be such that $\Upsilon(a_n) = x$. Then,

$$\Gamma^{k_{a_1}}(\Upsilon(a_n)) = \Upsilon(\tilde{a}_n)$$

where $(\tilde{a}_n)$ is the sequence defined by $\tilde{a}_n = a_{n+1}$ for all $n \in \mathbb{N}$. This implies, in particular, that $\Gamma^{k_{a_1}}(x) \in \Delta_2$ by the way Δ_2 has been defined. Hence we have that for every $x \in \Delta_2$ either $\Gamma^{k_1}(x)$ or $\Gamma^{k_2}(x)$ is also in Δ_2. If, by contradiction, it would exist n_0 such that $\Gamma^n x \in J$ for every $n \geq n_0$, for sure we could find $n_1 \geq n_0$ such that $y = \Gamma^{n_1} x \in \Delta_2 \cap J$. Since $\Delta_2 \subseteq K_1 \cup K_2$ it would follow that $y \in \partial K_1 \cup \partial K_2$ which is absurd by the way Δ_2 has been defined. ■

Theorem 7 *Assume that* Γ *is almost* (I,J)*-stable with an at most countable subset of points in* I *never entering inside* J*. Then*

$$\frac{\gamma_k}{2^k} \leq \binom{k + 2\mathbf{N} - 1}{2\mathbf{N} - 1} e^{\frac{\mathbf{N}}{e}} \qquad \forall k \geq 1 \ . \tag{87}$$

Proof Decompose the set $\mathcal{X}_P$ into maximal subfamilies $\mathcal{X}_P^1, \mathcal{X}_P^2, \dots, \mathcal{X}_P^m$ in such a way that two principal states belong to the same family if and only if there exists a loop in $\mathcal{G}$ connecting them. Also we can assume the families are ordered in such a way that if there exists a path from $\mathbf{x}_1 \in \mathcal{X}_P^i$ to $\mathbf{x}_2 \in \mathcal{X}_P^j$, then $i \leq j$. Let N_i be the cardinality of $\mathcal{X}_P^i$. We thus have $\mathbf{N} = \sum_{i=1}^m N_i$.

Given now any path ν of length k inside the graph $\mathcal{G}$ starting from a principal state, we can always split it as

$$\nu = \nu_1 \mu_1 \nu_2 \mu_2 \cdots \nu_m \mu_m$$

where ν_i is a path connecting two principal states in $\mathcal{X}_P^i$ while ν_i is a path only consisting of non-principal states. Assume ν_i has length k_i^1 and that μ_i has length k_i^2. We thus have

$$k = \sum_{i=1}^m k_i^1 + \sum_{i=1}^m k_i^2$$

The number of ways we can split k in the sum above is equal to

$$\binom{k + 2m - 1}{2m - 1} .$$

Once the number k_i^1 and k_i^2 have been fixed, we notice that the path ν_i can be chosen in N_i distinct ways corresponding to the ways we can choose the initial principal state. This follows from the fact that from any principal state in $\mathcal{X}_P^i$ there is exactly one path reaching another element in $\mathcal{X}_P^i$ because otherwise there would be two distinct simple loops in $\mathcal{G}$ contradicting the result in Lemma 8. Notice that using the fact that $\sum_{i=1}^m N_i = \mathbf{N}$, by Lemma 3, the number of ways we can chose the family of paths $\nu_1, \nu_1, \ldots, \nu_m$ is bounded from above by

$$\prod_{i=1}^{m} N_i \leq \left(\frac{\mathbf{N}}{m}\right)^m \leq e^{\frac{\mathbf{N}}{e}} .$$

Once all the paths ν_i have been chosen, the remaining paths μ_i can be chosen in at most $2^{k_i^2}$ distinct ways. Hence, the number of ways we can chose the family of paths $\mu_1, \mu_1, \ldots, \mu_m$ is bounded from above by 2^k. We thus have the thesis. ■

8 Conclusions

In this paper some stabilizing quantized feedback strategies are proposed and their different properties in terms of performance and communication requirements are compared. These strategies are based on nesting one base quantized feedback. The performance, defined as the expected time needed to get from a big initial state set into a smaller target state set, is analyzed by using the concept of Perron-Frobenius operator associated with a nonlinear transformation.

The second part of the paper is devoted to the search of general bounds which could highlight the trade-off existing between performance and information flow required by a quantized control technique. This investigation is based on a symbolic representation of the closed loop nonlinear system. In this way the system is described by a Markov chain with possibly infinite states. Counting the paths on the graph which represents the Markov chain, it is possible to obtain bounds on the performance which yield to some interesting trade-off relations. This method is based on a technical result which is expressed in terms of general Markov chains and its proof, though quite long, is based on basic combinatorial relations.

It is our hope that, as information theory has been a successful symbolic technique to treat digital communication, a symbolic technique will be the right tool to deal with digital control as well. In fact, although the present paper deals only with the static control of linear scalar systems, the symbolic method proposed here seems to be very promising for treating more general situations. In [10] the same method is applied for treating both the case in which a memory structure is allowed on the controller and the case in which the system is multidimensional. We hope that this method will be useful to solve also other questions which remain open. In our opinion the most important ones are the following:

1. In most of the contributions on control with communication constraint proposed in the literature it is assumed that the channels are digital with finite rate but noiseless. In the future investigations it will be important to allow the presence of errors in the data exchange between the plant and the controller.

2. In our opinion more attention has to be devoted to the control problem with communication constraint in those situations in which there are more interacting agents to be controlled to achieve an joint control objective. In this case the communication constraint have to be imposed on the data which are exchanged by the differently located agents.

3. In this paper we have been able to analyze the performance of some simple quantized feedback strategies. It remains to obtain an algorithm able to provide an approximate performance evaluation for any given specific quantized feedback. In our opinion a promising method could be based on the approximation of the Perron-Frobenius operator by a finite state Markov chain which is connected with the so called Ulam's conjecture (see [19] and the references therein).

A Appendix: Some useful elementary combinatorics

In the paper we use some elementary properties of the binomials. The first one is the following

$$\sum_{j=0}^{m} \binom{l+j}{j} = \binom{m+l+1}{m} \tag{88}$$

which follows by iterating the elementary identity

$$\binom{m+l+1}{m} = \binom{m+l}{m} + \binom{m+l}{m-1} \ .$$

Another useful formula follows by comparing the binomial coefficients of the term z^k in the polynomial identity

$$(1+z)^{n_1}(1+z^{-1})^{n_2} = (1+z)^{n_1+n_2} z^{-n_2}$$

which yields

$$\sum_{j=0}^{(n_1-k)\wedge n_2} \binom{n_1}{k+j}\binom{n_2}{j} = \binom{n_1+n_2}{k+n_2} \ . \tag{89}$$

Another useful formula is given the following series of inequalities which holds true for all $n, m \geq 1$ [1, pag. 113]

$$\begin{aligned} \binom{n+m}{m} &\leq \sqrt{\frac{1}{2\pi}\left(\frac{1}{n}+\frac{1}{m}\right)} \left(1+\frac{n}{m}\right)^m \left(1+\frac{m}{n}\right)^n \leq \\ &\leq \sqrt{\frac{1}{2\pi}\left(\frac{1}{n}+\frac{1}{m}\right)} \left(1+\frac{n}{m}\right)^m e^m \leq \sqrt{\frac{1}{2\pi}\left(\frac{1}{n}+\frac{1}{m}\right)} e^{n+m} \ . \end{aligned} \tag{90}$$

From (90) we can argue that for all $n \geq 0$ and $m \geq 1$

$$\binom{n+m}{m} \leq \frac{1}{\sqrt{\pi}} \left(1+\frac{n}{m}\right)^m e^m \ . \tag{91}$$

Finally consider the function

$$f(x) := \left(\frac{A}{x}\right)^{Bx} \ .$$

This a unimodal function having a unique maximum in $x_M = \frac{A}{e}$. This implies that for all $\bar{x} > 0$ we have

$$\max_{0<x\leq\bar{x}} f(x) = f(\bar{x} \wedge x_M) = \left(\frac{A}{\bar{x} \wedge \frac{A}{e}}\right)^{B\left(\bar{x}\wedge\frac{A}{e}\right)} \ . \tag{92}$$

Observe moreover that for all $\hat{x} > 0$ we have

$$\left(\frac{A}{x}\right)^{B(x\wedge\hat{x})} \leq \left(\frac{A}{x\wedge\hat{x}}\right)^{B(x\wedge\hat{x})}$$

which implies that

$$\max_{0<x\leq\bar{x}} \left(\frac{A}{x}\right)^{B(x\wedge\hat{x})} \leq \max_{0<x\leq\bar{x}} \left(\frac{A}{x\wedge\hat{x}}\right)^{B(x\wedge\hat{x})} = \max_{0<x\leq\bar{x}\wedge\hat{x}} f(x) = \left(\frac{A}{\bar{x}\wedge\hat{x}\wedge\frac{A}{e}}\right)^{B(\bar{x}\wedge\hat{x}\wedge\frac{A}{e})} \ . \tag{93}$$

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