Convergence analysis of a distributed voltage support strategy for optimal reactive power compensation ⋆

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Abstract: We consider the problem of commanding the electronic power interfaces of the microgenerators in a low voltage microgrid for the task of optimal reactive power compensation. In this work, we analyze the convergence of the strategy proposed by Tenti et al. (2012). The proof of convergence gives some additional insight on the behavior of the algorithm and allows the characterization of its rate of convergence as a function of the microgrid parameters.

Keywords: Networked control systems, distributed control, distributed optimization, reactive power, distribution networks, smart microgrids.

1. INTRODUCTION

Recent technological advances, together with environmental and economic motivations, have been driving the appearance of small power generation devices in the power distribution grid. These microgenerators are generally powered by renewable energy sources, and they are interconnected to the low voltage grid via power electronic interfaces (inverters). The availability of a large number of inverters in the grid can yield relevant benefits for the network operation, which go beyond the availability of clean, inexpensive electrical power. They can be used for providing of a number of ancillary services that are of great interest for the management of the grid: voltage support, harmonic compensation, reactive power compensation, among others (Katiraei and Iravani, 2006; Prodanovic et al., 2007). We focus here on the problem of reactive power compensation for the minimization of distribution losses.

In order to exploit these available resources, the distribution grid must be provided with an ICT architecture which today is not present. One possible approach for deploying such an architecture is the concept of smart microgrids. A microgrid is a portion of the power distribution network which is managed autonomously from the rest of the grid, in order to improve the quality of the service, achieve economic savings, and increase the hosting capacity. There are different ways in which the electronic power interfaces inside a microgrid can be commanded for solving the problem of optimal reactive power compensation. For example via a *centralized* solution, where a microgrid supervisor collects all the field measurements, knows the grid topology and the grid parameters, and dispatch the power inverters via a communication channel (Zhao et al., 2005; Lavaei et al., 2011). Another option consists in deriving some *purely local* control strategies, in which every microgenerator decide its behavior based on its own measurements (Turitsyn et al., 2011). Because there is no cooperation, suboptimal results are expected in this case. Another option is the approach of *networked control systems*. Each microgenerator is provided with some computing, communication, and measurement capabilities. They also have some local knowledge of the microgrid electrical topology, while there is no supervising unit that knows the entire system parameters. Via local exchange of information between microgenerators, a cooperative global behavior can be achieved. Increased robustness, automatic reconfiguration, and reduced communication requirements are also notable benefits of this type of solutions.

This approach has been explored only recently in the field of power systems. One example is the work by Bolognani and Zampieri (2011a). A different distributed algorithm, based on a voltage support strategy, has been proposed by Tenti et al. (2012), motivated mainly by a simulative study. In the following, we formally prove the convergence of the algorithm by Tenti et al. to the desired optimal solution. The analytic study is also useful because it can give some extra insight about the global behavior of the system, starting from a characterization of the rate of convergence of the algorithm with respect to some system and design parameters.

2. MICROGRID MODEL

For the purpose of this paper, we model a micro-grid as a radial directed graph $\mathcal{G}$ (see Figure 1), in which the edges (whose set is denoted by $\mathcal{E}$) represent the power lines, and whose nodes (whose set is denoted by $\mathcal{V}$) represent loads, micro-generators, and also the point of connection of the micro-grid to the transmission grid (called point of common coupling, or PCC).

We limit our study to the steady state behavior of the system, when all voltages and currents are sinusoidal signals at the same frequency. They can therefore be represented via a complex number whose absolute value corresponds

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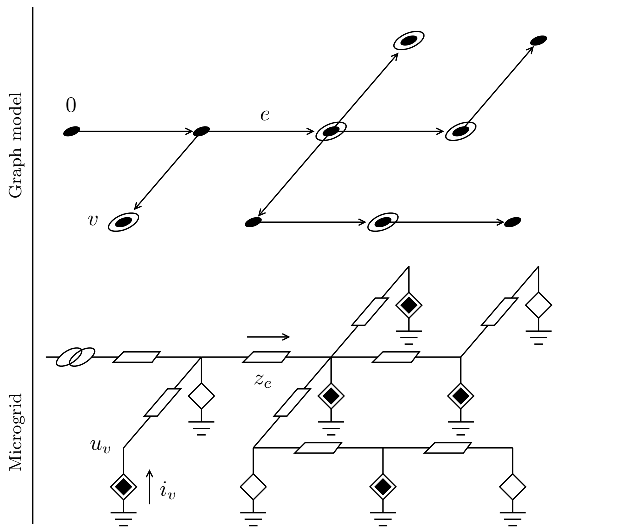

Fig. 1. The lower panel is a circuit representation of a microgrid, while the upper panel is the corresponding graph model. Black diamonds (below) and circled nodes (above) correspond to microgenerators connected to the microgrid. Node 0 corresponds to the point where the microgrid connects to the utility (PCC). The other nodes are the microgrid loads.

to their root-mean-square value, and whose angle corresponds to their phase with respect to an absolute time reference. In this notation, the steady state of the grid is described by the variables:

- $u_v \in \mathbb{C},\ v \in \mathcal{V}$, are the grid voltages at the points where the nodes are connected to the grid;
- $i_v \in \mathbb{C},\ v \in \mathcal{V}$, are the currents injected by the nodes into the grid;
- $\xi_e \in \mathbb{C},\ e \in \mathcal{E}$, are the currents flowing on the edges of the grid.

We denote by $z_e \in \mathbb{C},\ e \in \mathcal{E}$, the impedance of the power lines.

Micro-generators form a subset of the nodes of the microgrid. They exhibit the following features:

- they are connected to the grid via electronic interfaces (power inverters) which can be commanded in order to inject into the grid the desired amount of active and reactive power, as in Lopes et al. (2006); Green and Prodanović (2007);
- they can measure their own voltage and current via on-board synchronized phasor measurement units (Phadke and Thorp, 2008);
- they have some basic computational capabilities;
- they can communicate with other micro-generators via power line communication (PLC), i.e. by using the electric grid to convey data messages to the units that are sufficiently close (Galli et al., 2011).

Also the PCC, whose corresponding node is indexed as 0, is provided with the same measurement and communication capabilities. We denote by $\mathcal{C} \subset \mathcal{V}$ the subset of nodes corresponding to the microgenerators, and by $\mathcal{C}_0 = \mathcal{C} \cup \{0\}$ the subset that also includes the PCC.

We introduce the vectors u and i in which the voltages u_v and the currents i_v are stacked, respectively. With no loss of generality, we adopt the block decomposition

$$u = \begin{bmatrix} u_0 \\ u' \\ u'' \end{bmatrix}, \quad i = \begin{bmatrix} i_0 \\ i' \\ i'' \end{bmatrix}, \tag{1}$$

where u' and i' corresponds to the nodes in $\mathcal{C}$ (microgenerators), while u'' and i'' corresponds to the nodes in $\mathcal{V} \backslash \mathcal{C}_0$ (the loads).

Each node v of the micro-grid (the PCC, the loads, and the microgenerators) is characterized by a law relating its injected current i_v with its voltage u_v. We model the node corresponding to the PCC (node 0) as a constant voltage generator at the nominal voltage U_N, with fixed angle ψ

$$u_0 = U_N e^{j\psi}.$$

We assume instead that the voltage u_v and the current i_v of every node v except the PCC, satisfy the following law

$$u_v \bar{i}_v = s_v \left| \frac{u_v}{U_N} \right|^{\eta_v}, \quad \forall v \in \mathcal{V} \backslash \{0\}, \tag{2}$$

where $s_v = p_v + jq_v$ is the *nominal complex power* (p_v and q_v being the *active* and *reactive* nominal power, respectively), and η_v is a characteristic parameter of the node v. The model (2) is called *exponential model* and is widely adopted in the literature on power flow analysis (Haque, 1996). Notice that s_v is the complex power that the node would inject into the grid if the voltage at its point of connection were the nominal voltage U_N. Microgenerators fit in this model with $\eta_v = 0$, as they are commanded via a complex power reference and they are capable of injecting it independently from the voltage at their point of connection.

3. REACTIVE POWER COMPENSATION PROBLEM

The problem of optimal reactive power compensation in smart microgrid consists in deciding where to inject the correct amount of reactive power that satisfies some optimality criterion. The reactive power can be provided by the microgenerators (nodes in $\mathcal{C}$) and by the utility, via the PCC. For this reason, we denote the nodes in $\mathcal{C}_0$ as *compensators*. The commands that are available in order to fulfill this task are the reactive power references for the power inverters that equip the microgenerators, and therefore the reactive power injections by the nodes in $\mathcal{C}$.

Power distribution losses has been chosen in Tenti et al. (2012) as a metric for optimality. This choice yields the following nonlinear optimization problem.

$$\min_{q_h, h \in \mathcal{C}} \sum_{e \in \mathcal{E}} \Re(z_e) |\xi_e|^2 \tag{3}$$

where $q_h, h \in \mathcal{C}$ are the amounts of reactive power injected into the grid by the microgenerators, and $\Re(z_e)$ stands for the real part of z_e. The way in which the power line currents ξ_e depend on the reactive power injections q_h depends on the physical laws that govern the microgrid, and will be investigated in Section 5.1.

4. ALGORITHM BY Tenti et al. (2012)

The following distributed algorithm has been proposed in Tenti et al. (2012) for solving (3). In order to formally describe the algorithm, we need the following definitions.

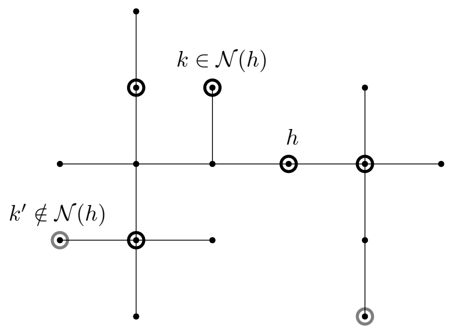

Fig. 2. An example of neighbor compensators. Circled nodes (both gray and black) are compensators. Nodes circled in black belong to the set $\mathcal{N}(h) \subset \mathcal{C}_0$. Node circled in gray are compensators which do not belong to the set of neighbors of h. For each compensator $k \in \mathcal{N}(h)$, the path that connects h to k does not include any other compensator besides h and k themselves.

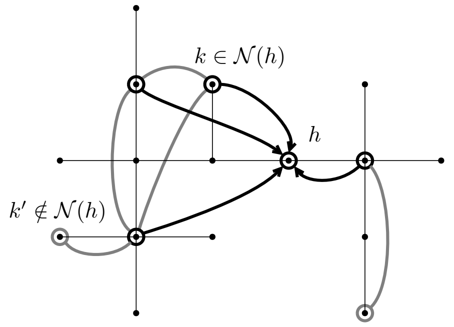

Fig. 3. The proposed communication architecture. The thick lines are the available communication links between the compensators (the circled nodes). Each node h is capable of collecting data messages only from the *neighbor compensators*.

Path: Let $h, k \in \mathcal{V}$ be two nodes of the graph $\mathcal{G}$. The path $\mathcal{P}_{hk} = (v_1, \ldots, v_\ell)$ is the sequence of nodes, without repetitions, that satisfies

- $v_1 = h$
- $v_\ell = k$
- for each $i = 1, \ldots, \ell - 1$, the nodes v_i and v_{i+1} are connected by an edge.

Notice that, as the microgrid topology is radial, there is only one path connecting a pair of nodes $h, k \in \mathcal{V}$.

Neighbor compensators: Let h be a node belonging to $\mathcal{C}$ (i.e. a microgenerator). The set of compensators that are neighbors of h, denoted as $\mathcal{N}(h)$, is the subset of $\mathcal{C}_0$ defined as

$$\mathcal{N}(h) = \{k \in \mathcal{C}_0 \mid \mathcal{P}_{hk} \cap \mathcal{C}_0 = \{h, k\}\}.$$

Figure 2 gives an example of such set. It is assumed in Tenti et al. (2012) that every microgenerator $h \in \mathcal{C}$ knows its set of neighbors $\mathcal{N}(h)$ and can communicate with them (see Figure 3). Notice that this communication architecture can be constructed by each compensator in an initialization phase, by exploiting the PLC channel (as suggested for example in Costabeber et al. 2011). This allows also a plug-and-play implementation of such architecture, where new microgenerators can be connected to the grid with automatic reconfiguration.

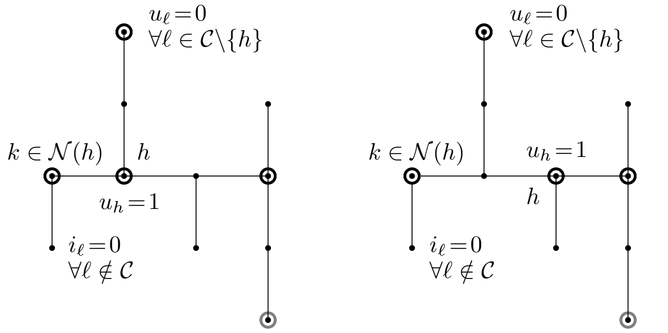

Fig. 4. A representation of how the elements G_{kh} are defined. Notice that in the configuration of the left panel, as the paths from h to its neighbors $k \in \mathcal{N}(h)$ do not share any edge, the gains G_{kh} corresponds to the path admittances $1/Z_{kh}$.

It is also assumed that each microgenerator $h \in \mathcal{C}$ knows the electric topology that connects it to its neighbor compensators. As suggested in Costabeber et al. (2011), also these data can be estimated in an initialization phase via some ranging technologies over the PLC channel. Alternatively, this limited amount of knowledge of the microgrid topology can be stored in the microgenerator computational units at the deployment time.

G-parameters: Consider the *hybrid network matrix* (Desoer and Kuh, 1984) G that satisfies

$$\begin{bmatrix} i_0 \\ i' \\ u'' \end{bmatrix} = G \begin{bmatrix} u_0 \\ u' \\ i'' \end{bmatrix}. \tag{4}$$

In particular, the elements of G corresponding to the pair $k \in \mathcal{C}_0, h \in \mathcal{C}$ correspond to

$$G_{kh} = i_k \Big|_{\substack{u_h = 1 \\ u_\ell = 0,\ \ell \in \mathcal{C} \backslash \{h\} \\ i_\ell = 0,\ \ell \notin \mathcal{C}}}. \tag{5}$$

i.e. the current injected by the compensators k when

- microgenerator h is replaced with a unitary voltage generator;
- all other compensators are replaced by short circuits;
- all loads are replaced by open circuits.

Notice that the matrix G depends only on the microgrid electric topology, and that

$$G_{kh} \neq 0 \quad \text{if and only if} \quad k \in \mathcal{N}(h). \tag{6}$$

Notice moreover that

$$G_{hh} + \sum_{k \neq h} G_{kh} = 0. \tag{7}$$

Figure 4 gives a representation of this definition. Notice that, in the special case in which the paths from compensator h to its neighbor compensators are all disjoint paths, then $G_{kh} = 1/Z_{kh}$, where Z_{kh} is the corresponds to the impedance of the electric path connecting h to k. This special case is the configuration considered in Tenti et al. (2012), and in this sense this definition allows to extend those results to more general situations.

The algorithm proposed in Tenti et al. (2012) corresponds to the repeated execution of the following steps.

ALGORITHM

At each algorithm iteration t

1. a microgenerator h is activated;
2. all other microgenerators keep their reactive power injection constant;
3. microgenerator h collects phasorial voltage measurements from its neighbor compensators
$$u_k, \quad k \in \mathcal{N}(h)\backslash\{h\};$$
4. microgenerator h computes the *target voltage*
$$u_h^* = \frac{\sum_{k\in\mathcal{N}(h)\backslash\{h\}} G_{kh}\, u_k}{\sum_{k\in\mathcal{N}(h)\backslash\{h\}} G_{kh}}; \tag{8}$$
5. microgenerator h updates the amount of injected reactive power q_h to q_h^+ so that the distance between u_h^* and the resulting voltage u_h^+ is minimized. This is achieved via the update
$$q_h^+ = q_h + \delta$$
with
$$\delta = -\Im\left(\bar{u}_h \frac{u_h^* - u_h}{Z_h^{\text{eq}}}\right), \tag{9}$$
where $\bar{u}_h$ is the complex conjugate of u_h, $\Im$ stands for the imaginary part, and Z_h^{eq} is the equivalent impedance of the grid seen by the microgenerator h.

5. CONVERGENCE ANALYSIS

In order to study the convergence of the algorithm, we first need to introduce a model for the microgrid power flows.

5.1 Approximated power flow equations

Assumption 1. We assume that all the micro-grid power line impedances have the same inductance/resistance ratio, i.e. for each edge $e \in \mathcal{E}$ we have
$$z_e = |z_e| e^{j\theta}.$$

We introduce the symmetric matrix X as done in Bolognani and Zampieri (2011b), which depends on the electrical topology of the micro-grid and satisfies
$$(\mathbf{1}_h - \mathbf{1}_k)^T X (\mathbf{1}_h - \mathbf{1}_k) = |Z_{hk}|, \quad h,k \in \mathcal{C} \tag{10}$$
$$X\mathbf{1}_0 = 0, \tag{11}$$
where $\mathbf{1}_h$ is the vector with 1 in position h and 0 elsewhere. The matrix X allows to write the exact relation between node voltages and node currents
$$u = e^{j\theta} X i + \mathbf{1} U_N e^{j\psi}, \tag{12}$$
where $\mathbf{1}$ is the vector of all ones.

The approximated model proposed in Bolognani and Zampieri (2011b) is based on the fact that the micro-grid operating point, in its regular regime, is characterized by a relatively high nominal voltage compared to the voltage drops across the power lines, and by relatively small power distribution losses, compared to the power delivered to the loads. According to this analysis, node voltages are approximated (up to an error term which is smaller than $1/U_N$ for large nominal voltage U_N) by an affine expression of the injected complex powers
$$u_v \approx e^{j\psi}\left(U_N + \frac{1}{U_N} e^{j\theta} \mathbf{1}_v^T X \bar{s}\right), \tag{13}$$
where s is the vector whose elements are the nominal complex node powers $s_v, v \in \mathcal{V}\backslash\{0\}$, augmented with $s_0 = 0$. We consider the vector $q = \Im(s)$ with the same block structure as in (1)
$$q = \begin{bmatrix} 0 \\ q' \\ q'' \end{bmatrix},$$
where q' corresponds to the reactive power injections by the microgenerators, while q'' corresponds to the reactive power injections by the loads.

Following Bolognani and Zampieri (2011b), the problem of optimal reactive power injection at the compensators can therefore be expressed as a convex, quadratic, linearly constrained problem, in the form
$$\min_{q'} \quad J(q), \qquad \text{where} \quad J(q) = \frac{\cos\theta}{2} q^T X q. \tag{14}$$

5.2 Algorithm analysis

Given the model presented in the previous section, it is possible to rewrite the steps of the algorithm proposed in Tenti et al. (2012) as follows.

We rewrite the target voltage u_h^* in (8) as
$$u_h^* = b_h^T u, \tag{15}$$
where the elements of the vector b_h are, for $kin\mathcal{C}_0$,
$$[b_h]_k = \begin{cases} 0 & \text{for } k = h \\ \dfrac{G_{kh}}{\sum_{k\in\mathcal{N}(h)\backslash\{h\}} G_{kh}} = -\dfrac{G_{kh}}{G_{hh}} & \text{for } k \neq h, \end{cases} \tag{16}$$
where we used the properties (6) and (7) of the weights G_{hk}. We also rewrite step 5 of the algorithm as
$$q^+ = q + \mathbf{1}_h \delta.$$
The resulting voltage u_h^+ can be rewritten, via the approximate equation (13), as
$$\begin{aligned} u^+ &= e^{j\psi}\left(U_N \mathbf{1} + \frac{1}{U_N} e^{j\theta} X(p - jq - \mathbf{1}_h\delta)\right) \\ &= u - je^{j\psi}\frac{1}{U_N} e^{j\theta} X \mathbf{1}_h \delta, \end{aligned}$$
and in particular
$$u_h^+ = u_h - je^{j\psi}\frac{1}{U_N} e^{j\theta} X_{hh}\delta. \tag{17}$$

According to step 5 of the algorithm, via (9), the value of δ can be expressed as
$$\delta = -\Im\left[\bar{u}_h \frac{b_h^T - \mathbf{1}_h^T}{X_{hh} e^{j\theta}} u\right],$$
where we also used the fact that, according to the proposed model, the equivalent impedance of the grid seen by microgenerator h is equal to
$$Z_h^{\text{eq}} = Z_{h0} = X_{hh} e^{j\theta}.$$
By using the expression (13) for the microgrid voltages, we have that
$$\begin{aligned} \delta &= -\Im\left[\left(U_N + \frac{1}{U_N} e^{-j\theta} \mathbf{1}_h^T X s\right) \frac{b_h^T - \mathbf{1}_h^T}{X_{hh} e^{j\theta}} \left(\frac{1}{U_N} e^{j\theta} X \bar{s}\right)\right] \\ &= \frac{b_h^T - \mathbf{1}_h^T}{X_{hh}} X q + \tilde{\delta}, \end{aligned}$$
where $\tilde{\delta}$ is infinitesimal for large nominal voltage U_N.

It is possible to show that, according to the approximated model, this choice of δ is indeed the one that minimizes the distance between u_h^* and u_h^+ as defined in (15) and (17), respectively. We omit the proof here, which is based on standard complex algebra steps.

The resulting discrete time systems that describes the algorithm behavior is therefore

$$\begin{aligned} q(t+1) &= q(t) + \mathbf{1}_h \delta \\ &= \left(I - \frac{\mathbf{1}_h (\mathbf{1}_h - b_h)^T X}{X_{hh}} \right) q(t), \end{aligned}$$

or equivalently

$$q(t+1) = F(t) q(t), \tag{18}$$

where $F(t)$ depends on which microgenerator h has been activated at time t

$$F(t) = F_h = I - \frac{\mathbf{1}_h \mathbf{1}_h^T (I - B) X}{X_{hh}},$$

B being the matrix whose rows are the vectors b_h. Via the same block decomposition as before, B results to be divided as

$$B = \begin{bmatrix} 0 & 0 & 0 \\ * & B' & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{19}$$

where $*$ is a generic nonzero block.

5.3 Convergence result

In order to study the convergence of the algorithm, we introduce the variable

$$x(t) = q'(t) - q'_{\text{opt}},$$

where q'_{opt} is the solution of the optimization problem (14). By standard optimization arguments, we can characterize q'_{opt} via the condition

$$X q_{\text{opt}} = X \begin{bmatrix} 0 \\ q'_{\text{opt}} \\ q'' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix},$$

meaning that the optimal solution is a stationary point with respect to the decision variables. It is easy to see that q_{opt} is an equilibrium for (18). Indeed, for any $h \in \mathcal{C}$, using the block structure (19) of B, we have

$$\begin{aligned} F_h q_{\text{opt}} &= q_{\text{opt}} - \frac{\mathbf{1}_h \mathbf{1}_h^T (I - B) X}{X_{hh}} q_{\text{opt}} \\ &= q_{\text{opt}} - \frac{\mathbf{1}_h \mathbf{1}_h^T (I - B)}{X_{hh}} \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix} = q_{\text{opt}}. \end{aligned}$$

By also introducing the block structure of the matrix X

$$X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & X' & * \\ 0 & * & * \end{bmatrix},$$

we can thus refer for the stability analysis to the reduced discrete-time system

$$x(t+1) = F'(t) x(t), \tag{20}$$

where

$$F'(t) = F'_h = I - \frac{\mathbf{1}_h \mathbf{1}_h^T (I - B') X'}{X'_{hh}}. \tag{21}$$

The following lemma provides a convenient expression for B', for studying the convergence of the system (20).

Lemma 2. Let B' be the block of B as in (19). We have

$$B' = I - \left(\operatorname{diag}\{|G_{hh}|, h \in \mathcal{C}\}\right)^{-1} (X')^{-1}$$

where $\operatorname{diag}\{|G_{hh}|, h \in \mathcal{C}\}$ is the diagonal matrix whose elements are the elements $|G_{hh}|, h \in \mathcal{C}$.

Proof. It follows from (16), as B' is the block of B corresponding to the nodes in $\mathcal{C}$, that

$$B' = I - \left(\operatorname{diag}\{G_{hh}, h \in \mathcal{C}\}\right)^{-1} (G')^T,$$

where we also have partitioned G as

$$G = \begin{bmatrix} * & * & * \\ * & G' & * \\ * & * & * \end{bmatrix}.$$

By recalling (4), together with (12), we have

$$\begin{bmatrix} i_0 \\ i' \\ u'' \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & G' & * \\ * & * & * \end{bmatrix} \begin{bmatrix} u_0 \\ u' \\ i'' \end{bmatrix},$$

$$\begin{bmatrix} u_0 \\ u' \\ u'' \end{bmatrix} = e^{j\theta} \begin{bmatrix} 0 & 0 & 0 \\ 0 & X' & * \\ 0 & * & * \end{bmatrix} \begin{bmatrix} i_0 \\ i' \\ i'' \end{bmatrix} + \mathbf{1} u_0.$$

It is easy to see that, in the configurations in which $u' = \mathbf{1}_h$, for some $h \in \mathcal{C}$, $u_0 = 0$, $i'' = 0$,

$$i' = G u' \quad \text{and} \quad u' = e^{j\theta} X' i',$$

and therefore G' is symmetric and equals

$$G' = (e^{j\theta} X')^{-1}.$$

We conclude observing that $G_{hh} = e^{j\theta} |G_{hh}|$. □

The following technical lemma is also needed before stating the main convergence result.

Lemma 3. For any $h \in \mathcal{C}$ we have $|G_{hh}| X_{hh} \geq 1$.

Proof. The statement descends from circuit-theory considerations. Remember that $e^{j\theta} X_{hh}$ corresponds to the equivalent impedance of the grid seen from node h when the PCC is replaced by a short circuit, and all the other microgenerators and loads are replaced with open circuits. The value of G_{hh}, on the other hand, is the equivalent admittance of the grid seen by node h when the PCC *and* all the microgenerators are replaced by short circuits, while the loads are replaced again with open circuits. It then follows that $X_{hh} \geq |G_{hh}|^{-1}$, and then $|G_{hh}| X_{hh} \geq 1$. □

Theorem 4. Assume that the microgenerators $\mathcal{C}$ are activated according to a random i.i.d. sequence, each of them with nonzero probability of being activated. Then

$$\lim_{t \to \infty} q(t) = q_{\text{opt}}.$$

Proof. By using Lemma 2, we have that the matrices F_h in (21) can be rewritten as

$$F_h = I - \frac{\mathbf{1}_h \mathbf{1}_h^T \left(\operatorname{diag}\{|G_{hh}|, h \in \mathcal{C}\}\right)^{-1}}{X'_{hh}}.$$

They are therefore diagonal matrices whose elements are

$$[F_h]_{vv} = \begin{cases} 1 & \text{for } v \neq h \\ 1 - (|G_{hh}| X_{hh})^{-1} & \text{for } v = h. \end{cases}$$

Via Lemma 3, we have that, for all $h \in \mathcal{C}$,

$$0 \leq 1 - (|G_{hh}| X_{hh})^{-1} < 1$$

Therefore, assuming that each matrix F_h, $h \in \mathcal{C}$, has a constant nonzero probability of being chosen, the discrete

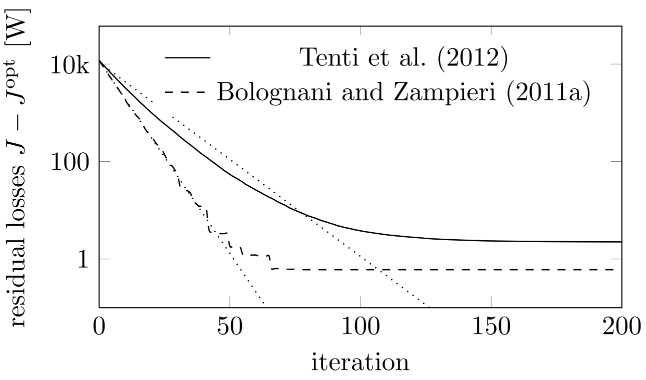

Fig. 5. The algorithm behavior (averaged over 1000 realizations), compared to the algorithm proposed in Bolognani and Zampieri (2011a). The dotted rays represent the corresponding analytic bounds on the rate of convergence.

time system (20) converges to zero. By the definition of $x(t)$, this means that $q(t)$ converges to q_{opt}. $\square$

The same convergence result can be extended, with minor modifications, to more general ways for activating the microgenerators, as long as some very weak conditions are satisfied (namely, the probability of choosing a microgenerator h need not vanish in time).

6. SIMULATIONS

We refer the interested reader to Tenti et al. (2012) for a simulative study of the algorithm. We present in Figure 5 a comparison of the algorithm performance with the alternative algorithm presented in Bolognani and Zampieri (2011a) (from which we adopt the simulation testbed). The following can be observed.

- The two algorithms present different performances, and the strategy by Bolognani and Zampieri (2011a) prevails. However, they have different requirements in terms of coordination of the microgenerators. While in the algorithm by Tenti et al. (2012) only one microgenerator actuates the system at every iteration, in Bolognani and Zampieri (2011a) the microgenerators update their reactive power injection in pairs.
- Both algorithms drive the microgrid to the minimum losses (J^{opt}, which has been calculated by a numerical solver), up to a negligible error.
- The analytic study presented in this paper for the algorithm by Tenti et al. (2012) yields also the tools for studying its convergence speed. Following the same methods proposed in Bolognani and Zampieri (2011a), it is possible to show how the algorithm slowest dynamic is a function of
$$\max_{h \in \mathcal{C}} |G_{hh}| X_{hh}$$
and therefore can be evaluated analytically and can provide some useful design tips. The expected rate of convergence has been plotted in Figure 5.

7. CONCLUSIONS

The main contribution of this work is the analytic proof of convergence of the algorithm proposed in Tenti et al. (2012) for the problem of optimal distributed reactive power compensation. The proposed analysis allows also to study the dynamic behavior of the algorithm and to characterize its rate of convergence, and, by providing a common mathematical framework, to compare different strategies.

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