Abstract

Driven by fundamental technological achievements like the digital revolution in communication and computing and the miniaturization of electronic components, a growing interest in the field of networked systems has appeared, in recent years, in different scientific areas like physics, communication and control engineering, economy, and lately, mathematics. Besides differences among the various approaches, the basic model consists of a certain number of agents, namely, systems living in a common environment and which communicate among each other according to some pre-specified communication pattern. Such a pattern may be fixed or varying according to their physical position, to their internal state and can possibly be affected by noises. The evolution law of each system typically depends on the information obtained through its communication links and is, in general, modeled by a difference or differential equation. For this type of models, the interest is, in general, in studying time evolution, asymptotic behaviors, and in formulating and solving related control problem. In particular, a fundamental issue is understanding the emerging of a group behavior from the individual dynamics and the communication pattern. While many models have already been proposed, the mechanisms which determine how the various individual actions get reflect in the group behavior are, in general, hard to understand and, up to now, few results are known.
In this thesis we will focus on the so-called *consensus problem*, where the group of agents has to reach an agreement on key pieces of information or on a common decision (represented by scalar or vector values) that enable them to cooperate in a coordinate fashion. We will consider a standard algorithm proposed in literature to solve this problem. We will provide some theoretical developments by

- characterizing the speed of convergence of this algorithm for particular communication pattern exhibiting symmetries,
- investigating the realistic and practical situation in which the systems can communicate each other only through digital channels and hence can exchange only quantized information,
- proposing the application of this algorithm to a problem of distributed estimation.

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Contents

Chapter 1

Introduction

In recent years there has been an increasing interest in studying dynamical systems which can be modeled as an interconnection of a large number of identical subsystems [98, 141, 2, 50]. Typically the subsystems exhibit simple dynamics while the overall behavior of the aggregated dynamical units has the complexity features depending on the way the interconnection is built up. The scientific fields where these models appear are manifold.

On the one hand, interesting dynamical systems arise in biological networks at multiple levels of resolution, from interactions among molecules and cells [87] to the behavioral ecology of animal groups [95]. The brain is a network of nerve cells connected by axons, while cells are networks of molecules connected by biochemical reactions. Flocks of birds and schools of fish can travel in formation and act as one unit (see [105] and Figures 1.2 and 1.1), allowing these animals to defend themselves against predator and protect their territories. Wildebeest and other animals exhibit complex collective behaviors when migrating, such as obstacle avoiding, leader election, and formation keeping (see [126, 66] and Figure 1.3). Certain foraging behaviors include animals partitioning their environment into nonverlapping zones (see [12] and Figure 4). Honey bees [122], gorillas [133], and white-faced capuchins [17] exhibit synchronized group activities such as initiation of motion and change of travel direction.

All these social animal behavior has been investigated since many years: see [95, 105] and [126, 66, 12, 122, 133, 17] for specific examples of animal species and [38] and [41] for general studies. Remarkably, it has been noticed that all the above coherent behaviors (individuals typically maintain a fixed distance from neighbors) are obtained by individuals following simple decentralized rules and often with clear bio-mechanical limits to locomotion and

Figure 1.1: School of fish. Groups of animals can act as one unit apparently without following a group leader.

Figure 1.2: Flock of snow geese. Self-organized behaviors emerge in biological groups, even though no individual has global knowledge of the group state. Snow geese fly in formation during migration.

information exchange (an individual changes its trajectory only on the basis of the behavior of its closest neighbors). In other words, these coordination behaviors emerge despite the fact that each individual lacks global knowledge of the network state and can plan its motion by observing only its closest neighbors.

Figure 1.3: Herd of wildebeest in the Serengeti National Park, Tanzania. Wildebeest and other animals exhibit complex coordinated behaviors when migrating, such as obstacle avoiding, leader election, and formation keeping.

At the same time, recently, technological advances in communication and computation have paved the way for the development of large scale multi-agent systems, in particular groups of embedded systems, such as multivehicle and sensor networks. The ambitious goal in this context is to determine local individual rules which under a certain communication exchange pattern make it possible to reach certain global goals. In particular in few years, groups of autonomous agents with computing, communication, and mobility capabilities are expected to become economically feasible and perform a variety of spatially distributed sensing tasks, such as search and rescue, surveillance, environmental monitoring, and exploration.

In general, there are technological limitations on the amount of computation and communication that each unit can perform and the solution will have to take into consideration these constraints. This has stimulated the growth of a new area in control theory known has cooperative control: many identical input/output subsystems are connected through a communication network and each individual has at disposal, at every instant, the information on the state of only a limited number of other units and must implement a local feedback rule only based of this information. Different models are possible depending on the choice of the communication pattern: the set of neighbor systems with which each system exchange information can be fixed (a typically wired situation) or may be time-varying depending on the mutual positions (a typically wireless situation). Moreover, this in-

formation can be exact, but often instead quantized if we are using a digital channel with limited capacity and also possibly corrupted by noise. Moreover, there may be exhogeneous disturbances in the system. The control goals can be of classical type (as stabilization, rendezvous, tracking of a reference trajectory) or of new type (as shape formation), often the goal can be multi-objective: maintaining formation while moving in a certain direction and avoiding obstacles.

A fundamental issue both on the 'natural' and 'artificial' side is to quantitatively explain the emerging of group behavior from the individual dynamics and communication pattern (experimentally deduced in biology or chosen in artificial systems). Though many models have already been proposed, the mechanisms explaining how the group behavior arises from the various individual actions are, in general, hard to understand and, up to now, few results are known. In the cooperative control context however other issues show up. We want in this case to be able to explicitly synthesize control laws for which provable desired behaviors can be achieved and also understand which are the fundamental limitations: indeed a trade-off between performance of the overall system measured through some suitable cost functional and the amount of exchanged information allowed (capacity of the digital channels, number of individual connections, etc.) is expected.

A first intermediate goal toward a comprehensive understanding of how the dynamics of the individual systems can give rise to a group behavior, is to study in full depth some simple problem for N units whose state evolution is described by a simple difference or differential equation (of degree 1 or 2 typically). In this thesis, we will focus on the so-called *consensus problem* where a group of systems must communicate with its neighbors to reach agreement (consensus) on key pieces of information or on a common decision (represented by scalar or vector values) that enable them to cooperate in a coordinate fashion. In general, the systems start with some different initial decisions and communicate between them locally, under some constraints on connectivity and inter-agent information exchange. The *averaging problem* is a special case in which the goal is to compute the exact average of the initial values of the systems. A natural and widely studied consensus algorithm, proposed and analyzed by Tsitsiklis [138] and Tsitsiklis and al. [139], involves, at each time step, every system taking a weighted average of its own value with values received from some of the other systems. To be more precise, we consider a set V of N agents, numbered 1 through N, which will henceforth be referred to as *nodes* (we will see later that the nodes can represent different objects, autonomous agents, sensors, processors, vehicles, clocks according to the different application we are considering). Each node

i starts with a scalar value $x_i(0)$. At each nonnegative integer time t, node i receives from some of the other nodes j a message with the value of $x_j(t)$, and updates its value according to:

$$x_i(t+1) = \sum_{j=1}^{N} P_{ij}(t)x_j(t)$$

where the $P_{ij}(t)$ are weights with the property that $P_{ij}(t) \neq 0$ only if node i receives information from node j at time t. We use the notation $P(t)$ to denote the *weighted matrix* $\{P_{ij}(t)\}_{i,j=1,\dots,N}$. Given a matrix P, let $\mathcal{E}(P)$ denote the set of directed edges (j,i), including self-edges (i,i), such that $P_{ij} \neq 0$. At each time t, the nodes' connectivity can be represented by the directed graph $\mathcal{G}(t) = (V, \mathcal{E}(P(t)))$. The goal is to study the convergence of the iterates $x_i(t)$ to some common value $\bar{x}$, as t approaches infinity. In the case of the *average consensus problem* this value $\bar{x}$ is the average of the initial values, $\bar{x} = x_{ave} := \frac{1}{N}\sum_{i=1}^{N} x_i(0)$. In this thesis we will focus particularly on the *average consensus* algorithm.

The consensus problem has been massively addressed in the last years by many researchers. Indeed the consensus problem arises in a number of applications including distributed algorithms in Multi-vehicle Cooperative Control, information processing in sensor networks, load balancing, and distributed optimization (e.g. agreeing on the estimates of some unknown parameters). In the following, we will briefly review the applications of the algorithms of consensus.

Distributed Algorithms in Multi-Vehicle Cooperative control

The recent advances in miniaturizing of computing, communication, sensing, and actuation, that we have already cited in the Abstract and along the Introduction, have made it feasible to envision large numbers of autonomous vehicles (air, ground and water) working cooperatively to accomplish an objective. Compared to autonomous vehicles that perform solo missions, greater efficiency and operational capability can be realized from teams of autonomous vehicles operating in a coordinated fashion. Cooperative control of multivehicle systems has potential impact in numerous civilian, homeland security, and military applications. Potential civilian applications include monitoring forest fires, oil fields, pipelines, and tracking wildlife. Potential homeland security applications include border patrol and monitoring the perimeter of nuclear power plants. For the military, applications include surveillance, reconnaissance, and battle damage assessment. To enable these

applications, various cooperative control capabilities need to be developed, including formation control, rendezvous, attitude alignment, flocking, foraging, task and role assignment, payload transport, air traffic control and cooperative search. Many of these capabilities require that the team of autonomous vehicles reach the agreement on key pieces of information or on a common decision in order to operate in a coordinating fashion. This agreement can be obtain by the standard consensus algorithm, briefly described above.

Load Balancing

In load balancing [44] the nodes are processors, computers, and the edges as physical connections among them. The corresponding communication graph presents in general some nice symmetry (e.g. a line, a ring, a torus, a hypercube, etc) and also symmetry with respect to communication exchange (if i and j are connected by an edge, it means that i can send data to j and viceversa). In most situations the communication graph is fixed. The measure x_i at each node is in this case the number of tasks which the processor i has to accomplish. The idea is that, in order to speed up the whole computation, processors should exchange tasks along the available edges in order to balance as much as possible the tasks among the various processors. The natural goal is that each processor will have at the end the same quantity of tasks to work on, namely a quantity of task close to the average.

Clock Synchronization

The recent advances in technology have also made low cost, low power wireless sensors a reality. For several applications of a wireless sensor networks, such as mobile target tracking, event detection, efficient TDMA scheduling, and sleep scheduling with very low duty cycle, it is essential that the nodes act in a coordinated and synchronized fashion. All these applications required global clock synchronization, that is, all the nodes of the network need to refer to a common notion of time. For instance, consider the problem of tracking a moving target using proximity sensors, where some nodes are deployed in the environment and their proximity sensors detect when the moving object passes in their vicinity [94]. Assuming that the position of the sensors is known, it is essential that the instants of detection are precisely time-stamped for determining the trajectory (direction and speed) of the moving object. Clearly, the precision of the tracking algorithm based on this system is limited by the accuracy of the clock synchronization. Other interesting ap-

plications, which need a time-synchronization service, are habitat monitoring [136], power scheduling and TDMA communication schemes in wireless sensor networks [72], and rapid synchronized coordination of powerlines nodes in electric power distribution networks for catastrophic power-outage prevention [6]. Recently novel consensus-based protocol, for synchronizing a wireless has been elaborated, see for instance [120, 125, 32].

Decentralized estimation

Wireless sensor networks (WSN) have broad applications in surveillance and environmental monitoring, collaborative processing and information, and gathering scientific data from spatially distributed sources for environmental modeling and protection. Dealing with sensor networks, distributed estimation and tracking is one of the most fundamental collaborative information processing problem. Multi-sensor data fusion and tracking problems have a long history in signal processing, control theory, and robotics [1]. Moreover, estimation issues in wireless networks with fully distributed protocol of communication and elaboration data, i.e., without any hierarchical structure, have been the center of much attention only lately [96, 130]. Due to the huge number of devices that many applications could involve, collecting measurements from distributed wireless sensors nodes at a single location for on-line data processing may not be feasible due to several reasons among which long packet delay (e.g. due to multi-hop transmission) and/or limited bandwidth of the wireless network (e.g. due to energy consumption requirements). This problem is particularly relevant in wireless ad-hoc sensor networks where information needs to be multi-hopped from one to another using closer neighbors.
It turns out that a fundamental problem in sensor networks is to solve detection and estimation problem using *scalable algorithms*. This requires development of novel distributed algorithms for estimation and in particular for Kalman filtering. Recently, new scalable sensor fusion schemes requiring fusion of sensor measurements combined with local Kalman filtering have been proposed [130]. The key component of this approach is the introduction of a distributed filter that allows the nodes of a sensor network to track the average of N sensor measurements using an average consensus based distributed filter called *consensus filter* [102].

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1.1 Overview and contributions of the Thesis

The dissertation is organized as follows:

- **Chapter 2**. In this chapter we review the fundamental features of the *consensus algorithm* and of the *average consensus algorithm*, both for the time-varying case and for the time-invariant case. In particular, we summarize the main results concerning the convergence and we introduce the concept of speed of convergence toward the consensus.

- **Chapter 3**. In this chapter we derive bounds on the convergence rate to the average consensus for set of systems that exchange information over time-invariant communication networks with symmetries (the Cayley symmetries). We show that, in time-invarinat networks, symmetries yield rather slow convergence to the average consensus. In particular for such networks we have computed a tight bound for the convergence rate.

- **Chapter 4**. In this chapter we focus on the time-varying consensus problem, by addressing, in particular, the analysis of the so-called *randomized consensus algorithms*. Precisely, in this chapter, we assume that the consensus matrices constitute a sequence of i.i.d. matrix valued random variables. We review the concepts of *probabilistic consensus* and *average probabilistic consensus*. In the first part of the chapter, we introduce two random strategies, that illustrate a remarkable property of the randomized consensus algorithms, i.e., they allow to achieve better performance than deterministic ones with comparable complexity. In the second part of the chapter we review a well-known random consensus algorithm: the symmetric gossip. In particular, we will provide an interesting characterization of the symmetric gossip algorithm over Cayley graphs.

- **Chapter 5**. In this chapter we consider the more realistic and practical situation in which the communication network between the systems is constituted of only rate-constrained digital links. This, in general, prevents the nodes from having a precise knowledge about the state of the other nodes. Indeed, through digital channels the nodes can exchange only symbolic data in a finite alphabet and using this information they can build at most an estimate of their neighbors. Here, we assume that the nodes quantize their information before transmitting it. In particular we introduce two quantizers, well-known in the literature: the *deterministic* uniform quantizer and the *probabilistic* uniform

quantizer. Moreover, we restrict to the time-invariant average consensus problem. Since in the literature, remarkable results regarding the time-invariant quantized average consensus problem via probabilistic quantizers are already present, we focus mainly to the study of the time-invariant quantized average consensus problem via deterministic quantizers. The main contribution is the introduction of a simple and effective adaptation of the standard average consensus algorithm which does not converge to an asymptotic agreement, but is able to preserve the average of states and to drive the systems reasonably near to the consensus. We analyze this scheme by means of a worst-case model and a probabilistic model showing favorable convergence properties and providing performance bounds for the limit points of the iterates generated.

- **Chapter 6**. In this chapter we analyze the effects due to the presence of a communication network constituted of only digital channels on the symmetric gossip algorithm introduced in Chapter 4. We introduce two particular strategies, the *partially quantized* strategy and the *globally quantized* strategy, depending whether the systems use exact information regarding their own state or not to update their states. We will analyze these strategies both via the deterministic quantizer and via the probabilistic quantizer. We show that the *globally quantized* strategy both via the deterministic quantizer and via the probabilistic quantizer ensures that, almost surely, the consensus is reached. The drawback of this strategy is that it does not preserve the average of the initial conditions. On the other hand, the *partially quantized* strategy maintains the initial average at each iteration of the algorithm, but does not guarantee that the consensus is reached in general. However we show that the *partially quantized* drives asymptotically all the states very close to the initial average (we quantify how close).

- **Chapter 7**. Also in this chapter we assume that the nodes can communicate between them only through digital channels. As in Chapter 5, we consider, here, the time-invariant case. In the previous two chapters, in order to face with the effects due to the forced quantization, we have elaborated strategies which either preserves the average of the state but do not converge to a consensus or do not preserve the average but converge to a consensus which, in general, does not coincide with the initial average. The main contribution of this chapter is to introduce a novel quantized strategy that permits both to maintain the initial average and to reach it asymptotically. More precisely we adapt

coding/decoding strategies, that were proposed for centralized control and communication problems, to the distributed consensus problem. In particular, we present two coding/decoding strategies, one based on the exchange of logarithmically quantized information, the other on a zoom in - zoom out strategy (this latter involves the use of uniform quantizers). We provide analytical and simulative results illustrating the convergence properties of these strategies. In particular we show that the convergence factors depend smoothly on the accuracy parameter of the quantizers used and that, remarkably, that the critical quantizer accuracy sufficient to guarantee convergence is independent from the network dimension.

- **Chapter 8**. In this chapter we deal with a possible application of the consensus ideas to the wide field of the distributed estimation. In particular we study a prototypical problem of distributed estimation for sensor networks; the state of a scalar linear system is estimated via a two stage procedure which consists in (i) a standard (and decentralized) Kalman-like update and (ii) information propagation using consensus strategies. To this purpose, two design parameters, i.e. the Kalman gain and the consensus matrix have to be designed. This choice is made by optimizing the steady state prediction (or estimation) error. We discusse, under specific circumstances, the behavior of the "optimal" parameters.

- **Chapter 9**. In this chapter we summarize the result found and we gather out our conclusions.

In the appendices, we include some reference material for the reader, in particular

- **Appendix A**. In this appendix, we review some concepts on harmonic analysis on finite groups which are useful for the analysis, illustrated in Chapter 3, of the spectral properties of the Cayley matrices

- **Appendix B**. In this appendix, we recall some notation and concepts on directed and undirected graphs which are useful throughout all the thesis.

- **Appendix C**. In this appendix, we collect some algebraic results regarding the stability of discrete time linear parameter varying (LPV) systems and the solvability of a particular Lyapunov equation. The proofs of some theorem stated in this thesis, are based on these algebraic results.

Notation

We let $\prod_{i\in\{1,\dots,N\}} S_i$ denote the Cartesian product of sets $S_1, \dots, S_N$.

We let $\mathbb{C}$, $\mathbb{R}$ denote the set of complex numbers and the set of real numbers, respectively. The set of natural numbers is denoted by $\mathbb{N}$.

For $x \in \mathbb{R}^d$, we denote by $\|x\|$ (or $\|x\|_2$), by $\|x\|_\infty$ and by $\|x\|_F$ the Euclidean norm, the ∞-norm and the Frobenius norm of x, respectively.

Given a vector $x \in \mathbb{C}^d$ we denote by x^* its conjugate transpose. We use the same notation even if $x \in \mathbb{R}^d$.

We define the vectors $\mathbf{0} = [0, \dots, 0]^*$ and $\mathbb{1} = [1, \dots, 1]^*$ in $\mathbb{R}^d$.

Given a vector $x = [x_1, \dots, x_d]^*$ belonging to $\mathbb{R}^d$ or to $\mathbb{C}^d$, $\operatorname{diag}\{x\}$ or $\operatorname{diag}\{x_1, \dots, x_N\}$ mean a diagonal matrix having the components of x as diagonal elements.

Given a matrix $M \in \mathbb{R}^{d\times d}$, with $\sigma(M)$ we denote the spectrum of M, i.e., the set of its eigenvalues.

Let M any matrix belonging to $\mathbb{R}^{d\times d}$. With $\operatorname{tr} M$ we denote the trace of M, i.e., the sum of the diagonal entries. With M^* we denote the transpose of M.

For $f, g : \mathbb{N} \to \mathbb{R}$, we say that $f \in O(g)$ if there exist $N_0 \in \mathbb{N}$ and $k > 0$ such that $|f(N)| \leq k|g(N)|$ for all $N \geq N_0$. For $f, g : \mathbb{N} \to \mathbb{R}$, we say that $f \in o(g)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

Let Ω be a random variable. Then, $\mathbb{E}[\Omega]$ denotes the expectation of Ω.

For notations related to graphs, we refer the reader to the Appendix A.

Chapter 2

The Consensus Problem

2.1 Introduction

This chapter overviews the fundamental discrete-time consensus algorithm in which a scalar information state is updated by each system using a first-order difference equation [101, 74, 112, 88, 139]. The information flow between the systems is modeled by a direct graph. No constraints of bandwidth are considered. Both fixed and dynamically changing interaction topologies are analyzed. The main results regarding the convergence of the consensus algorithm, are summarized. The concept of speed of convergence toward the consensus is introduced and a optimization problem is formulated.

In this chapter we will refer to the notions of graph theory collected in Appendix B.

2.2 Problem Formulation

Consider $N > 1$ systems whose dynamics are coupled by the following discrete time state equations

$$x_i(t+1) = \sum_{j=1}^{N} P_{ij}(t) x_j(t) \qquad i = 1, \dots, N\,, \tag{2.1}$$

where $x_i(t) \in \mathbb{R}$ is the state of the i-th system at time t and $P_{ij}(t) \in \mathbb{R}$ are coefficients which vary with the time t. More compactly we can write

$$x(t+1) = P(t)x(t), \tag{2.2}$$

where $x(t) \in \mathbb{R}^N$ and $P(t) \in \mathbb{R}^{N\times N}$. The sequence $P(t)$ is said to achieve the *consensus* if the following conditions are satisfied:

(a) If $x(0) \in \mathcal{I}$ then $x(t) = x(0)$ for every $t \in \mathbb{N}$, where $\mathcal{I}$ is the subspace generated by the N- dimensional vector $\mathbb{1} := [1, \dots, 1]^*$.

(b) For any $x(0) \in \mathbb{R}^N$, there exists a scalar α such that

$$\lim_{t\to\infty} x(t) = \alpha\mathbb{1}.$$

Moreover, if $\alpha = N^{-1}\mathbb{1}^*x(0)$, we say that *average consensus* is achieved.

It is worth noting that condition (a) is imposed in order to avoid meaningless updating rules. Indeed if $x(\bar{t}) = \alpha\mathbb{1}$ at some time instant $\bar{t}$, then condition (a) guarantees that $x(t) = \alpha\mathbb{1}$ for all $t \geq \bar{t}$. Moreover it is possible to see that condition (a) implies also that

$$P(t)\mathbb{1} = \mathbb{1} \tag{2.3}$$

for every $t \in \mathbb{N}$. A matrix satisfying (2.3) is called *quasi-stochastic* [128].

From now on, we assume that this condition holds for the sequence of matrices $P(t)$.

Assumption 2.1 *The matrix $P(t)$ in (2.2) is a quasi-stochastic matrix for each time instant $t \in \mathbb{N}$.*

It is worth providing now other three definitions that will be useful throughout all this thesis and that specify the concept of *quasi-stochastic* matrix for some particular case. If the *quasi-stochastic* matrix $P(t)$ satisfies also the condition $\mathbb{1}^*P(t) = \mathbb{1}^*$ then it is called *quasi-doubly stochastic*. Moreover if we restrict the *quasi-stochastic* (respectively the *quasi-doubly stochastic*) matrix $P(t)$ to have all the elements nonnegative, namely $P_{ij}(t) \geq 0$ for all pair i, j, then $P(t)$ is said to be a *stochastic* (respectively a *doubly stochastic*) matrix [123].

Note now that, the fact that in the matrix $P(t)$ the element in position i, j is different from zero, means that the system i needs the state of the system j in order to update its state. This implies that we need to communicate the state $x_j(t)$ from the system j to the system i. In this context, a good description of the information flow required by a specific matrix $P(t)$ is given by the directed graph $\mathcal{G}_{P(t)}$ with set of vertices $\{1, \dots, N\}$ in which there is an arc from j to i whenever in the matrix $P(t)$ the element $P_{ij}(t) \neq 0$.

The graph $\mathcal{G}_{P(t)}$ is said to be the *communication graph* associated with $P(t)$. Conversely, given any directed graph $\mathcal{G}$ with set of vertices $\{1, \ldots, N\}$, we say that a matrix $P(t)$ is *compatible* with $\mathcal{G}$ if $\mathcal{G}_{P(t)}$ is a subgraph of $\mathcal{G}$ (we will use the notation $\mathcal{G}_{P(t)} \subseteq \mathcal{G}$). For the sake of the clarity, it is worth noting that if a graph $\mathcal{G}$ contains the self loop (i, i) it means that the i-th agent has access to its own state.

We say that the consensus problem is solvable on a sequence of graphs $\mathcal{G}(t)$ if there exists a sequence of matrices $P(t)$ compatible with the sequence $\mathcal{G}(t)$ and satisfying conditions (a) and (b).

Observe now that, if both $P_{ij}(t)$ and $P_{ji}(t)$ are different from zero, then both the edges (j, i) and (i, j) belongs to $\mathcal{G}_{P(t)}$. In such case the interaction topology between the systems can be modeled by an undirected graph. However, in general, an undirected graph can be viewed as a special case of a direct graph. A direct graph is useful because it takes into account the general case where information flow may be unidirectional.

Note that consensus algorithm (2.1) is distributed in the sense that each system needs only the information from its neighbors. Intuitively, the information state of each system is updated as the weighted average (with possible negative weights) of its current state and the current states of its neighbors. Also note that the weights $P_{ij}(t)$ in (2.1) may be time-varying to represent both the time-varying relative confidence of each'system information state and the fact that the interaction topology may be changing dynamically due to unreliable transmission or a limited communication/sensing range.

Remark 2.2 The consensus problem first appeared in [141]. It is worth noting that the Vicsek model can be viewed as a special case of (2.1) by letting

$$P_{ij} = \begin{cases} \frac{1}{1+|\mathcal{N}_i|} & \text{if } (j,i) \in \mathcal{G}(t) \text{ or } i = j \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{N}_i(t) = \{j \in V : j \neq i,\ (j, i) \in \mathcal{G}(t)\}$ represents the set of neighbors of system i at time t and $|\mathcal{N}_i(t)|$ its cardinality, that is, each system simply averages its own information state with those that are communicated to it. Note that the matrix $P(t)$ built in this way is a stochastic matrix but not in general a doubly stochastic matrix, even if $\mathcal{G}$ is an undirected graph. Instead the simplified Vicsek model used in [74] which is obtained by letting $P_{ij}(t) = 1/k$ if $(j, i) \in \mathcal{G}(t)$ and $P_{ij}(t) = 0$ otherwise, $\forall\, j \neq i$, and $P_{ii}(t) = 1 - \sum_{j \neq i} P_{ij}(t)$, where $k > N$ is a constant, leads to a doubly stochastic matrix in the case $\mathcal{G}$ is undirected.

Remark 2.3 When considering a team of vehicles cooperating together, a

nice interpretation of the consensus problem (called in this case also rendezvous problem), closer to the classical control theory, is the following. Let us rewrite (2.1) as a discrete time single-integrator

$$x_i(t+1) = x_i(t) + u_i(t) \qquad i = 1, \dots, N\,, \tag{2.4}$$

where $u_i(t) \in \mathbb{R}$ is the control input that the i-th vehicle applies at time t. More compactly, we can write that

$$x(t+1) = x(t) + u(t)\,, \tag{2.5}$$

where $x(t), u(t) \in \mathbb{R}^N$. In the rendezvous problem, the goal will be to design a sequence of feedback control laws

$$u(t) = K(t)x(t), \qquad K(t) \in \mathbb{R}^{N\times N}$$

yielding the consensus of the states, namely a sequence of feedback control laws control such that the closed loop system

$$x(t+1) = (I + K(t))x(t) \tag{2.6}$$

yields

$$\lim_{t\to\infty} x(t) = \alpha \mathbb{1}, \tag{2.7}$$

where α is a scalar depending only on $x(0)$ and on the sequences $K(t)$. Note that if we define

$$P(t) := I + K(t)$$

then we obtain (2.2). Conversely, we can rewrite (2.1) as

$$x_i(t+1) = x_i(t) + \sum_{j=1}^{N} P_{ij}(t)\,(x_j(t) - x_i(t))\,,$$

and defining

$$u_i(t) = \sum_{j=1}^{N} P_{ij}(t)\,(x_j(t) - x_i(t))$$

or, more compactly,

$$u(t) = (P(t) - I)x(t) \tag{2.8}$$

we obtain (2.4) and (2.5).

We conclude this section by formulating a first preliminary algebraic result on the convergence of the consensus algorithm. The proof can be found in [112].

Theorem 2.4 *Discrete-time algorithm (2.1) achieves consensus if and only if*

$$P(t)P(t-1)\cdots P(2)P(1)P(0) \to \mathbb{1}c^*, \tag{2.9}$$

as $t \to \infty$, where c is a N-dimensional vector.

2.2.1 Convergence Analysis of Consensus Algorithms

In this subsection we summarize the results known about the conditions ensuring the solvability and the convergence of the consensus algorithm. First we will consider the general case of Time-varying Communication Topologies, then we will specialize to the case of Time-invariant Communication Topologies and weights P_{ij} constant.

Convergence Analysis for Time-varying Communication Topologies

We start by considering the case of direct switching interaction topologies. The next two theorems, whose proof can be found in [88, 25], rely mainly on the notion of connectivity of a node to an another node, provided at the end of Appendix B.

Theorem 2.5 *Let $P(t)$, $t \in \mathbb{N}$, be a sequence of stochastic matrices, such that $P_{ii}(t) \neq 0$, $i = 1, \ldots, N$, for all $t \in \mathbb{N}$. Consider the sequence of directed graphs $\mathcal{G}_{P(t)}$, associated to the sequence $P(t)$. Assume the existence of real numbers $0 < m \leq M$ such that $m \leq P_{ij}(t) \leq M$ for all $t \in \mathbb{N}$ and $(j, i) \in \mathcal{G}_{P(t)}$. If there exists a duration $T \geq 0$ such that for all $t_0 \in \mathbb{N}$ the graph*

$$\mathcal{G}_{P(t_0+1)} \cup \cdots \cup \mathcal{G}_{P(t_0+T)}$$

contains a node connected to all the other nodes, then

$$\lim_{t\to\infty} x(t) = c\mathbb{1},$$

where $c \in \mathbb{R}$ is a constant depending only on the initial condition $x(0)$ and on the sequence $P(t)$.

Theorem 2.5 gives a general result about stochastic matrices that are not necessarily symmetric. The following theorem presents a convergence result for the case of symmetric matrices (i.e., undirected graphs) under connectivity requirements that are weaker than the ones stated in the previous Theorem 2.5.

Theorem 2.6 *Let $P(t)$, $t \in \mathbb{N}$, be a sequence of symmetric stochastic matrices, such that $P_{ii}(t) \neq 0$, $i = 1, \dots, N$, for all $t \in \mathbb{N}$. Consider the sequence of undirected graphs $\mathcal{G}_{P(t)}$, associated to the sequence $P(t)$, $t \in \mathbb{N}$. Assume the existence of real numbers $0 < m \leq M$ such that $m \leq P_{ij}(t) \leq M$ for all $t \in \mathbb{N}$ and $(j, i) \in \mathcal{G}_{P(t)}$. If for all $t_0 \in \mathbb{N}$ the graph*

$$\bigcup_{t \geq t_0} \mathcal{G}_{P(t)}$$

is connected, then

$$\lim_{t \to \infty} x(t) = c\mathbb{1},$$

where $c \in \mathbb{R}$ is a constant depending only on the initial condition $x(0)$ and on the sequence $P(t)$.

Remark 2.7 Note that the above two Theorems provide conditions ensuring the convergence of the consensus algorithms, only for sequence of stochastic matrices. To the best of our knowledge, there are no similar result for the more general case of quasi-stochastic matrices.

Remark 2.8 For the sake of the completeness it is worth noting that the authors in [113] and [112] provide very similar results. In particular, given a sequence of stochastic matrices graphs (*respectively* a sequence of symmetric stochastic matrices) $P(t)$, they relate the convergence of the consensus problem to the following two facts:

(i) $P_{ij}(t) \in \bar{\mathcal{P}}$, for all $t \in \mathbb{N}$, where $\bar{\mathcal{P}}$ is a finite set of nonnegative numbers that are no larger than one;

(ii) there exists an infinite sequence of contiguous, nonempty, uniformly bounded time intervals $[t_j, t_{j+1})$, $j = 1, 2, \dots$ starting at $t_1 = 0$, with the property that the union of the directed graphs associated to the matrices (*respectively* undirected graphs) across each such interval has a directed spanning tree (*respectively* is connected).

If the above two facts yield true, then

$$\lim_{t \to \infty} x(t) = c\mathbb{1},$$

where $c \in \mathbb{R}$ is a constant depending only on the initial condition $x(0)$ and on the sequence $P(t)$.

Convergence Analysis for Time-Invariant Communication Topologies

In this subsection, we investigate conditions under which the information state of consensus algorithm converges when the communication topology is time invariant and the gains P_{ij} are constant. In this case (2.1) and (2.2) become

$$x_i(t+1) = \sum_{j=1}^{N} P_{ij} x_j \qquad i = 1, \dots, N\,, \tag{2.10}$$

and

$$x(t+1) = Px(t) \tag{2.11}$$

where $P \in \mathbb{R}^{N\times N}$.

We state immediately the following proposition which provides an algebraic characterization of the quasi-stochastic matrices achieving the consensus (see [35]).

Proposition 2.9 *Let P be any quasi-stochastic matrix, namely $P\mathbb{1} = \mathbb{1}$. Then P yields asymptotically the consensus if and only if the following three conditions hold :*

(A) 1 *is the only eigenvalue of P on the unit circle centered in* 0*;*

(B) the eigenvalue 1 *has algebraic multiplicity one (namely it is a simple root of the characteristic polynomial of P) and $\mathbb{1}$ is its eigenvector;*

(C) all the other eigenvalues are strictly inside the unit disk centered in 0*.*

Now, it is worth noting that a matrix P satisfying the above three properties achieves the average consensus if and only if $\mathbb{1}^*P = \mathbb{1}^*$, that is if and only if P is a quasi- doubly stochastic matrix. To the best of our knowledge, there are no other conditions regarding the quasi-stochastic matrices (or quasi-doubly stochastic matrices) characterizing the convergence of the consensus algorithm (2.11). Something more can be said if we restrict to nonnegative matrices, i.e., to stochastic (or doubly stochastic) matrices. The following result is the straightforward consequence of standard results on stochastic matrices [63, pag. 88 and pag. 95].

Theorem 2.10 *Let $\mathcal{G}$ be a directed graph and assume that $\mathcal{G}$ contains all loops (i, i). The following conditions are equivalent:*

(i) the consensus problem is solvable on $\mathcal{G}$;

(ii) $\mathcal{G}$ contains a node which is connected to all the other node.

Moreover, if the above conditions are satisfied, any stochastic matrix P such that $\mathcal{G}_P = \mathcal{G}$ and $P_{ii} \neq 0$ for every $i = 1, \dots, n$ solves the consensus problem.

We observe that requiring that a graph $\mathcal{G}$ has a node connected to all the other nodes is equivalent to requiring that the graph $\mathcal{G}$ has a *spanning tree*. Moreover it is worth also noting that, as shown in [115], if (2.11) converges asymptotically, then the consensus equilibrium will be equal to a weighted average of the initial conditions of those systems that have a direct path to all other systems. Hence we can hope that, if $\mathcal{G}$ is strongly connected, adding probably some further condition, also the average consensus problem can be solved. Indeed the next proposition, shows that, differently from the time-varying case, for the time-invariant consensus algorithm it is also possible provide conditions guaranteeing the solvability of the average consensus problem (see [101]). Also the next theorem is the straightforward of standard results on stochastic matrices.

Theorem 2.11 *Let $\mathcal{G}$ be a directed graph and assume that $\mathcal{G}$ contains all loops (i, i). The following conditions are equivalent:*

(i) the average consensus problem is solvable on $\mathcal{G}$;

(ii) $\mathcal{G}$ is strongly connected.

Moreover, if the above conditions are satisfied, any doubly stochastic matrix P such that $\mathcal{G}_P = \mathcal{G}$ and $P_{ii} \neq 0$ for every $i = 1, \dots, n$ solves the average consensus problem.

We conclude this subsection by stating the following result which is a direct consequence of the above Theorem and characterizes the undirected communication graphs and symmetric matrices P. Moreover we provide two interesting and simple rules for choosing the weights of the symmetric matrix P, often used in the literature: the *Maximum-degree weights* and the *Metropolis weights* (see [149]).

Theorem 2.12 *Let $\mathcal{G}$ be an undirected graph. Assume that $\mathcal{G}$ is connected and it contains all loops (i, i). Then any symmetric stochastic matrix P such that $\mathcal{G}_P = \mathcal{G}$ and $P_{ii} \neq 0$ for every $i = 1, \dots, N$ achieves asymptotically the average consensus.*

Example 2.13 [Maximum-degree weights]
Given an undirected graph $\mathcal{G} = (V, \mathcal{E})$, let $d \in \mathbb{R}$ be such that

$$d \geq \max_{1 \leq i \leq N} \{d_i\}$$

where $d_i = |\mathcal{N}_i|$, with $\mathcal{N}_i = \{j \in V \,|\, j \neq i, (j,i) \in \mathcal{E}\}$. Here we use the constant weight $1/(d+1)$ on all the edges, and choose the self-weights so that the sum of weights at each node is 1:

$$P_{ij} = \begin{cases} \frac{1}{d+1} & \text{if } (j,i) \in \mathcal{E} \text{ and } i \neq j \\ 1 - \frac{d_i}{d+1} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Example 2.14 [Metropolis weights]
Let $\mathcal{G}$ be as in the previous example. The Metropolis weights are defined as

$$P_{ij} = \begin{cases} \frac{1}{1+\max\{d_i, d_j\}} & \text{if } (j,i) \in \mathcal{E} \\ 1 - \sum_{(i,k) \in \mathcal{E} \setminus \{(i,i)\}} P_{ik} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

where d_i is as in the above example. With Metropolis weights, the weight on each edge is one over one plus the larger degree at its two incident vertices, and the self-weights are chosen so the sum of weights at each node is 1.

2.2.2 Consensus Synthesis

In some applications, consensus algorithms must satisfy given requirements or optimize a suitable performance index. The simplest control performance index is the rate of convergence toward the consensus equilibrium. We follow the approach proposed in [147]. Given a sequence of matrices $\mathbb{P}(t),\ t \in \mathbb{N}$, yielding the consensus, i.e satisfying (2.9), we define the *asymptotic convergence factor* as

$$r_{asym} = \sup_{x(0)} \limsup_{t \to \infty} \left(\|x(t) - x_{ave}(t)\|_2 \right)^{\frac{1}{t}} .$$

where $x_{ave}(t) = 1/N \mathbb{1}^* x(t)$.

From this choice of performance, the problem one would like to solve is the following.

Problem: Given a sequence of graphs $\mathcal{G}(t),\ t \in \mathbb{N}$, find a sequence of matrices $P(t),\ t \in \mathbb{N}$, such that $\mathcal{G}_{P(t)} \subseteq \mathcal{G}(t)$ and minimizing r_{asym}.

The solution of the above problem is very hard in general. Some interesting results can be found in the literature, if we restrict to the time-invariant average consensus. In order to illustrate them, we need first to introduce the concept of *essential spectral radius*. Given any quasi-stochastic matrix P, we define the quantity

$$\rho_{ess}(P) = \begin{cases} 1 & \text{if } \dim \ker(P-I) > 1 \\ \max\{|\lambda| : \lambda \in \sigma(P) \setminus \{1\}\} & \text{if } \dim \ker(P-I) = 1\,, \end{cases} \tag{2.12}$$

which is called the *essential spectral radius* of the matrix P.
It is easy to see that a quasi-stochastic matrix P yields asymptotically the consensus, namely satisfies the conditions (A), (B), (C) stated in Proposition 2.9, if and only if $\rho_{ess}(P) < 1$.
Consider now the time-invariant average consensus algorithm

$$x(t+1) = Px(t),$$

where P is a quasi-doubly stochastic matrix such that $\rho_{ess}(P) < 1$. The following result relates the asymptotic convergence to the essential spectral radius of P (see [25]).

Theorem 2.15

$$\rho_{ess}(P) = r_{asym}(P).$$

Using this result the above problem can be formulated as the following minimization problem:

$$\begin{array}{ll} \text{minimize} & \rho_{ess}(P) \\ \text{subject to} & P \in \mathcal{P} \end{array} \tag{2.13}$$

where $\mathcal{P} = \{\mathcal{P} : \mathcal{G}_{\mathcal{P}} \subseteq \mathcal{G}, \mathcal{P}\mathbb{1} = \mathbb{1}, \mathbb{1}^*\mathcal{P} = \mathbb{1}^*\}$. The authors in [147] refer to this problem as the *fastest distributed linear averaging problem (FDLA)*. It is worth noting that $\rho_{ess}(P) = \rho(P - 1/N\mathbb{1}\mathbb{1}^*)$, where given a matrix $M \in \mathbb{R}^{N\times N}$, $\rho(M)$ denotes the spectral radius of M. Hence the above problem can be seen as a spectral radius minimization problem. Even though the constraint in this problem are linear equalities, the solution of the problem in general is very hard. The main reason is that the objective function, i.e. the spectral radius of a matrix, is not a convex function.
Now suppose we add the additional constraint that weights are symmetric, i.e., $P_{ij} = P_{ji}$. In this case the problem can be posed as

$$\begin{array}{ll} \text{minimize} & \rho_{ess}(P) \\ \text{subject to} & P \in \mathcal{P},\ P = P^* \end{array} \tag{2.14}$$

which is a convex problem. This represent the symmetric version of the FDLA problem. In [147] it is shown that this problem can be cast as a semidefinite program, and therefore efficiently and globally solved.

Remark 2.16 When we are dealing with average consensus algorithm it is meaningful to consider the displacement from the average of the initial conditions

$$\Delta(t) := x(t) - \left(\frac{1}{N}\mathbb{1}^T x(0)\right)\mathbb{1}\,. \tag{2.15}$$

It is immediate to check that, if P satisfies condition $P\mathbb{1} = \mathbb{1}$ then $\Delta(t)$ satisfies the recursive equation

$$\Delta(t+1) = P\Delta(t)\,. \tag{2.16}$$

Moreover, if P also satisfies condition $\mathbb{1}^* P = \mathbb{1}^*$, then

$$\Delta(t) = x(t) - \left(\frac{1}{N}\mathbb{1}^T x(t)\right)\mathbb{1}\,.$$

Notice finally that the initial conditions $\Delta(0)$ are such that

$$\mathbb{1}^T\Delta(0) = 0\,. \tag{2.17}$$

Hence the asymptotic behavior of our average consensus problem can equivalently be studied by looking at the evolution (2.16) on the hyperplane characterized by the condition (2.17).

We conclude this section by noting that the FDLA problem (2.13) is closely related to the problem of finding the fastest mixing Markov chain on a graph [20]. This last problem can be posed as

$$\begin{array}{ll}\text{minimize} & \rho_{ess}(P)\\ \text{subject to} & P \in \tilde{\mathcal{P}},\end{array} \tag{2.18}$$

where $\tilde{\mathcal{P}} = \{P : \mathcal{G}_P \subseteq \mathcal{G}, \mathcal{P} \text{ is doubly stochastic}\}$. Note that minimizing the essential spectral radius $\rho_{ess}(P)$ it is equivalently to maximizing $1 - \rho_{ess}(P)$(which is called the spectral gap of the associated Markov chain). Recently some very effective algorithms have been proposed for this maximization limited, however, to the case in which P is a symmetric matrix [20]. The only difference in the two problem formulations is that in the FDLA problem, the weights can be (and the optimal ones often are) negative, hence faster convergence could be achieved compared with the fastest mixing Markov chains on the same graph.

2.3 Conclusion

In this chapter we have overviewed the standard discrete-time consensus algorithm. In particular we have summarized the main results regarding the convergence. It is worth now ending by mentioning the numerous recent directions of research on consensus and averaging: continuous-time consensus algorithms [101, 88, 80, 81], characterization of the convergence rates and time complexity [103, 104, 77, 35, 27, 110], consensus over random networks [70, 145, 109, 137, 108, 106, 107, 58, 57], consensus in finite-time [40, 135], consensus algorithms for general functions [13, 39, 84, 135], quantized consensus problems [29, 30, 75, 152, 150, 148, 33, 92, 8, 61], connections with the heat equation and partial difference equation [60], spatially-decaying interactions [42], convergence in time- delayed and asynchronous settings [16, 7, 59], consensus on manifolds [118, 119, 73], applications to distributed signal processing [151, 31, 130, 149, 100]. Numerous interesting results are reported in the recent PhD theses [79, 83, 26, 71]. Finally, we point out two recent surveys [99, 114] and the texts [113, 25].

Chapter 3

The symmetries in the consensus problem

3.1 Introduction

As explained in the previous chapter, the connectivity properties of the *communication graph* influence the convergence properties of the consensus algorithm (2.2). A natural question now arises: " How much does the amount of information exchanged by the systems in the consensus algorithm influence the rate of convergence towards the asymptotic agreement?"

This chapter aims to provide an answer to the above question. Intuitively one should expect that the larger is the communication effort between the systems, the better are the performance achievable by the consensus algorithms. The main result of the chapter is a mathematical characterization of this fact for a particular class of graphs and matrices exhibiting symmetries.

We have pointed out in the previous chapter how quasi-doubly stochastic matrices, whose associated communication graphs satisfy some assumptions, described in Theorem 2.5, in Theorem 2.6, in Theorem 2.10, in Theorem 2.11 and in Theorem 2.12, are matrices that guarantee the solvability of the average consensus, with a degree of efficiency which is related to the spectral properties of such matrices (see Theorem 2.15). Moreover we recall that, if we restrict to the case in which the weights are nonnegative number, then the consensus matrices are doubly stochastic and hence regarding them as Markov chains allows to relate the consensus convergence rate to the mixing rate of the chain [14]. The problem of bounding the mixing rate of a Markov chain is a large research area: see for instance the survey [68] and references

there in.

Spectral properties of doubly stochastic matrices can be characterized in a easier way if we impose some symmetries on the matrices themselves, and, as consequence, on the associated communication graph. Both Markov chains and graphs satisfying symmetries, which are called Cayley graphs, are widely studied in the literature [10, 89, 140]. It is known that symmetries described by Abelian groups yield rather poor convergence rates [4].

In this chapter, by modelling the communication networks as Cayley graphs defined on Abelian groups we are able to extend the available bounds on the mixing time of Markov chains defined on such groups [48, 14, 117]. In particular, in Section 3.3, we derive a bound that is a function of the number of agents and the incoming arcs in each vertex. The main result shows that, if we impose symmetries in the communication network and we keep the number of incoming arcs in each vertex bounded, then the convergence rate degrades as the number of agents increases. Moreover, we show that the proposed bound is tight. It is worth noting that the idea of imposing symmetries on the communication graph is not new [45, 111, 127]. One of the reasons to impose symmetries in the communication graph derives from the fact that graphs with symmetries allow a much more compact representation, which can be relevant when a very large number of agents is considered. In particular in [127] the authors show, for particular symmetries, how it is possible to obtain better performance by increasing the number of incoming arcs on each vertex. In this chapter we extend the result to a broader class of graphs with symmetries.

The chapter is organized as follows. In Section 3.2 we will formulate the problem. In particular our goal will be to minimize the *essential spectral radius*, quantity that we have defined in (2.12). We will characterize the solution for a first class of matrices and graphs exhibiting circulant symmetries. In Section 3.3 we will extend our results to the broader class of Cayley graphs and Cayley matrices. Finally we will gather out our conclusions in Section 3.4.

3.2 Problem formulation

In this section we consider the minimization problem (2.18) formulated in Section 2.2.2. For the sake of the clarity we briefly reformulate it. Suppose that there are N system connected in a network. The network's communication topology is given by the direct graph $\mathcal{G} = (V, \mathcal{E})$, where $V = \{1, \dots, N\}$

is the nodes set and $\mathcal{E} \subseteq V \times V$ is the edges set. Let $x_i(t)$ denote the state information of the i-th system at time t. We define $x(t) = [x_1(t), \dots, x_N(t)]^*$ to be the global information state. We have that $x(t)$ is updated according to the following recursive equation

$$x(t+1) = Px(t),$$

where $P \in \mathbb{R}^{N \times N}$ is a doubly stochastic matrix, such that $\mathcal{G}_P \subseteq \mathcal{G}$. Let $\rho(P)$ be the *essential spectral radius* of P[1]. Moreover, given a graph $\mathcal{G}$ let

$$\rho_{\mathcal{G}} = \min \{\rho(P) \,|\, \text{P is doubly stochastic}, \mathcal{G}_P \subseteq \mathcal{G}\}.$$

Given a graph $\mathcal{G}$, an interesting issue should be that of understand how the connectivity of $\mathcal{G}$ influences the best performance achievable by the consensus algorithm on $\mathcal{G}$, which is represented by $\rho_{\mathcal{G}}$. Intuitively we expect $\rho_{\mathcal{G}}$ to be sensitive to the communication effort. Precisely we conjecture that if $\mathcal{G}_1 \subset \mathcal{G}_2$ then $\rho_{\mathcal{G}_1} > \rho_{\mathcal{G}_2}$. However we have been not able to prove this so far. This analysis becomes more treatable if we limits our considerations to graphs $\mathcal{G}$ and matrices P exhibiting symmetries. We will show how these symmetries limit the achievable performance in terms of convergence rate.
We start our analysis in the next subsection by limiting ourselves to the cyclic symmetry. Then we will extend our results to a broader class of symmetries: the Cayley symmetries.

3.2.1 Circulant symmetries

In order to introduce cyclic symmetry, we define the following map

$$p : \{1, \dots, N\} \to \{1, \dots, N\} : i \mapsto i + 1 \bmod N$$

The matrix P is said to be symmetric with respect to p if

$$P_{i,j} = P_{p(i),p(j)} \quad \forall\, i, j \in \{1, \dots, N\}.$$

This condition is equivalent to impose that

$$P = \sum_{i=0}^{N} p_i \Pi^i$$

[1]Since there is no risk of confusion, for the sake of the notational convenience, in this chapter we denote the *essential spectral radius* of P by $\rho(P)$ instead of $\rho_{ess}(P)$ as in the previous chapter.

where

$$\Pi := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

These matrices are called circulant [46]. Notice that, in this case, since $\Pi\mathbb{1} = \mathbb{1}$ and $\mathbb{1}^*\Pi = \mathbb{1}^*$, then circulant matrices satisfy condition $P\mathbb{1} = \mathbb{1}$ if and only if $\sum_i p_i = 1$. Moreover condition $\mathbb{1}^*P = \mathbb{1}^*$ is also satisfied, and thus such matrices drive the state to the average of the initial conditions. Consequently, if we choose P to be nonnegative, then P is always doubly stochastic. The spectral properties of circulant matrices are particularly simple. Indeed, it can be shown that

$$\sigma(P) = \left\{ P\left(e^{j\frac{2\pi}{N}h}\right) : h = 0, 1, \ldots, N-1 \right\}$$

where $P(z) := \sum_{i=0}^{N} p_i z^i$. Notice that $P(e^{j0}) = 1$. Hence,

$$\rho(P) = \max\left\{ \left| P\left(e^{j\frac{2\pi}{N}h}\right) \right| : h = 1, \ldots, N-1 \right\} .$$

Moreover, the corresponding eigenvectors v_h's form an orthonormal basis and $v_0 = (1/N)\mathbb{1}$. Notice that, in order to have consensus stability in this context, it is sufficient to impose that

$$\left| P\left(e^{j\theta}\right) \right| < 1 \quad \forall\ \theta \neq 0\,. \tag{3.1}$$

This condition is slightly stronger than consensus stability, however it provides a stability condition independent of the number of systems N.

Circulant solutions to the consensus problem exist if the graph $\mathcal{G}$ admits an analogous symmetry. Indeed, consider a strongly connected graph $\mathcal{G}$ on $\{1, \ldots, N\}$ containing all the self loops (i, i) and that is symmetric with respect to p, in the sense that if there is an arc from i to j, there is also an arc from $p(i)$ to $p(j)$. Then, it is immediate to find a circulant matrix P such that $\mathcal{G}_P = \mathcal{G}$ and which solves the consensus problem. Indeed if $j_1, \ldots, j_\mu$ are the incoming arcs of vertex 1 in $\mathcal{G}$, it is sufficient to choose weights $p_0, \ldots, p_\mu$ such that $p_i > 0$ for $i = 0, \ldots, \mu$, and $\sum_i p_i = 1$. If we consider $P = \sum p_i \Pi^i$, then it is clear that condition (3.1) is satisfied.

The particular spectral structure of circulant matrices allows to obtain asymptotic results on the behavior of the essential spectral radius $\rho(P)$ and

therefore on the rate of convergence of the corresponding consensus scheme. Define

$$\rho_{\mathcal{G}}^{\text{circ}} = \inf\{\rho(P) \mid P \text{ circulant and stochastic}\,,\ \mathcal{G}_P \subseteq \mathcal{G}\}\,.$$

Let us start from some examples.

Example 3.1 Suppose $\mathcal{G}$ is described by the arcs $i \leftarrow i+1 \pmod N$ and all the self-loops (i,i). We can choose therefore $P(z) = p_0 + p_1 z$, where $p_0, p_1 \in \mathbb{R}$. In this case we have that

$$x(t+1) = \{p_0 I + p_1 \Pi\} x(t)\,.$$

The condition $P(1) = p_0 + p_1 = 1$ implies that $P(z) = p + (1-p)z$ for some $p \in \mathbb{R}$. In this case it can be shown that we have consensus stability if and only if $0 < p < 1$ and that the rate of convergence is

$$\rho(P) = \left((1-p)^2 + p^2 - 2p(1-p)\cos\left(\frac{2\pi}{N}\right)\right)^{\frac{1}{2}}\,.$$

The p that minimizes $\rho(P)$ is $p = 1/2$ and yields

$$\rho_{\mathcal{G}}^{\text{circ}} = \left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{N}\right)\right)^{\frac{1}{2}} \simeq 1 - \frac{\pi^2}{2}\frac{1}{N^2}$$

where the last approximation is meant for $N \to \infty$.

Example 3.2 Suppose $\mathcal{G}$ is described by the arcs $i \leftarrow i-1$ and $i \leftarrow i+1 \pmod N$ and all the self-loops (i,i). For the sake of simplicity we assume that N is even; very similar results can be obtained for odd N. We can choose in this case $P(z) = p_0 + p_1 z + p_{-1} z^{-1}$, where $p_0, p_1, p_{-1} \in \mathbb{R}$. In this case we have that

$$x(t+1) = \{I + p_0 I + p_1 \Pi + p_{-1}\Pi^{-1}\} x(t)\,.$$

The condition $P(1) = 1$ becomes in this case $p_0 + p_1 + p_{-1} = 1$. Symmetry and convexity arguments [23, 19] allow to say that a minimum of $\rho(P)$ is for sure of the type $p_1 = p_{-1}$. With this assumption the cost functional reduces to

$$\rho(P) = \max\left\{\left|1 - 2p_1\left(1 - \cos\left(\frac{2\pi}{N}\right)\right)\right|, |p_0 - 2p_1|\right\}\,.$$

The minimum is achieved for

$$p_0 = 1 - \frac{2}{3 - \cos\left(\frac{2\pi}{N}\right)}\,, \quad p_1 = p_{-1} = \frac{1}{3 - \cos\left(\frac{2\pi}{N}\right)}$$

and we have

$$\rho_{\mathcal{G}}^{\text{circ}} = \frac{1+\cos\left(\frac{2\pi}{N}\right)}{3-\cos\left(\frac{2\pi}{N}\right)} \simeq 1 - 2\pi^2 \frac{1}{N^2}$$

where the last approximation is meant for $N \to \infty$.

Notice that in the first example the optimality is obtained when all the nonzero elements of π are equal. This is not a general feature since the same does not happen in the second example. Notice moreover that in this example, as N tends to infinity, the optimal solution tends to $p_0 = 0, p_1 = p_{-1} = 1/2$.

The case of communication exchange with two neighbors (Example 3.2) offers a better performance compare to the case with one neighbor (Example 3.1). However, in both cases $\rho_{\mathcal{G}}^{\text{circ}} \to 1$ for $N \to +\infty$. This fact is more general: if we keep bounded the number of incoming edges in a vertex, the essential spectral radius will always converge to 1. This is very easy to see in the case when there is only one incoming edge. Indeed, in this case, by repeating the arguments of Example 3.1 we have the following result.

Proposition 3.3 *Consider a strongly connected graph $\mathcal{G}$ on $\{1,\dots,N\}$, containing all the self-loops (i,i). Assume that $\mathcal{G}$ is symmetric with respect to p and assume there is only one incoming edge in any vertex. Then,*

$$\rho_{\mathcal{G}}^{\text{circ}} \geq 1 - \frac{\pi^2}{2}\frac{1}{N^2}\,.$$

In the general situation a much more careful analysis permits to obtain the following bound.

Theorem 3.4 *Consider a strongly connected graph $\mathcal{G}$ on $\{1,\dots,N\}$, containing all the self-loops (i,i). Assume that $\mathcal{G}$ is symmetric with respect to p and let ν be the number of incoming edges in any vertex. Then,*

$$\rho_{\mathcal{G}}^{\text{circ}} \geq 1 - C\frac{1}{N^{2/\nu}}\,,$$

where C is a constant independent of the chosen graph.

We omit the proofs of the above two results, since they are particular cases of Theorem 3.7, that we will state and prove in the next Section. Moreover, in order to avoid confusion, we remark that, given a node i, we do not encounter the self-loop (i,i) in the set of the incoming arcs to this node i.

Notice that in Example 3.2 we have that $\nu = 2$ and we have an asymptotic behavior $\rho_{\mathcal{G}}^{\text{circ}} \simeq 1 - 2\pi^2 N^{-2}$, while the lower bound of Theorem 3.4 is, in this case, $1 - 2\pi^2 N^{-1}$. Now we can wonder whether it is possible to achieve the bound performance. In other words, we would like to understand whether the lower bound we have just found is tight or not. In the following example we will show that this is the case.

Example 3.5 Suppose that $N = M^\nu$ and that

$$P = \frac{1}{\nu+1} \sum_{i=0}^{\nu-1} \Pi^{M^i}$$

The matrix P has eigenvalues

$$\lambda_h = p_\nu \left(e^{j\frac{2\pi}{N}h} \right) \qquad h = 1, \dots, N-1$$

where

$$p_\nu(z) := \frac{1}{\nu+1} \left(1 + \sum_{i=0}^{\nu-1} z^{M^i} \right)$$

We will show that, for all $h = 1, \dots, N-1$ we have that

$$\left| p_\nu(e^{j\frac{2\pi}{M^\nu}h}) \right| \leq 1 - \frac{1}{\nu+1}\frac{1}{M^2}$$

This fact will be shown by induction on ν. The fact that the assertion holds for $\nu = 1$ follows from Example 3.1. Assume now that the assertion holds for $\nu - 1$. Let h_0, h_1 such that $0 \leq h_0 \leq M-1$, $0 \leq h_1 \leq M^{\nu-1} - 1$ and $h = h_0 + Mh_1$. If $h_0 \neq 0$ then

$$\begin{aligned}
\left| p_\nu \left(e^{j\frac{2\pi}{N}h} \right) \right| &\leq \frac{1}{\nu+1} \left| 1 + e^{j\frac{2\pi}{M^\nu}M^{\nu-1}h} \right| + \frac{1}{\nu+1} \left| \sum_{i=0}^{\nu-2} e^{j\frac{2\pi}{M^\nu}M^i h} \right| \\
&\leq \frac{1}{\nu+1} \left| 1 + e^{j\frac{2\pi}{M}h_0} \right| + \frac{\nu-1}{\nu+1} \\
&= \frac{2}{\nu+1} \left| p_1(e^{j\frac{2\pi}{M}h}) \right| + \frac{\nu-1}{\nu+1} \\
&\leq \frac{2}{\nu+1} \left(1 - \frac{1}{2}\frac{1}{M^2} \right) + \frac{\nu-1}{\nu+1} \\
&\leq 1 - \frac{1}{\nu+1}\frac{1}{M^2}
\end{aligned}$$

If $h_0 = 0$, then $h = Mh_1$ and so

$$\begin{aligned}\left|p_\nu(e^{j\frac{2\pi}{N}h})\right| &= \frac{1}{\nu+1}\left|1+\sum_{i=0}^{\nu-1} e^{j\frac{2\pi}{M^{\nu-1}}M^i h_1}\right| \\ &= \frac{1}{\nu+1}\left|2+\sum_{i=0}^{\nu-2} e^{j\frac{2\pi}{M^{\nu-1}}M^i h_1}\right| \\ &= \frac{\nu}{\nu+1}\left|p_{\nu-1}(e^{j\frac{2\pi}{M}h_1})\right| + \frac{1}{\nu+1} \\ &\leq \frac{\nu}{\nu+1}\left(1-\frac{1}{\nu}\frac{1}{M^2}\right)+\frac{1}{\nu+1} \\ &\leq 1-\frac{1}{\nu+1}\frac{1}{M^2}\end{aligned}$$

This bound proves that there exists a circulant graph $\mathcal{G}$ with ν incoming edges in any vertex such that

$$\rho_{\mathcal{G}}^{\text{circ}} \leq 1-\frac{1}{\nu+1}\frac{1}{N^{2/\nu}}\,.$$

proving in this way that the bound proposed by the previous theorem is tight.

3.3 Cayley symmetries

The analysis carried out in the previous section can be extended to graphs $\mathcal{G}$ and matrices P exhibiting more general symmetries.

In order to treat symmetries on a graph $\mathcal{G}$ in a general setting, we need to introduce the concept of Cayley graphs defined on Abelian groups [10, 4]. Let G be any finite Abelian group of order $|G| = N$, and let S be a subset of G containing zero. The Cayley graph $\mathcal{G}(G, S)$ is the directed graph with vertex set G and arc set

$$\mathcal{E} = \{(g,h) : h-g \in S\}\,.$$

Note that the fact that S contains zero implies that all the self loops (g, g) belong to $\mathcal{E}$. From now on, along this section, when dealing with a graph $\mathcal{G}$ we will assume that it contains all the self-loops.
Notice that a Cayley graph is always in-regular, namely the in-degree of each vertex is equal to $|S|$. Notice also that strongly connectivity can be checked

algebraically. Indeed, it can be seen that a Cayley graph $\mathcal{G}(G, S)$ is strongly connected if and only if the set S generates the group G, which means that any element in G can be expressed as a finite sum of (not necessarily distinct) elements in S. If S is such that $-S = S$ we say that S is inverse-closed. In this case the graph obtained is undirected.

Symmetries can be introduced also on matrices. Let G be any finite Abelian group of order $|G| = N$. A matrix $P \in \mathbb{R}^{G\times G}$ is said to be a Cayley matrix over the group G if

$$P_{i,j} = P_{i+h,j+h} \qquad \forall\ i, j, h \in G\,.$$

It is clear that for a Cayley matrix P there exists a $\pi : G \to \mathbb{R}$ such that $P_{i,j} = \pi(i - j)$. The function π is called the generator of the Cayley matrix P. Notice that, if π and π' are generators of the Cayley matrices P and P' respectively, then $\pi + \pi'$ is the generator of $P + P'$ and $\pi * \pi'$ is the generator of PP', where $(\pi * \pi')(i) := \sum_{j\in G} \pi(j)\pi'(i-j)$ for all $i \in G$. This shows that P and P' commute. Notice finally that, if P is a Cayley matrix generated by π, then $\mathcal{G}_P$ is a Cayley graph with $S = \{h \in G : \pi(h) \neq 0\}$.

It is clear that the graph $\mathcal{G}_P$ supporting a Cayley matrix P is a Cayley graph with

$$S := \{i : \pi(i) \neq 0\} \cup \{0\}.$$

Moreover it is easy to see that for any Cayley matrix P we have that $P\mathbb{1} = \mathbb{1}$ if and only if $\mathbb{1}^T P = \mathbb{1}^T$. This implies that a Cayley stochastic matrix is automatically doubly stochastic. In this case the function π associated with the matrix P is a probability distribution on the group G. Among the multiple possible choices of the probability distribution π, there is one which is particularly simple, namely $\pi(g) = 1/|S|$ for every $g \in S$.

Example 3.6 Let us consider the group $\mathbb{Z}_N$ of integers modulo N and the Cayley graph $\mathcal{G}(\mathbb{Z}_N, S)$ where $S = \{-1, 0, 1\}$. Notice that in this case S is inverse-closed. Consider the uniform probability distribution

$$\pi(0) = \pi(1) = \pi(-1) = 1/3$$

The corresponding Cayley stochastic matrix is given by

$$P = \begin{pmatrix} 1/3 & 1/3 & 0 & 0 & \cdots & 0 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 1/3 & 0 & 0 & 0 & \cdots & 0 & 1/3 & 1/3 \end{pmatrix}.$$

Notice that in this case we have two symmetries. The first is that the graph is undirected and the second that the graph is circulant. These symmetries can be seen in the structure of the transition matrix P that, indeed, turns out to be both symmetric and circulant.

Given a Cayley graph $\mathcal{G}$ we can define

$$\rho_{\mathcal{G}}^{\text{Cayley}} = \min\{\rho(P) \mid P \text{ Cayley stochastic}, \mathcal{G}_P \subseteq \mathcal{G}\}\,.$$

It will turn out that $\rho_{\mathcal{G}}^{\text{Cayley}}$ can be evaluated or estimated in many cases. Moreover, it clearly holds that $\rho_{\mathcal{G}}^{\text{Cayley}} \geq \rho_{\mathcal{G}}$. Before continuing we give a short analysis of the spectral properties of the Cayley stochastic matrices on finite Abelian groups, which is the basis of our main results.

3.3.1 Cayley stochastic matrices on finite Abelian groups

In this subsection we will refer to notions on group characters and on harmonic analysis on groups which are illustrated in Appendix A.

Fix now a Cayley matrix P on the Abelian group G generated by the function $\pi : G \to \mathbb{R}$. The spectral structure of P is very simple. To see this, first notice that P can be interpreted as a linear function from $\mathbb{C}^G$ to itself simply by considering, for $f \in \mathbb{C}^G$, $(Pf)(g) := \sum_h P_{gh} f(h)$. Notice that the trivial character χ_0 corresponds to the vector $\mathbb{1}$ having all components equal to 1. For every $\chi \in \hat{G}$, it holds

$$(P\chi)(g) = \sum_{h\in G} P_{gh}\chi(h) = \sum_{h\in G} \pi(g-h)\chi(h) = \sum_{h\in G} \pi(h)\chi(g-h) = \hat{\pi}(\chi)\chi(g)\,.$$

Hence, χ is an eigenfunction of P with eigenvalue $\hat{\pi}(\chi)$. Since the characters form an orthonormal basis it follows that P is diagonalizable and its spectrum is given by

$$\sigma(P) = \{\hat{\pi}(\chi) \mid \chi \in \hat{G}\}\,.$$

We can interpret a character as a linear function from $\mathbb{C}$ to $\mathbb{C}^G$ as follows

$$\chi : \ \mathbb{C} \ \to \mathbb{C}^G \ : z \mapsto z\chi\,.$$

Its adjoint is the linear functional

$$\chi^* : \ \mathbb{C}^G \ \to \mathbb{C} \ : f \mapsto < f, \chi > \ .$$

With this notation, $N^{-1}\chi\chi^*$ is a linear function from $\mathbb{C}^G$ to itself, projecting $\mathbb{C}^G$ on the eigenspace generated by χ. In this way, P can be represented as

$$P = \sum_{\chi \in \hat{G}} \hat{\pi}(\chi) N^{-1} \chi\chi^* .$$

Conversely, it can easily be shown that, given any $\hat{\theta} : \hat{G} \to \mathbb{C}$, the matrix

$$P = \sum_{\chi \in \hat{G}} \hat{\theta}(\chi) N^{-1} \chi\chi^* ,$$

is a Cayley matrix generated by the Fourier transform $\hat{\theta}$.

Suppose now that P is the matrix of the system (2.11). The displacement from the $\Delta(t)$, defined in (2.15), can be represented as

$$\Delta(t) = (I - N^{-1}\chi_0\chi_0^*)x(t) .$$

As we had already remarked, Δ is governed by the dynamics $\Delta^+ = P\Delta$ and the initial condition $\Delta(0)$ is characterized by $< \Delta(0), \chi_0 >= 0$. Notice that

$$\Delta(t) = P^t\Delta(0) = \frac{1}{N}\sum_{\chi \in \hat{G}} \hat{\pi}(\chi)^t \chi < \Delta(0), \chi >= \frac{1}{N}\sum_{\chi \neq \chi_0} \hat{\pi}(\chi)^t \chi < \Delta(0), \chi > .$$

Hence,

$$||\Delta(t)||^2 = \frac{1}{N}\sum_{\chi \neq \chi_0} |\hat{\pi}(\chi)^{2t}| < \Delta(0), \chi > |^2 .$$

This shows in a very simple way, in this case, the role of $\rho(P) = \max_{\chi \neq \chi_0} |\hat{\pi}(\chi)|$ in the rate of convergence.

3.3.2 The essential spectral radius of Cayley matrices.

The particular spectral structure of Cayley matrices allows to obtain asymptotic results on the behavior of the essential spectral radius $\rho(P)$ and therefore on the rate of convergence of the corresponding consensus algorithm.
In Section 3.2.1 we have proved the following fact for circulant graphs: if we keep bounded the number of incoming arcs in a vertex, the essential spectral radius for stochastic circulant matrices always converges to 1, for $N \to \infty$. The next result provides a bound which proves that this negative behavior is a general feature also of the broader class of the Abelian stochastic Cayley matrices. This slow convergence rate has already been noticed, for some specific case, in the literature [19, 127, 97, 24].

Theorem 3.7 *Let G be any finite Abelian group of order N and let $S \subseteq G$ be a subset containing zero. Moreover let $\mathcal{G}$ be the Cayley graph associated with G and S. If $|S| = \nu + 1$, then*

$$\rho_{\mathcal{G}}^{\text{Cayley}} \geq 1 - CN^{-2/\nu}\,, \tag{3.2}$$

where $C > 0$ is a constant independent of G and S.

In order to prove Theorem 3.7 we need the following technical lemma.

Lemma 3.8 *Let $\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong [-1/2, 1/2[$. Let $0 \leq \delta < 1/2$ and consider the hypercube $V = [-\delta, \delta]^k \subseteq \mathbb{T}^k$. For every finite set $\Lambda \subseteq \mathbb{T}^k$ such that $|\Lambda| \geq \delta^{-k}$, there exist $\bar{x}_1$, $\bar{x}_2 \in \Lambda$ with $\bar{x}_1 \neq \bar{x}_2$ such that $\bar{x}_1 - \bar{x}_2 \in V$.*

Proof: For any $x \in \mathbb{T}$ and $\delta > 0$, define the following set

$$L(x, \delta) = [x, x + \delta] + \mathbb{Z} \subseteq \mathbb{T}\,.$$

Observe that for all $y \in \mathbb{T}$, $L(x, \delta) + y = L(x + y, \delta)$. Now let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_k) \in \mathbb{T}^k$ and define

$$L(\bar{x}, \delta) = \prod_{i=1}^{k} L(\bar{x}_i, \delta)\,.$$

Also in this case we observe that $L(\bar{x}, \delta) + \bar{y} = L(\bar{x} + \bar{y}, \delta)$ for every $\bar{y} \in \mathbb{T}^k$. Consider now the family of subsets

$$\{L(\bar{x}, \delta),\ \ \bar{x} \in \Lambda\}\,.$$

We claim that there exist $\bar{x}_1$ and $\bar{x}_2$ in Λ such that $\bar{x}_1 \neq \bar{x}_2$ and such that $L(\bar{x}_1, \delta) \cap L(\bar{x}_2, \delta) \neq \emptyset$. Indeed, if not, we would have that

$$1 \geq m\left(\bigcup_{\bar{x} \in \Lambda} L(\bar{x}, \delta)\right) = \sum_{\bar{x} \in \Lambda} m\left(L(\bar{x}, \delta)\right) = |\Lambda|\delta^k \geq 1$$

where $m(\cdot)$ is the Lebesgue measure on $\mathbb{T}^k$ and where we used the hypothesis $|\Lambda| \geq \delta^{-k}$. However, since all $L(\bar{x}_1, \delta)$ are closed, it is not possible that $m\left(\bigcup_{\bar{x} \in \Lambda} L(\bar{x}_1, \delta)\right) = 1$. Notice finally that

$$L(\bar{x}_1, \delta) \cap L(\bar{x}_2, \delta) \neq \emptyset \ \Leftrightarrow\ L(0, \delta) \cap L(\bar{x}_2 - \bar{x}_1, \delta) \neq \emptyset \ \Leftrightarrow\ \bar{x}_2 - \bar{x}_1 \in V\,.$$

■

We are able now to provide the proof of Theorem 3.7.

Proof: From Theorem A.9 stated in Appendix A, we can assume, with no loss of generality, that

$$G = \mathbb{Z}_{N_1} \oplus \ldots \oplus \mathbb{Z}_{N_r} .$$

Assume we have fixed a probability distribution π supported on S. Let P be the corresponding stochastic Cayley matrix. It follows from previous considerations that the spectrum of P is given by

$$\sigma(P) = \left\{ \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \cdots \sum_{k_r=0}^{N_r-1} \pi(k_1, \ldots, k_r) e^{i\frac{2\pi}{N_1}k_1\ell_1} e^{i\frac{2\pi}{N_2}k_2\ell_2} \cdots e^{i\frac{2\pi}{N_r}k_r\ell_r} : \right.$$
$$\left. : \ell_1 \in \mathbb{Z}_{N_1}, \ldots, \ell_r \in \mathbb{Z}_{N_r} \right\}$$

Denote by $\bar{k}^j = (k_1^j, \ldots, k_r^j)$, for $j = 1, \ldots, \nu$, the non-zero elements in S, and consider the subset

$$\Lambda = \left\{ \left(\sum_{i=1}^{r} \frac{k_i^1 \ell_i}{N_i}, \ldots, \sum_{i=1}^{r} \frac{k_i^\nu \ell_i}{N_i} \right) + \mathbb{Z}^\nu \ : \ \ell_1 \in \mathbb{Z}_{N_1}, \ldots, \ell_r \in \mathbb{Z}_{N_r} \right\} \subseteq \mathbb{T}^\nu .$$

Let $\delta = (\prod_i N_i)^{-1/\nu}$ and let V be the corresponding hypercube in $\mathbb{T}^\nu$ defined as in Lemma 3.8. We want to show that there exists $\bar{\ell} = (\ell_1, \ldots \ell_r) \in \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_r}$, $\bar{\ell} \neq 0$ such that

$$\left(\sum_{i=1}^{r} \frac{k_i^1 \ell_i}{N_i}, \ldots, \sum_{i=1}^{r} \frac{k_i^\nu \ell_i}{N_i} \right) + \mathbb{Z}^\nu \in V .$$

We consider two cases.

(i) If there exists $\bar{\ell} = (\ell_1, \ldots \ell_r) \in \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_r}$, $\bar{\ell} \neq 0$ such that

$$\left(\sum_{i=1}^{r} \frac{k_i^1 \ell_i}{N_i}, \ldots, \sum_{i=1}^{r} \frac{k_i^\nu \ell_i}{N_i} \right) + \mathbb{Z}^\nu = 0 \in V \tag{3.3}$$

then clearly we are done.

(ii) Assume now there are no $\bar{\ell} = (\ell_1, \ldots \ell_r) \in \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_r}$, $\bar{\ell} \neq 0$ satisfying condition (3.3). In this case it can be shown that two different $\bar{\ell}', \bar{\ell}'' \in \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_r}$ yield

$$\left(\sum_{i=1}^{r} \frac{k_i^1 \ell_i'}{N_i}, \ldots, \sum_{i=1}^{r} \frac{k_i^\nu \ell_i'}{N_i} \right) + \mathbb{Z}^\nu \neq \left(\sum_{i=1}^{r} \frac{k_i^1 \ell_i''}{N_i}, \ldots, \sum_{i=1}^{r} \frac{k_i^\nu \ell_i''}{N_i} \right) + \mathbb{Z}^\nu ,$$

namely different elements in $\mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_r}$ always lead do distinct elements in Λ. This implies that $|\Lambda| = \prod_i N_i = \delta^{-\nu}$ and so we are in a position to apply Lemma 3.8 and conclude that there exist two different $\bar{\ell}', \bar{\ell}'' \in \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_r}$ such that

$$\left[\left(\sum_{i=1}^{r} \frac{k_i^1 \ell_i'}{N_i}, \ldots, \sum_{i=1}^{r} \frac{k_i^\nu \ell_i'}{N_i}\right) + \mathbb{Z}^\nu\right] - \left[\left(\sum_{i=1}^{r} \frac{k_i^1 \ell_i''}{N_i}, \ldots, \sum_{i=1}^{r} \frac{k_i^\nu \ell_i''}{N_i}\right) + \mathbb{Z}^\nu\right] \in V$$

and hence

$$\left(\sum_{i=1}^{r} \frac{k_i^1 \ell_i}{N_i}, \ldots, \sum_{i=1}^{r} \frac{k_i^\nu \ell_i}{N_i}\right) + \mathbb{Z}^\nu \in V\,,$$

where $\bar{\ell} = \bar{\ell}' - \bar{\ell}'' \neq 0$.

Consider now the eigenvalue

$$\begin{aligned}\lambda &= \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \cdots \sum_{k_r=0}^{N_r-1} \pi(k_1, \ldots, k_r) e^{i(\frac{2\pi}{N_1} k_1 \ell_1 + \frac{2\pi}{N_2} k_2 \ell_2 + \cdots + \frac{2\pi}{N_r} k_r \ell_r)} \\ &= \pi(0, \ldots 0) + \sum_{j=1}^{\nu} \pi(k_1^j, \ldots, k_r^j) e^{i(\frac{2\pi}{N_1} k_1^j \ell_1 + \frac{2\pi}{N_2} k_2^j \ell_2 + \cdots + \frac{2\pi}{N_r} k_r^j \ell_r)}\,.\end{aligned}$$

Its norm can be estimated as follows

$$\begin{aligned}|\lambda| &\geq \pi(0, \ldots 0) + \sum_{j=1}^{\nu} \pi(k_1^j, \ldots, k_r^j) \cos\left(\frac{2\pi}{N_1} k_1^j \ell_1 + \frac{2\pi}{N_2} k_2^j \ell_2 + \cdots + \frac{2\pi}{N_r} k_r^j \ell_r\right) \\ &\geq \pi(0, \ldots 0) + \sum_{j=1}^{\nu} \pi(k_1^j, \ldots, k_r^j) \left(1 - 2\pi^2 \left(\frac{k_1^j}{N_1} \ell_1 + \frac{k_2^j}{N_2} \ell_2 + \cdots + \frac{k_r^j}{N_r} \ell_r\right)^2\right) \\ &\geq \pi(0, \ldots 0) + \sum_{j=1}^{\nu} \pi(k_1^j, \ldots, k_r^j) \; - \sum_{j=1}^{\nu} \pi(k_1^j, \ldots, k_r^j) 2\pi^2 \frac{1}{N^{2/\nu}} \\ &\geq 1 - 2\pi^2 \frac{1}{N^{2/\nu}}\end{aligned}$$

and so we can conclude.

■

Theorem 3.7 in particular implies that, if we consider a sequence of Abelian Cayley graphs $\mathcal{G}(G_N, S_N)$ such that $|G_N| = N$ and $|S_N|$ grows less then logarithmically in N and we consider a sequence of Cayley stochastic

matrices P_N compatible with $\mathcal{G}(G_N, S_N)$, then, necessarily, $\rho(P_N)$ converges to 1. This had already been shown, for adjacency matrices, in [4].

Notice that Example 3.5 shows that also the bound we have just found is tight. However we provide now another interesting example confirming this.

Consider the group $(\mathbb{Z}_N^r,+)$, with $r \in \mathbb{N}$ and $r \geq 1$, and the Cayley graph $\mathcal{G}(\mathbb{Z}_N^r, S)$, where $S = \{0, e_1, \ldots, e_r\}$, where e_j is the vector with all elements equal to 0 except a 1 in position j. Consider the probability distribution π on S described by

$$\pi(0) = k_0\,, \quad \pi(e_j) = k_j\,, \ \forall j = 1, \ldots, r$$

with $k_j \geq 0$ and $\sum_{j=0}^{r} k_j = 1$. The Fourier transform of π is

$$\hat{\pi}(\chi_{\ell_1}, \ldots, \chi_{\ell_r}) = \sum_{g \in S} \chi(-g)\pi(g) = k_0 + \sum_{j=1}^{r} k_j e^{-i\frac{2\pi}{N}\ell_j}$$

for $\ell_j = 0, 1, \ldots, N-1\,,\ j = 1, \ldots, r\,$. We thus have

$$\rho_{\mathcal{G}}^{\text{Cayley}} = \min_{\substack{k_j \geq 0 \\ \sum k_j = 1}} \max_{\substack{0 \leq \ell_j \leq N-1 \\ (\ell_1, \ldots, \ell_r) \neq (0, \ldots, 0)}} \left| k_0 + \sum_{j=1}^{r} k_j e^{-i\frac{2\pi}{N}\ell_j} \right| .$$

We have the following result.

Proposition 3.9 *The above min-max is reached by* $(\ell_1, \ldots, \ell_r) = (1, \ldots, 1)$ *or by* $(\ell_1, \ldots, \ell_r) = (0, \ldots, 1, \ldots, 0)$ *and* $k_0 = k_1 = \ldots = k_r = 1/(r+1)$, *which yield*

$$\rho_{\mathcal{G}}^{\text{Cayley}} = \left(1 - \frac{2r}{(r+1)^2}\left(1 - \cos\frac{2\pi}{N}\right)\right)^{\frac{1}{2}} \simeq 1 - \frac{4\pi^2 r}{(r+1)^2}\frac{1}{N^2}$$

In order to prove the above proposition we need the following technical Lemma.

Lemma 3.10 *Suppose that* $k_0, \ldots, k_r$ *are fixed. Then*

$$\max_{\substack{0 \leq \ell_j \leq N-1 \\ (\ell_1, \ldots, \ell_r) \neq (0, \ldots, 0)}} \left| k_0 + \sum_{j=1}^{r} k_j e^{-i\frac{2\pi}{N}\ell_j} \right| = \max_{\substack{\ell_j \in \{0,1\} \\ (\ell_1, \ldots, \ell_r) \neq (0, \ldots, 0)}} \left| k_0 + \sum_{j=1}^{r} k_j e^{-i\frac{2\pi}{N}\ell_j} \right|$$

Proof: Consider a r-tuple $(\ell_1, \ldots, \ell_r) \neq (0, \ldots, 0)$ and let $\ell \neq 0$ be a value taken by some of the ℓ_j. Let $J \subseteq \{1, \ldots, r\}$ be the nonempty index set such that $\ell_j = \ell$ if and only if $j \in J$. Consider the new r-tuple $(\ell'_1, \ldots, \ell'_r)$ defined as follows

$$\ell'_j = \begin{cases} 1 \text{ if } j \in J \\ 0 \text{ otherwise} \end{cases}$$

We want to show that

$$\left| k_0 + \sum_{j=1}^{r} k_j e^{-i\frac{2\pi}{N}\ell_j} \right| \leq \left| k_0 + \sum_{j=1}^{r} k_j e^{-i\frac{2\pi}{N}\ell'_j} \right| \tag{3.4}$$

Indeed, we have that

$$\begin{aligned}
&\left| k_0 + \sum_{j=1}^{r} k_j e^{-i\frac{2\pi\ell_j}{N}} \right|^2 = \\
&\quad = \left| k_0 + \sum_{j=1}^{r} k_j \cos\frac{2\pi\ell_j}{N} - i\left(\sum_{j=1}^{r} k_j \sin\frac{2\pi\ell_j}{N}\right) \right|^2 = \\
&\quad = \left(k_0 + \sum_{j=1}^{r} k_j \cos\frac{2\pi\ell_j}{N} \right)^2 + \left(\sum_{j=1}^{r} k_j \sin\frac{2\pi\ell_j}{N} \right)^2 \\
&\quad = \sum_{j=0}^{r} k_j^2 + 2k_0 \sum_{j=1}^{r} k_j \cos\left(\frac{2\pi\ell_i}{N}\right) + \\
&\qquad + 2\sum_{i=1}^{r-1}\sum_{j=i+1}^{r} k_i k_j \cos\left(\frac{2\pi\ell_i}{N}\right)\cos\left(\frac{2\pi\ell_j}{N}\right) + \\
&\qquad + 2\sum_{i=1}^{r-1}\sum_{j=i+1}^{r} k_i k_j \sin\left(\frac{2\pi\ell_i}{N}\right)\sin\left(\frac{2\pi\ell_j}{N}\right) \\
&\quad = \sum_{j=0}^{r} k_j^2 + 2k_0 \sum_{j=1}^{r} k_j \cos\left(\frac{2\pi\ell_i}{N}\right) + 2\sum_{i=1}^{r-1}\sum_{j=i+1}^{r} k_i k_j \cos\left(\frac{2\pi(\ell_j - \ell_i)}{N}\right)
\end{aligned}$$

Observe now that

$$\cos\left(\frac{2\pi\ell_i}{N}\right) \leq \cos\left(\frac{2\pi\ell'_i}{N}\right)$$

for all j, and that

$$\cos\left(\frac{2\pi(\ell_j - \ell_i)}{N}\right) \leq \cos\left(\frac{2\pi(\ell'_j - \ell'_i)}{N}\right)$$

for all i, j. This yields (3.4) and proves the lemma. ■

We can now prove Proposition 3.9.

Proof:

We start by observing that, if $\ell_j \in \{0, 1\}$, then

$$\left| k_0 + \sum_{j=1}^{r} k_j e^{-i\frac{2\pi}{N}\ell_j} \right|^2 = 1 - 2\left(\sum_{j\in J} k_j\right)\left(1 - \sum_{j\in J} k_j\right)\left(1 - \cos\frac{2\pi}{N}\right)$$

where $J \subseteq \{1, \ldots, r\}$ is such that $j \in J$ if and only if $\ell_j = 1$. Maximizing this quantity over the non identically zero vectors $(\ell_1, \ldots, \ell_r) \in \{0, 1\}^r$ is equivalent to maximize over all the possible nonempty sets J. From this it is not difficult to see that

$$\max_{\substack{0\le\ell_j\le N-1 \\ (\ell_1,\ldots,\ell_r)\neq(0,\ldots,0)}} \left| k_0 + \sum_{j=1}^{r} k_j e^{-i\frac{2\pi}{N}\ell_j} \right|^2 = 1 - 2m(1-m)\left(1 - \cos\frac{2\pi}{N}\right)$$

where $m := \min\{k_0, k_1, \ldots, k_r\}$. Since $m \le 1/2$, then in order to minimize $1 - 2m(1-m)\left(1 - \cos\frac{2\pi}{N}\right)$ over the possible $k_0, k_1, \ldots, k_r$, we need to maximize m. This is obtained by choosing $k_0 = k_1 = \ldots = k_r = 1/(r+1)$, which yield $m = 1/(r+1)$ and

$$\rho_{\mathcal{G}}^{\text{Cayley}} = \left(1 - \frac{2r}{(r+1)^2}\left(1 - \cos\frac{2\pi}{N}\right)\right)^{\frac{1}{2}} \simeq 1 - \frac{4\pi^2 r}{(r+1)^2}\frac{1}{N^2}\,.$$

■

From previous result we see that, by keeping N fixed and by varying r we obtain a sequence of matrices for which the rate of convergence tends to 1 logarithmically in the number of systems and for which the degree of the associated graph grows logarithmically in the number of systems.

The previous example is based on a Cayley graph over the group $\mathbb{Z}_N^r$. The same behavior exhibited in this example can also be obtained starting from a Cayley graph over the cyclic group $\mathbb{Z}_N$. Indeed if we take $\mathbb{Z}_{N^r}$ and the subset

$$S := \{0, 1, N, N^2, \ldots, N^{r-1}\}\,,$$

we can construct a Cayley stochastic matrix with an essential spectral radius that is asymptotically equivalent to the one obtained in the previous example.

More in general, it can be proved that, if in the family of groups $\mathbb{Z}_N^r$ we maintain N fixed (prime) and we vary r, there exists a constant $c < 1$ such that for every r there exists $S_r \subseteq \mathbb{Z}_N^r$ such that $|S_r| \simeq cr$ and such that

$$\rho(P_r) \leq d < 1\,.$$

where $P_r = |S_r|^{-1} A_r$ and where A_r is the adjacency matrices of the corresponding Cayley graph. Such Cayley graphs are constructed using the theory of channel codes over finite fields [4]. Extensions to non-prime N are likely to be possible considering the theory of group codes.

The question at this point is the following: Is the Cayley structure on the matrix or the Cayley structure on the graph that prevents to obtain good performance? In other words, do there exist stochastic matrices supported by Abelian Cayley graphs that exhibit better performance than what imposed by the bound (3.2)? Notice that, in order to make fair comparisons, we need to limit to doubly stochastic matrices. We conjecture that for doubly stochastic matrices supported on Abelian Cayley graphs the bound (3.2) continues to hold.

What about other graphs? An easy way to restrict to doubly stochastic matrices is by imposing that they are symmetric and so that the corresponding graphs are undirected. If A is the adjacency matrix of a ν-regular undirected graph, then, $P = \nu^{-1} A$ is doubly stochastic. For these graphs, we recall a basic asymptotic lower bound by Alon and Boppana [3] on the second eigenvalue

$$\liminf_{N \to +\infty} \rho(P) \geq \frac{2\sqrt{\nu - 1}}{\nu}\,,$$

where the lim inf is intended to be performed along the family of all ν-regular undirected graphs having N vertices.

Ramanujan graphs (see [89] and references therein) are those ν-regular undirected graphs achieving the previous bound, namely such that $\rho(P) = 2\nu^{-1}\sqrt{\nu - 1}$. Hence, through these graphs, it would be possible to keep the essential spectral radius bounded away from 1, while keeping the degree fixed. In fact, there are plenty of Ramanujan graphs (for instance any complete graph), but it is still an open problem if for any N and ν there exists a Ramanujan graph with N vertices and degree ν. There are only partial results in this direction. For example it is possible to prove that, if ν is such that $\nu - 1$ is the power of a prime, then there exist a sequence of Ramanujan graphs with a growing number of vertices and of fixed degree ν. Moreover, when available, these constructions are quite complicated and the fact that they strictly depend on the choice of particular number of vertices makes

them not so interesting from our point of view. However, it is interesting to notice that graphs behaving similarly to the Ramanujan ones are not so unlikely. Indeed Friedman [62] showed that, by averaging the essential spectral radius of the adjacency matrices A of all undirected ν-regular Cayley graphs having N vertices, we obtain for $P = \nu^{-1}A$

$$\mathbb{E}(\rho(P)) \leq \frac{2\sqrt{\nu-1}}{\nu}\left(1 + \frac{\ln \nu/2}{\sqrt{\nu}} + O\left(\frac{1}{\sqrt{\nu}}\right)\right) + O\left(\frac{\nu^{1/2}\ln\ln N}{\ln N}\right).$$

Of course the previous bound has to be interpreted in an asymptotic sense for $N \to +\infty$ and $\nu \to +\infty$. As a consequence we have in particular that, if we fix ν sufficiently large, in the average, $\rho(P)$ will remain bounded away from 1 as $N \to +\infty$.

3.4 Conclusions

We have derived bounds on the convergence rate to the average consensus for a set of systems that exchange information over time-invariant communication networks with symmetries. We have showed that, in time-invariant networks, symmetries yield rather slow convergence to the average consensus. In particular for such networks we have computed a tight bound for the convergence rate.

Chapter 4

Randomized Consensus Algorithms

4.1 Introduction

In the previous chapter we have considered the time-invariant average consensus algorithm (2.10). In this chapter we will focus on the more general time-varying case by addressing, in particular, the analysis of the so-called *randomized consensus algorithms*. Roughly speaking a randomized algorithm works as follows. Assume we have a family $\mathcal{P}$ of quasi-stochastic matrices. Then, at each iteration of the consensus algorithm, the consensus matrix $P(t)$ is chosen randomly inside this set of matrices, according to a certain probability distribution probability. For a randomized algorithm, convergence is considered in a probabilistic sense and performance are studied in mean square sense.

The use of the *random algorithms* is quite appealing, since they take into consideration some fundamental limitations arising in the realistic implementation of the consensus algorithms. Indeed, it should be pointed out that, in many practical applications, a node can not simultaneously receive data from two different neighbor nodes (for instance collision can delete messages in wireless environment) and in some applications it cannot simultaneously transmit to more than a node (this happens for instance for processors nets).

In this chapter, we consider three interesting examples of randomized consensus algorithms. The first two emphasize a remarkable feature of the randomized consensus algorithms, i.e., that they allow to achieve better performance than the deterministic ones with comparable complexity. This com-

parison is done by considering the consensus algorithms having as communication graphs the Abelian Cayley graphs presented in the previous chapter. The last example reviews the symmetric gossip algorithms introduced in [22] and [24] (other random linear schemes under the name of gossip algorithms have been studied for instance in [51] and [76]). As underlined in [24], the gossip algorithms are mainly motivated by applications to sensors, peer-to-peer and *ad hoc* networks. In general, these networks are constrained by the following operational characteristics: i) they may not have a centralized entity for facilitating computation, communication, and time-synchronization, ii) the network topology may not be completely known to the nodes of the network, iii) nodes may join or leave the network (even expire), so that the network topology itself may change, and iv) in the case of sensor networks, the computational power and energy resources may be very limited. These constraints have motivated the design of gossip algorithms: schemes which distribute the computational burden and in which a node communicates with a randomly chosen neighbor.

The Chapter is organized as follows. In Section 4.2, we will introduce the *randomized consensus algorithms* and we will adopt, from [58], the concept of *probabilistic consensus*. In Section 4.3, following [58], we will review some algebraic conditions ensuring the achievement of the probabilistic consensus. Moreover, we will introduce two particular randomized strategies: the *time-varying strategy with bounded in-degree* and the *time-varying Cayley graphs strategy*. In Section 4.4, we will provide some criteria in order to evaluate the performance of randomized consensus algorithms. We will analyze, from this point of view, the time-varying strategy with bounded in-degree and the time-varying Cayley graphs strategy. In Section 4.5, we will introduce the symmetric gossip algorithms, particularizing them for the class of Cayley graphs. Finally, in Section 4.6, we will summarize the main conclusions.

4.2 Problem formulation

Consider the time varying consensus problem, as defined in Section 2.2. Namely, we have N coupled linear dynamical systems

$$x_i(t+1) = \sum_{j=1}^{N} P_{ij}(t) x_j(t) \qquad i = 1, \ldots, N \tag{4.1}$$

where $x_i(t) \in \mathbb{R}$ is the state of the i-th system at time t and $P_{ij}(t) \in \mathbb{R}$ coefficients which vary with the time t. More compactly, we can write

$$x(t+1) = P(t)x(t),$$

where $x(t) \in \mathbb{R}^N$ and $P(t) \in \mathbb{R}^{N\times N}$. We recall that the sequence $P(t)$ is said to achieve consensus if conditions (a) and (b) stated in Section 2.2 are satisfied.

In this chapter, we will assume to have statistical information on the matrices $P(t)$ and we will adopt the probabilistic approach introduced in [58], instead of a worst case analysis considered in [138, 74, 88, 112, 16]. More precisely, in this chapter we will assume that $P(t)$ is a sequence of i.i.d. matrix valued random variables and $x(t)$ is the stochastic process which is the solution of the equation (4.1). We say that the sequence $P(t)$ achieves the *probabilistic consensus* if condition (a) holds while (b) is replaced by

(b') For any $x(0) \in \mathbb{R}^N$, there exists a scalar random variable α such that

$$\lim_{t\to\infty} x(t) = \alpha\mathbb{1} \qquad \text{almost surely.} \tag{4.2}$$

If $\alpha = N^{-1}\mathbb{1}^*x(0)$ almost surely, we talk about *probabilistic average consensus.*

In this chapter, we will restrict to cases in which $P(t)$ are stochastic matrices. Notice that condition (a) is then clearly automatically satisfied. If, moreover, $P(t)$ is doubly stochastic, then the average is invariant, namely $N^{-1}\mathbb{1}^*x(t) = N^{-1}\mathbb{1}^*x(0)$ for every t and hence in this case, consensus implies average consensus. Let now

$$Q(t) = P(t-1)\cdots P(0),$$

so that we can write $x(t) = Q(t)x(0)$. The random variable α in (4.2) is a linear function of the initial condition $x(0)$ so that we can write $\alpha = \rho^*x(0)$ for some random variable ρ taking values in $\mathbb{R}^N$ and such that $\mathbb{1}^*\rho = 1$. Therefore probabilistic consensus can be equivalently expressed by saying that there exists a random variable ρ taking values in $\mathbb{R}^N$ such that

$$\lim_{t\to\infty} Q(t) = \mathbb{1}\rho^* \tag{4.3}$$

almost surely. Moreover notice that $\mathbb{1}\rho^*$ is a matrix whose rows are all equal to ρ^*. Notice that $x(\infty) = \rho^*x(0)$. We have probabilistic average consensus exactly when $\rho = N^{-1}\mathbb{1}$ almost surely.

4.3 Conditions for the probabilistic consensus

In this section we briefly review some conditions which ensures the probabilistic consensus. In doing this, we follow the approach adopted in [58], where the authors show that the probabilistic consensus turns out to be an easily checkable property, namely as easily checkable as the deterministic consensus in the time-invariant case.

We immediately state the following result which appears in [37].

Theorem 4.1 *The algorithm $P(t)$ achieves probabilistic consensus if and only if for every pair of systems i, j we have that*

$$\mathbb{P}[\mathcal{E}_{ij}] = 1$$

where

$$\mathcal{E}_{ij} = \{\exists\, k,\, \exists\, t \;\; | Q_{ik}(t) Q_{jk}(t) > 0\}\,.$$

To obtain a more handy condition, the authors in [58] impose a hypothesis which is however always satisfied in all the cases which are commonly considered in literature. In the following proposition they assume that all the diagonal elements of $P(t)$ are nonzero with probability 1. The simple condition, ensuring probabilistic consensus, that they have found is a direct consequence of the above Theorem and is based on the expected value of $P(t)$ which is denoted as $\bar{P} = \mathbb{E}[P(t)]$. We have the following result, whose proof is in [58].

Proposition 4.2 *Assume that for any i we have that $P(t)_{ii} > 0$ almost surely. If $\mathcal{G}_{\bar{P}}$ is strongly connected, then $P(t)$ achieves probabilistic consensus.*

Remark 4.3 Notice, as remarked in [58], that there is a sort of weak converse to previous result. Indeed, assume that the sequence $P(t)$ achieves probabilistic consensus. From the almost sure convergence $Q(t) \to \mathbb{1}\rho^*$ and Lebesgue dominated convergence theorem it follows that $\mathbb{E}[Q(t)] \to \mathbb{1}\mathbb{E}[\rho]^*$ and so $\bar{P}^t \to \mathbb{1}\mathbb{E}[\rho]^*$. In other words, $\bar{P}$ achieves consensus. It can very well happen that $\mathbb{E}\rho = \mathbb{1}$. even if ρ is not equal to $\mathbb{1}$ almost surely. In other terms, even if $\bar{P}$ achieves average consensus, not necessarily $P(t)$ will also achieve average probabilistic consensus. This will appear in the two cases we will consider in the next section.

We proceed now by introducing two interesting examples of randomized consensus algorithms. We will prove that in both these cases the probabilistic consensus occurs.

4.3.1 Time-varying strategy with bounded in-degree

In this example we consider a time-varying consensus algorithm in which we assume that each system receives the state of ν systems chosen randomly and independently between all the other system. Because of this it can happen that the resulting communication graph has multiple arcs connecting the same pair of nodes.

Fix $k_0, k_1, \dots, k_\nu \geq 0$ such that $\sum_j k_j = 1$. The consensus matrix is

$$P(t) = k_0 I + \sum_{i=1}^{\nu} k_i E_i(t) \tag{4.4}$$

where $E_i(t)$, $i = 1, \dots, \nu$, are ν independent sequences of independent random variables taking values on the set of matrices

$$\mathcal{E} := \{E \in \{0,1\}^{N \times N} \ : \ E\mathbb{1} = \mathbb{1}\}$$

and uniformly distributed in such a set. Notice that the set $\mathcal{E}$ is constituted by all matrices with entries 0 or 1 which have exactly one 1 in each row. The state $x(t)$ becomes a random variable which evolves according to

$$x(t) = \prod_{s=1}^{t} P(s)x(0)\,,$$

where $x(0)$ is a random variable independent of the processes $E_i(t)$. We have that

$$\begin{aligned} \bar{P} &= \mathbb{E}\left[k_0 I + \sum_{i=1}^{\nu} k_i E_i(t)\right] \\ &= k_0 I + \sum_{i=1}^{\nu} k_i \mathbb{E}\left[E_i(t)\right] \\ &= k_0 I + \frac{\sum_{i=1}^{\nu} k_i}{N} \mathbb{1}\mathbb{1}^*, \end{aligned}$$

where in the last equality we used the fact that $\mathbb{E}\left[E_i\right] = 1/N \mathbb{1}\mathbb{1}^*$. Note that $\mathcal{G}_{\bar{P}}$ is the complete graph and hence it is trivially strongly connected. Moreover in this case, if $k_0 > 0$, all the diagonal elements of $P(t)$ are nonzero with

probability 1. Applying Proposition 4.2 we can conclude that this algorithm yields the probabilistic consensus.

Finally notice that, in this example the matrices $P(t)$ are in general not doubly stochastic and so the average probabilistic consensus is not achieved. This, even if $\bar{P}$ is doubly stochastic.

4.3.2 Time-varying Cayley graphs strategy

In this example we consider a time-varying strategy similar to the one presented in the previous section. The difference is that here we impose the time-varying matrices to be Cayley.

Fix an Abelian group G and a number $\nu < |G|$. We consider a sequence of subsets $S_t \subseteq G$ that are randomly generated in the following way.

Let $\alpha_i(t)$, $i = 1, \dots, \nu$, be ν independent sequences of independent random variables taking value on G and uniformly distributed in such a set. We put

$$S_t = \{\alpha_0(t) = 0, \alpha_1(t), \dots, \alpha_\nu(t)\}\,.$$

Notice that in S_t there might be repetitions and so its cardinality may be less than $\nu + 1$.

Fix $k_0, k_1, \dots, k_\nu \geq 0$ such that $\sum_j k_j = 1$, and consider the sequence of probability distributions π_t on G supported on the sequence of sets S_t defined as

$$\pi_t(g) = \begin{cases} k_j \text{ if } g = \alpha_j(t) \\ 0 \quad \text{otherwise.} \end{cases}$$

Let $P(t)$ be the stochastic Cayley matrix associated with π_t. The state $x(t)$ becomes a random variable that evolves according to

$$x(t) = \prod_{s=1}^{t} P(s)x(0)\,,$$

where $x(0)$ is a random variable independent of the processes $\alpha_i(t)$. We know from Section 3.3.1 that we can represent

$$P(t) = \sum_{\chi \in \hat{G}} \hat{\pi}_t(\chi) N^{-1} \chi \chi^*$$

where

$$\hat{\pi}_t(\chi) = k_0 + \sum_{j=1}^{\nu} k_j \chi(-\alpha_j(t))\,,$$

It is immediate to verify that $\mathbb{E}[\chi(\alpha_j(t))] = 0$ when $\chi \neq \chi_0$. Therefore we have that

$$\mathbb{E}\left[\hat{\pi}_t(\chi)\right] = \begin{cases} k_0 & \text{if} \quad \chi \neq \chi_0 \\ \sum_{j=0}^{\nu} k_j & \text{if} \quad \chi = \chi_0 \end{cases}$$

and hence that

$$\begin{aligned} \bar{P} &= \mathbb{E}\left[\sum_{\chi \in \hat{G}} \hat{\pi}_t(\chi) N^{-1} \chi\chi^*\right] \\ &= \mathbb{E}\left[\hat{\pi}_t(\chi_0)\frac{1}{N}\chi_0\chi_0^*\right] + \mathbb{E}\left[\sum_{\chi \neq \chi_0} \hat{\pi}_t(\chi)\frac{1}{N}\chi\chi^*\right] \\ &= \frac{\sum_{j=0}^{\nu} k_j}{N}\mathbb{1}\mathbb{1}^* + k_0 \sum_{\chi \neq \chi_0} \frac{1}{N}\chi\chi^* \\ &= k_0 \sum_{\chi} \frac{1}{N}\chi\chi^* + \frac{\sum_{j=1}^{\nu} k_j}{N}\mathbb{1}\mathbb{1}^* \\ &= k_0 I + \frac{\sum_{j=1}^{\nu} k_j}{N}\mathbb{1}\mathbb{1}^* \end{aligned}$$

where in the last equality we have used the fact that $\sum_{\chi} \frac{1}{N}\chi\chi^* = I$. Observe that $\bar{P}$ is the same of the previous example. Therefore, if $k_0 > 0$, from Proposition 4.2 we can conclude that also this algorithm yields the probabilistic consensus. Moreover, since each $P(t)$ is a Cayley matrix and hence it is doubly stochastic, we have that in this case the average probabilistic consensus is achieved.

4.4 The measures of the performance

Assume now that we have a fixed random algorithm $P(t)$ achieving probabilistic consensus so that (4.3) is satisfied with a suitable ρ. The authors in [58] proposes of measuring the performance of the algorithm $P(t)$ by considering two figures. The first figure they consider is a normalized version of the *distance from the consensus*

$$d(t) = \frac{1}{N}\|x(t) - \mathbb{1}x_A(t)\|^2 = \frac{1}{N}\sum_{i=1}^{N} |x_i(t) - x_A(t)|^2 \tag{4.5}$$

where $x_A(t) = N^{-1}\mathbb{1}^* x(t)$ is the average of the components of $x(t)$. The second one is the *average displacement* from its initial value

$$\beta(t) = |x_A(t) - x_A(0)|^2. \tag{4.6}$$

Obviously in those situations where $P(t)$ is always doubly stochastic, we have that $\beta(t) = 0$ for every t. Notice moreover that

$$\frac{1}{N}\|x(t) - \mathbb{1}x_A(0)\|^2 = d(t) + \beta(t)$$

which shows that the evolution of $d(t)$ and $\beta(t)$ determines the evolution of $\frac{1}{N}\|x(t) - \mathbb{1}x_A(0)\|^2$. This coincides with the average distance from $x_i(t)$ to $x_A(0)$ and so it is the most important error parameter that typically one has to minimize.

We know that $d(t)$ converges to 0 almost surely. The main point is to estimate the speed of this convergence. In what follows, in order to evaluate the performance of the random strategies illustrated in the previous section, we will study the expectations of the two variables $d(t)$ and $\beta(t)$, i.e. $\mathbb{E}[d(t)]$ and $\mathbb{E}[\beta(t)]$, so providing a mean-square analysis of (4.1) in the case of *time-varying with bounded in-degree* and *time-varying Cayley graphs* strategies.

4.4.1 Mean square performance for time-varying with bounded in-degree strategy

We can prove the following result.

Theorem 4.4 *The quantity* $\mathbb{E}[d(t)]$ *satisfies the following recursive equation*

$$\mathbb{E}[d(t+1)] = \rho\mathbb{E}[d(t)] \tag{4.7}$$

where

$$\rho = k_0^2 + \frac{N-1}{N}\sum_{i=1}^{\nu} k_i^2.$$

Proof: We start by introducing the following variable

$$y(t) = x(t) - \frac{1}{N}\left(\mathbb{1}^* x(t)\right)\mathbb{1} = Yx(t)$$

where $Y = I - \frac{1}{N}\mathbb{1}\mathbb{1}^*$. A straightforward calculation shows that $YE_i = YE_iY$. Hence from (4.4) it is easy to see that $y(t)$ satisfies the following recursive

equation

$$y(t+1) = k_o y(t) + Y\left(\sum_{i=1}^{\nu} k_i E_i(t)\right) y(t). \tag{4.8}$$

Let now $W(t) = \mathbb{E}\left[y(t)y^*(t)\right]$. Notice that $\mathbb{E}[d(t)] = \text{tr}\left\{W(t)\right\}$. From (4.8) it results that

$$\begin{aligned}
&W(t+1) = \\
&= \mathbb{E}\left[\left(k_0 y(t) + Y\left(\sum_{i=1}^{\nu} k_i E_i(t)\right) y(t)\right)\left(k_0 y(t) + Y\left(\sum_{i=1}^{\nu} k_i E_i(t)\right) y(t) z\right)^*\right] \\
&= k_0^2 W(t) + k_0 \mathbb{E}\left[y(t)y^*(t)\left(\sum_{i=1}^{\nu} k_i E_i(t)\right)^*\right] Y + \\
&\quad + k_0 \mathbb{E}\left[Y\left(\sum_{i=1}^{\nu} k_i E_i(t)\right) y(t)\, y^*(t)\right] + \\
&\quad + Y\mathbb{E}\left[\left(\sum_{i=1}^{\nu} k_i E_i(t)\right) y(t)y^*(t)\left(\sum_{j=1}^{\nu} k_j E_j(t)\right)^*\right] Y \\
&= k_0^2 W(t) + k_0 \mathbb{E}\left[\mathbb{E}\left[y(t)y^*(t)\left(\sum_{i=1}^{\nu} k_i E_i(t)\right)^* \Big| E_1, \ldots, E_\nu\right]\right] Y + \\
&\quad + k_0 \mathbb{E}\left[\mathbb{E}\left[Y\left(\sum_{i=1}^{\nu} k_i E_i(t)\right) y(t)\, y^*(t) \Big| E_1, \ldots, E_\nu\right]\right] + \\
&\quad + Y\mathbb{E}\left[\mathbb{E}\left[\left(\sum_{i=1}^{\nu} k_i E_i(t)\right) y(t)y^*(t)\left(\sum_{j=1}^{\nu} k_j E_j(t)\right)^* \Big| E_1, \ldots, E_\nu\right]\right] Y \\
&= k_0^2 W(t) + k_0 W(t)\mathbb{E}\left[\left(\sum_{i=1}^{\nu} k_i E_i(t)\right)^*\right] Y + k_0 Y\mathbb{E}\left[\sum_{i=1}^{\nu} k_i E_i(t)\right] W(t) + \\
&\quad + Y\mathbb{E}\left[\left(\sum_{i=1}^{\nu} k_i E_i(t)\right) W(t)\left(\sum_{j=1}^{\nu} k_j E_j(t)\right)^*\right] Y
\end{aligned}$$

Since $\mathbb{E}[E_i] = \mathbb{1}\mathbb{1}^*/N$ and since $Y\,\mathbb{1}\mathbb{1}^* = \mathbb{1}\mathbb{1}^*\,Y = 0$ we have that the first two expectations in the previous equation are equal to zero. To compute the last expectation we need to distinguish two cases:

$i \neq j$: then $E_i(t)$, $E_j^*(t)$ and $W(t)$ are all independent and thus the expectation factorizes. Recall now that $\mathbb{E}[E_i] = \frac{1}{N}\mathbb{1}\mathbb{1}^*$. Two terms of the type $Y\,\mathbb{1}\mathbb{1}^*$ appear and thus for $i \neq j$ the expectation is zero,

$i = j$: then we have that

$$Y\mathbb{E}\left[k_i E_i(t)\, W(t)\, k_i E_i^*(t)\right] Y = k_i^2 Y\mathbb{E}\left[E_i(t) W(t) E_i^*(t)\right] Y$$

By straightforward calculations, it is possible to see that, for any $M \in \mathbb{R}^{N\times N}$ it holds

$$\mathbb{E}[E_i^T M E_i] = \frac{1}{N}\mathrm{tr}\,\{M\} I + \frac{1}{N^2}\mathbb{1}^* M \mathbb{1}(\mathbb{1}\mathbb{1}^* - I). \tag{4.9}$$

Hence

$$k_i^2 Y\mathbb{E}\left[E_i(t) W(t) E_i^*(t)\right] Y = \frac{k_i^2}{N}\mathrm{tr}\left(W(t)\right) Y - \frac{k_i^2}{N^2} Y\mathbb{1}^* W(t)\, \mathbb{1}Y.$$

We thus obtain that

$$W(t+1) = k_0^2 W(t) + \frac{1}{N}\sum_{i=1}^{\nu} k_i^2 \mathrm{tr}\left(W(t)\right) Y - \frac{1}{N^2}\sum_{i=1}^{\nu} k_i^2 Y\mathbb{1}^*\, W(t)\, \mathbb{1}Y.$$

Let $w(t) = \mathrm{tr}\left(W(t)\right)$, then we have

$$w(t+1) = k_0^2 w(t) + \frac{1}{N}\sum_{i=1}^{\nu} k_i^2\, \mathrm{tr}\left(\mathrm{tr}\left(W(t)\right) Y\right) - \frac{1}{N^2}\sum_{i=1}^{\nu} k_i^2\, \mathrm{tr}\left(\mathbb{1}^*\, W(t)\, \mathbb{1}Y\right).$$

The term $\mathrm{tr}\left(\mathrm{tr}\left(W(t)\right) Y\right) = (N-1)\,\mathrm{tr}\left(W(t)\right)$ since $\mathrm{tr}\left(Y\right) = N-1$ and the last term is zero since

$$\mathbb{1}^*\, W(t)\, \mathbb{1} = \sum_{i=1}^{N}\sum_{j=1}^{N} (W(t))_{ij} = 0\,.$$

We thus have the following difference equation

$$w(t+1) = \left(k_0^2 + \frac{N-1}{N}\sum_{i=1}^{\nu} k_i^2\right) w(t)\,. \tag{4.10}$$

By observing that $\mathbb{E}[d(t)] = w(t)$ we have proved the claim of the theorem. ■

Notice that the strongest exponential rate of convergence in (4.10) is given by

$$\min\left\{k_0^2 + \frac{N-1}{N}\sum_{i=1}^{\nu} k_i^2 \;\middle|\; k_0, k_1, \ldots, k_\nu \geq 0,\; k_0 + \sum_{i=1}^{\nu} k_i = 0\right\} = \frac{N-1}{N(\nu+1)-1}\,,$$

obtained by choosing

$$k_0 = \frac{N-1}{N(\nu+1)-1} \tag{4.11}$$

and

$$k_i = \frac{N}{N(\nu+1)-1} \qquad \forall i = 1, \dots, \nu. \tag{4.12}$$

As outlined previously, this random strategy will not reach the consensus at the average of the initial conditions. Therefore, in such case it is interesting to study also the behavior of $\mathbb{E}[\beta(t)]$ in particular to evaluate $\lim_{t\to\infty} \mathbb{E}[\beta(t)]$. We have the following result.

Proposition 4.5

$$\mathbb{E}[\beta(\infty)] = \alpha \mathbb{E}|| \left(I - N^{-1}\mathbb{1}\mathbb{1}^T\right) x(0)||^2 ,$$

where

$$\alpha = \frac{\sum_{i=1}^{\nu} k_i^2}{N[N(1-k_0^2) + (1-N)\sum_{i=1}^{\nu} k_i^2]} .$$

Proof: Consider $\Delta(t) := x(t) - N^{-1}\mathbb{1}\mathbb{1}^T x(0)$. One can show that the dynamics of $\Delta(t)$ is described by the equation $\Delta(t+1) = P(t)\Delta(t)$ where $P(t)$ is given in (4.4). By defining $W(t) := \mathbb{E}[\Delta(t)\Delta(t)^*]$, $w(t) = \operatorname{tr}(W(t)) = \mathbb{E}||\Delta(t)||^2$ and $s(t) = N^{-1}\mathbb{1}^* W(t)\mathbb{1}$, we obtain that

$$\begin{aligned}
&W(t+1) = \\
&= \mathbb{E}\left[\left(k_0 I + \sum_{i=1}^{\nu} k_i E_i(t)\right)\Delta(t)\Delta^*(t)\left(k_0 I + \sum_{i=1}^{\nu} k_i E_i(t)\right)^*\right] \\
&= k_0^2 W(t) + \mathbb{E}\left[\sum_{i=1}^{\nu} k_i E_i(t)\Delta(t)\Delta^*(t)k_0\right] + \mathbb{E}\left[k_0\Delta(t)\Delta^*(t)\sum_{i=1}^{\nu} k_i E_i^*(t)\right] + \\
&\quad + \mathbb{E}\left[\sum_{i=1}^{\nu} k_i E_i(t)\Delta(t)\Delta^*(t)\sum_{j=1}^{\nu} k_j E_j^*(t)\right] \\
&= k_0^2 W(t) + k_0\sum_{i=1}^{\nu} k_i\left(W(t)\frac{1}{N}\mathbb{1}\mathbb{1}^* + \frac{1}{N}\mathbb{1}\mathbb{1}^* W(t)\right) \\
&\quad + \sum_{i=1}^{\nu} k_i^2\left(\frac{1}{N}(\mathbb{1}^* W(t)\,\mathbb{1})\frac{1}{N}\mathbb{1}\mathbb{1}^* + \left(\frac{1}{N}\operatorname{tr} W(t) - \frac{1}{N^2}\mathbb{1}^* W(t)\,\mathbb{1}\right) I\right) + \\
&\quad + \sum_{\substack{i=1 \\ i\neq j}}^{\nu}\sum_{j=1}^{\nu} k_i k_j \frac{1}{N}(\mathbb{1}^* W(t)\mathbb{1})\frac{1}{N}\mathbb{1}\mathbb{1}^*
\end{aligned}$$

where in the last equality we used again the fact that $\mathbb{E}[E_i] = \frac{1}{N}\mathbb{1}\mathbb{1}^*$ and the property (4.9). Let us now define the following variables

$$w(t) = \operatorname{tr} W(t)$$
$$s(t) = \frac{1}{N}\,\mathbb{1}^* W(t)\mathbb{1}$$

then we have that

$$w(t+1) = \left(k_0^2 + \sum_{i=1}^{\nu} k_i^2\right) w(t) + \left(2k_0 \sum_{i=1}^{\nu} k_i + \sum_{\substack{i=1 \\ i\neq j}}^{\nu}\sum_{j=1}^{\nu} k_i k_j\right) s(t)$$

$$s(t+1) = \left((k_0^2 + 2k_0 \sum_{i=1}^{\nu} k_i + \frac{N-1}{N}\sum_{i=1}^{\nu} k_i^2 + \sum_{\substack{i=1 \\ i\neq j}}^{\nu}\sum_{j=1}^{\nu} k_i k_j\right) s(t) + \frac{1}{N}\sum_{i=1}^{\nu} k_i^2 w(t)\,.$$

Using the fact that $1 - k_0 = \sum_{i=1}^{\nu} k_i$ we have the following discrete-time system

$$\begin{pmatrix} w(t+1) \\ s(t+1) \end{pmatrix} = \underbrace{\begin{pmatrix} \sum_{i=0}^{\nu} k_i^2 & 1 - \sum_{i=0}^{\nu} k_i^2 \\ \frac{1}{N}\sum_{i=1}^{\nu} k_i^2 & 1 - \frac{1}{N}\sum_{i=1}^{\nu} k_i^2 \end{pmatrix}}_{A} \begin{pmatrix} w(t) \\ s(t) \end{pmatrix}\,.$$

In order to prove the theorem we need to compute $w(\infty) := \lim_{t\to\infty} w(t)$. For t that tends to infinity, the state $(w(t), s(t))^*$ will be aligned to the dominant eigenvector of A. The eigenvectors of A are

$$\lambda_1 = 1$$
$$\lambda_2 = k_0^2 + \frac{N-1}{N}\sum_{i=1}^{\nu} k_i^2\,.$$

Corresponding to the eigenvalues λ_1 and λ_2 we have the eigenvectors

$$a_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$a_2 = \begin{pmatrix} -\dfrac{N\left(1 - \sum_{i=1}^{\nu} k_i^2\right)}{\sum_{i=1}^{\nu} k_i^2} \\ 1 \end{pmatrix}$$

and thus we have that

$$\begin{pmatrix} w(t) \\ s(t) \end{pmatrix} = c_1 \lambda_1^t a_1 + c_2 \lambda_2^t a_2$$

from which we have $w(\infty) \approx c_1 a_1$. The constant c_1 is computed from the initial conditions

$$\begin{pmatrix} w(0) \\ s(0) \end{pmatrix} = c_1 a_1 + c_2 a_2 \,.$$

Note that

$$w(0) = \text{tr } \{W(0)\} = \mathbb{E}|| \left(I - N^{-1} \mathbb{1}\mathbb{1}^T\right) x(0)||^2$$

and

$$s(0) = \frac{1}{N} \mathbb{1}^* W(0) \mathbb{1} = \frac{1}{N} \mathbb{E} \left[\sum_{i=1}^{N} \left(x_i(0) - \frac{1}{N} \mathbb{1}^* x(0) \mathbb{1} \right) \right]^2 = 0.$$

Simple calculations yields to

$$c_1 = \frac{\sum_{i=1}^{\nu} k_i^2}{N \left[N(1 - k_0^2) + (1 - N) \sum_{i=1}^{\nu} k_i^2 \right]} w(0)$$

This concludes the proof. ■

If we use the control gains $k_0, k_1, \ldots, k_\nu$ as in (4.11) and (4.12), which yield the fastest convergence rate, then we have

$$\mathbb{E}[\beta(t)] = \frac{1}{N(N(1+\nu) - 1)} \mathbb{E} \|(I - N^{-1} \mathbb{1}\mathbb{1}^*) x(0)\|^2.$$

Notice that, if the initial states $x_i(0)$ of the systems are independent and $\mathbb{E}[x_i(0)^2]$ is the same for all i, then as $N \to \infty$ the mean square distance to the average of the initial conditions tends to zero.

4.4.2 Mean square performance for time-varying Cayley graphs strategy

Since in this case the matrix $P(t)$ is doubly stochastic for every $t \geq 0$, the only quantity we want to study is $\mathbb{E}[d(t)]$. In order to do so, we consider

the displacement from the average of the initial conditions $\Delta(t) := x(t) - \frac{1}{N}\mathbb{1}\mathbb{1}^T x(0) = (I - \frac{1}{N}\chi_0\chi_0^*)x(t)$, which is governed by

$$\Delta(t) = \prod_{s=1}^{t} P(s)\Delta(0)\,,$$

where $\Delta(0)$ is now a random variable taking values on $\mathbb{R}^G$ such that $< \Delta(0), \chi_0 >= 0$ and independent of the set of variables $\{\alpha_i(t)\}$. Clearly $\mathbb{E}[d(t)] = \mathbb{E}||\Delta(t)||^2$. For the sake of the clarity, we recall that χ_0 denotes the trivial character, as explained in Subsection 3.3.1. We have the following result.

Proposition 4.6

$$\mathbb{E}||\Delta(t)||^2 = \left(\sum_{j=0}^{\nu} k_j^2\right)^t \mathbb{E}||\Delta(0)||^2\,.$$

Proof: We know we can represent

$$P(t) = \sum_{\chi\in\hat{G}} \hat{\pi}_t(\chi) N^{-1}\chi\chi^*\,.$$

Hence,

$$\prod_{s=1}^{t} P(s) = \sum_{\chi\in\hat{G}} \left[\prod_{s=1}^{t} \hat{\pi}_s(\chi)\right] N^{-1}\chi\chi^*\,.$$

Let us study the average of the squared norm of the various eigenvalues.

$$\mathbb{E}\left[\left|\prod_{s=1}^{t} \hat{\pi}_s(\chi)\right|^2\right] = \prod_{s=1}^{t} \mathbb{E}\left[|\hat{\pi}_s(\chi)|^2\right]\,.$$

Since

$$\hat{\pi}_t(\chi) = k_0 + \sum_{j=1}^{\nu} k_j \chi(-\alpha_j(t))\,,$$

we obtain

$$\mathbb{E}\left[|\hat{\pi}_t(\chi)|^2\right] = k_0^2 + \sum_{j=1}^{\nu} k_0 k_j \left[\mathbb{E}\left[\chi(\alpha_j(t))\right] + \mathbb{E}\left[\chi(\alpha_j(t))^*\right]\right] + \\ + \sum_{j=1}^{\nu}\sum_{\ell=1}^{\nu} k_j k_\ell \mathbb{E}\left[\chi(\alpha_j(t))\chi(\alpha_\ell(t))^*\right]\,. \quad (4.13)$$

It is immediate to verify that $\mathbb{E}[\chi(\alpha_j(t))] = 0$ when $\chi \neq \chi_0$, that $\mathbb{E}[\chi(\alpha_j(t))\chi(\alpha_\ell(t))^*] = 0$ when $j \neq \ell$, and that $\mathbb{E}[|\chi(\alpha_j(t))|^2] = 1$. Substituting in (4.13) we then obtain

$$\mathbb{E}\left[|\hat{\pi}_t(\chi)|^2\right] = k_0^2 + \sum_{j=1}^{\nu} k_j^2 = \sum_{j=0}^{\nu} k_j^2\,, \quad \forall \chi \neq 0\,.$$

We finally have

$$\begin{aligned}\mathbb{E}||\Delta(t)||^2 &= \sum_{\chi\neq\chi_0} \mathbb{E}\left[\left|\prod_{s=1}^{t}\hat{\pi}(\chi)\right|^2\right] \frac{1}{N}\mathbb{E}|<\Delta(0),\chi>|^2 \\ &= \left(\sum_{j=0}^{\nu} k_j^2\right)^t \frac{1}{N}\sum_{\chi\neq\chi_0}\mathbb{E}|<\Delta(0),\chi>|^2 = \left(\sum_{j=0}^{\nu} k_j^2\right)^t \mathbb{E}||\Delta(0)||^2\,.\end{aligned}$$

■

Notice that

$$\min\left\{\sum_{j=0}^{\nu} k_j^2 \;\middle|\; k_j \geq 0\,,\ \sum_{j=1}^{\nu} k_j = 1\right\} = \frac{1}{\nu+1}$$

which is obtained by choosing $k_j = 1/(\nu+1)$ for all j. With such a choice we have thus obtained the following mean convergence result

$$\mathbb{E}||\Delta(t)||^2 = \left(\frac{1}{1+\nu}\right)^t \mathbb{E}||\Delta(0)||^2\,.$$

We noticed in the previous section that strategies with symmetries behave quite poorly. As suggested in the last part of the previous section, a way to overcome this difficulty is to resort to Ramanujan graphs or to undirected regular graphs generated randomly.

The results found in this subsection show that an alternative way to solve this problem while maintaining the symmetry of the matrices is by a time-varying strategy in which at every time instant the communication graph is chosen randomly in a set of Cayley graphs. Indeed such strategy yields a mean square convergence rate that is higher and, more importantly, independent of the number of systems.

Moreover notice that $\frac{1}{1+\nu} \geq \frac{N-1}{N(\nu+1)-1}$, which is the rate obtained through the time-varying strategy with bounded degree discussed before. However, for $N \to +\infty$, the two strategies yield the same rate.

Remark 4.7 From an implementation point of view this strategy has an evident drawback: the same random choice done at every time instance has to be done by all systems. This seems to require a supervised communication of this information to every system. A possible way to overcome this limitation is by imposing that each agent uses the same pseudorandom number generator starting from the same seed.

Remark 4.8 As any average result, it is not immediately evident how the average computation above reflects on the behavior of the system when we consider a generic sequence S_t of subsets chosen at random. A simple standard probabilistic argument however allows us to show that such a convergence rate is indeed achieved by almost every sequence S_t. Fix any $c > 1$ and notice that for every $\chi \in \hat{G}$,

$$P\left(\prod_{s=1}^{t} |\hat{\pi}_s(\chi)| \geq \left(\frac{c}{\nu+1}\right)^t\right) = P\left(\sum_{s=1}^{t} \ln |\hat{\pi}_s(\chi)| \geq t \ln \frac{c}{\nu+1}\right). \quad (4.14)$$

Notice that $Y_s = \ln |\hat{\pi}_s(\chi)|$ is a sequence of independent identically distributed random variables taking values on $[-\infty, 0]$. The idea is to apply Chebyschev inequality. To overcome the problem of possible unboundedness of Y_s, we consider two different cases. Suppose that there exist a $\chi \in \hat{G}$ such that $\hat{\pi}_s(\chi)$ assumes the value 0 with probability $p > 0$. In this case we can simply estimate

$$P\left(\prod_{s=1}^{t} |\hat{\pi}_s(\chi)| \geq \left(\frac{c}{\nu+1}\right)^t\right) \leq (1-p)^t. \quad (4.15)$$

If instead the event $\{\hat{\pi}_s(\chi) = 0\}$ has probability zero, then the random variable Y_s is bounded and can be estimated as follows. First notice that, using Jensen inequality, we have

$$\mathbb{E}[Y_s] = \mathbb{E}[\ln |\hat{\pi}_s(\chi)|] \leq \ln \mathbb{E}[|\hat{\pi}_s(\chi)|] = \ln \frac{1}{\nu+1}.$$

Let

$$\delta := \ln \frac{c}{\nu+1} - \mathbb{E}[Y_s] = \ln c + \ln \frac{1}{\nu+1} - \mathbb{E}[Y_s] \geq \ln c > 0.$$

We can now estimate

$$P\left(\sum_{s=1}^{t}\ln|\hat{\pi}_s(\chi)|\geq t\ln\frac{c}{\nu+1}\right) = P\left(\sum_{s=1}^{t}Y_s\geq t\ln\frac{c}{\nu+1}\right)$$
$$= P\left(\sum_{s=1}^{t}(Y_s-\mathbb{E}[Y_s])\geq t\delta\right) \qquad (4.16)$$
$$\leq \frac{\mathrm{Var}[Y_s]}{\delta^2 t^2}$$

A straightforward application of Borel-Cantelli lemma now allows to conclude from relations (4.15) and (4.16) that, for almost every sequence S_t of subsets,

$$\prod_{s=1}^{t}|\hat{\pi}_s(\chi)| < \left(\frac{c}{\nu+1}\right)^t \quad \text{for } t \text{ sufficiently large and} \quad \text{for every } \chi\in\hat{G}\,.$$

From this we also obtain that, for almost every sequence S_t,

$$||\Delta(t)||^2 \leq \left(\frac{c}{\nu+1}\right)^t ||\Delta(0)||^2 \quad \text{for } t \text{ sufficiently large}\,.$$

Considering that c can be chosen arbitrarily close to 1, this proves our claim. A similar performance result on generic random samples from the average behavior can be proved also for the previous strategy. The probabilistic tools needed become however a bit more refined: matrices are not simultaneously diagonalizable and we have to use Oseledec ergodic theorem for products of random matrices.

4.5 Symmetric Gossip algorithms

In this section we review the symmetric gossip algorithm as proposed in [24].

We start from an a priori fixed communication skeleton, namely a fixed underlying undirected graph $\mathcal{G} = (V, \mathcal{E})$, establishing which are the feasible communications among the agents. We assume that at every time instant a node among the N possible is chosen randomly. This node then chooses also randomly one of its neighbors, it establishes a bidirectional link with it and they average their quantity. Precisely, consider the instant time t. Assume i is the node chosen randomly and assume that j is the node chosen by the

node i. Then the updating rule is given by

$$\begin{aligned} x_i(t+1) &= \frac{1}{2}x_i(t) + \frac{1}{2}x_j(t) \\ x_j(t+1) &= \frac{1}{2}x_j(t) + \frac{1}{2}x_i(t) \end{aligned} \tag{4.17}$$

and

$$x_h(t+1) = x_h(t), \qquad \text{if } h \neq i \text{ and } h \neq j. \tag{4.18}$$

Now, let, for every $(i,j) \in \mathcal{E}$,

$$E^{(i,j)} = (e_i - e_j)(e_i - e_j)^* \tag{4.19}$$

where $e_i = [0, \dots, 0, 1, 0, \dots, 0]^*$ is a $N \times 1$ unit vector with the i-th component equal to 1 and let

$$R^{(i,j)} = I - \frac{1}{2}E^{(i,j)}. \tag{4.20}$$

Then we can write the Equation (4.17) and Equation (4.18) in vector form as

$$x(t+1) = P(t)x(t) \tag{4.21}$$

where $P(t) = R^{(i,j)}$.

If we assume that the node i is chosen with probability $1/N$ and that this node chooses randomly a node j among its ν_i neighbors with probability $1/\nu_i$, we have that

$$\mathbb{P}\left(P(t) = R^{(i,j)}\right) = \frac{1}{N}\left[\frac{1}{\nu_i} + \frac{1}{\nu_j}\right]$$

and hence that

$$\bar{P} = \sum_{(i,j)\,\in\,\mathcal{E}} \frac{1}{N}\left[\frac{1}{\nu_i} + \frac{1}{\nu_j}\right]\left[I - \frac{1}{2}(e_i - e_j)(e_i - e_j)^*\right].$$

Notice that, if $(h,k) \in \mathcal{E}$ and $h \neq k$,

$$\bar{P}_{hk} = \frac{1}{2N}\left[\frac{1}{\nu_h} + \frac{1}{\nu_k}\right].$$

where $\bar{P}_{hk}$ denotes the element in the h-th row and in the k-th column of the matrix $\bar{P}$. Since each $R^{(i,j)}$ has all the diagonal terms different from 0 and since $\mathcal{G}_{\bar{P}} = \mathcal{G}$ we have, from Proposition 4.2, that the symmetric gossip

consensus algorithm reaches, almost surely, the average consensus if and only if the graph $\mathcal{G}$ is connected.

We proceed now by providing an alternative way of introducing the symmetric consensus gossip algorithm that generalizes the case above described. Assume again that a graph $\mathcal{G} = (V, \mathcal{E})$ is given and assume that, at each time instant an edge is selected inside the set $\mathcal{E}$ with a probability $W^{(i,j)}$. The updating rule is again

$$x(t+1) = P(t)x(t)$$

where $\mathbb{P}\left[P(t) = R^{(i,j)}\right] = W^{(i,j)}$. Clearly $\sum_{(i,j)\in\mathcal{E}} W^{(i,j)} = 1$. Observe that this approach recovers the above case of the symmetric gossip algorithm by letting $W^{(i,j)} = \frac{1}{N}\left[\frac{1}{\nu_i} + \frac{1}{\nu_j}\right]$.
We have that

$$\bar{P} = \sum_{(i,j)\in\mathcal{E}} W^{(i,j)} R^{(ij)}$$

Clearly if $\mathcal{G} = (V, \mathcal{E})$ is connected and each edge $(i, j) \in \mathcal{E}$ can be selected with a strictly positive probability $W_{(i,j)}$, then $\mathcal{G}_{\bar{P}}$ is automatically connected. Moreover in this case all the diagonal elements of $P(t)$ are nonzero with probability 1. Applying Proposition 4.2 we can conclude that this algorithm yields the probabilistic consensus. Moreover, since the $P(t)$ are all symmetric we can conclude that in this case probabilistic average consensus is achieved.

It is worth noting that the fact that each $R^{(i,j)}$ has all the diagonal elements different from 0 implies also the following interesting property. Given $\mathcal{G} = (V, \mathcal{E})$ let $\tilde{\mathcal{G}} = (V, \tilde{\mathcal{E}})$ be the graph that we obtain from $\mathcal{G}$ where

$$\tilde{\mathcal{E}} = \{(i,j) : (i,j) \in \mathcal{E}\} \setminus \{(i,i) : (i,i) \in \mathcal{E}\},$$

namely $\tilde{\mathcal{E}}$ is the set $\mathcal{E}$ without all the possible self-loops. Assume that each edge $(i, j) \in \tilde{\mathcal{E}}$ can be selected with a strictly positive probability $\tilde{W}^{ij}$, where $\sum_{(i,j)\in\tilde{\mathcal{E}}} \tilde{W}^{ij} = 1$. Let $\bar{P}$ be the expected value of $P(t)$ constructed on the graph $\mathcal{G}$ and let $\tilde{P}$ be the expected value of $P(t)$ constructed on the graph $\tilde{\mathcal{G}}$. Then $\mathcal{G}_{\bar{P}} = \mathcal{G}_{\tilde{P}}$. Observe moreover that, if, at time instant t a self loop is selected then $P(t) = I$ and hence $x(t+1) = x(t)$, i.e. the state remains inalterated. In other words the self-loops are useless in order to reach the consensus. From now on, we will assume the following.

Assumption 4.9 *The graph $\mathcal{G} = (V, \mathcal{E})$ is a undirected connected graph without any self-loop and, at every time instant t, each edge $(i, j) \in \mathcal{E}$ can be selected with a strictly positive probability $W^{(i,j)}$.*

We end this subsection by defining the matrix $W \in \mathbb{R}^{N\times N}$ as the matrix having W^{ij} as element in the i-th row and j-th column and in the j-th row and i-th column, namely

$$W^{ij} = W^{ji} := W^{(i,j)}. \tag{4.22}$$

Note that, since from Assumption 4.9 we are considering graphs without self-loops, we have that $W_{ii} = 0$.

4.5.1 Analysis of the Symmetric Gossip Algorithm

Assume that an undirected graph $\mathcal{G}$ satisfying Assumption 4.9 is given and that W is the matrix defined in (4.22) at the end of Section 4.5. The objective of this subsection is to provide a mean-square analysis of the symmetric gossip algorithm as previously described, i.e.,

$$x(t+1) = P(t)x(t) \tag{4.23}$$

where $\mathbb{P}\left(P(t) = R^{(i,j)}\right) = W^{ij}$.
We start by assuming the following on the initial condition $x(0)$.

Assumption 4.10 *The initial condition $x(0)$ is a random vector variable such that $\mathbb{E}[x(0)] = 0$ and $\mathbb{E}[x(0)x^*(0)] = \sigma^2 I$ for some $\sigma^2 > 0$.*

As first step, we introduce the following variable

$$y(t) := \left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right) x(t)$$

Notice that $y(t) = x(t) - x_{ave}(t)\mathbb{1}$, where we recall that $x_{ave}(t)$ denotes the current average of the state $x(t)$, i.e., $x_{ave}(t) = 1/N\mathbb{1}^*x(t)$. Since $R^{(i,j)}$ is a doubly stochastic matrix for each pair (i,j), it follows that $x_{ave}(t) = x_{ave}(0)$ for all t and hence we can write also $y(t) = x(t) - x_{ave}(0)\mathbb{1}$. Moreover, this implies also that the variable $\beta(t) = 0$, defined in (4.6), is equal to 0 for all $t \geq 0$.

By observing that $R^{(i,j)}(I - 1/N\mathbb{1}\mathbb{1}^*) = (I - 1/N\mathbb{1}\mathbb{1}^*)R^{(i,j)}$ for each $R^{(i,j)}$, we obtain that the variable $y(t)$ satisfies the same recursive equation of $x(t)$, i.e.,

$$y(t+1) = P(t)y(t). \tag{4.24}$$

Let now

$$\Sigma(t) = \mathbb{E}\left[y(t)y^*(t)\right].$$

From Assumption 4.10 we have that

$$\Sigma(0) = \sigma^2 \left(I - \frac{1}{N} \mathbb{1}\mathbb{1}^* \right).$$

Now note that the quantity $d(t)$ defined in (4.5) is related to Σ and hence to y by

$$d(t) = \frac{1}{N} \text{tr} \left\{ \Sigma(t) \right\}.$$

We see now for a recursive expression for $\Sigma(t)$. From (4.24) we have that

$$\begin{aligned} \Sigma(t+1) &= \mathbb{E}\left[y(t+1) y^*(t+1) \right] \\ &= \mathbb{E}\left[P(t) y(t) y^*(t) P(t) \right] \\ &= \mathbb{E}\left[\mathbb{E}\left[P(t) y(t) y^*(t) P(t) | P(t) \right] \right] \\ &= \mathbb{E}\left[P(t)\, \mathbb{E}\left[y(t) y^*(t) \right] P(t) \right] \\ &= \mathbb{E}\left[P(t) \Sigma(t) P(t) \right] \end{aligned} \tag{4.25}$$

The calculation of the expectation $\mathbb{E}\left[P(t)\Sigma(t)P(t) \right]$ is not immediate. The next Theorem characterizes it. Before stating it, we provide some notational definitions that will be useful throughout the subsection. Given $X_1, X_2 \in \mathbb{R}^{N \times N}$, $X_1 \odot X_2$ denotes the component-wise product (Hadamard product) between X_1 and X_2, i.e. $(X_1 \odot X_2)_{hk} = (X_1)_{hk} (X_2)_{hk}$. Given a matrix $X \in \mathbb{R}^{N \times N}$ we have that diag $\{X\}$ means a diagonal matrix with the same diagonal elements of the matrix X. Given a N-dimensional column vector $v = [v_1, \ldots, v_N]^*$, either the symbol $\underline{\text{diag}}\ \{v_1, \ldots, v_N\}$ or the symbol $\underline{\text{diag}}\ \{v\}$ mean a diagonal matrix with $\left(\underline{\text{diag}}\ \{v_1, \ldots, v_N\} \right)_{ii} = v_i$. Finally, given the graph $\mathcal{G}$, the matrix A denotes its adjacency matrix, i.e

$$A_{ij} = A_{ji} = \begin{cases} 1 & \text{if} \quad (j,i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

Since in our work we are assuming that $\mathcal{G}$ is without self-loops, then $A_{ii} = 0, \ \ i = 1, \ldots, N$.

We can now state the following result.

Theorem 4.11 *Let $\mathcal{G} = (V, \mathcal{E})$ be a given graph satisfying Assumption* 4.9 *and let W be the matrix defined as in (4.22). Consider the equation (4.25).*

Then we have that

$$\begin{aligned}\Sigma(t+1) = \Sigma(t) &- \frac{1}{2}\underline{\text{diag}}\,\{W\mathbb{1}\}\,\Sigma(t) + \frac{1}{2}W\Sigma(t) - \frac{1}{2}\Sigma(t)\underline{\text{diag}}\,\{W\mathbb{1}\} \\ &+ \frac{1}{2}\Sigma(t)W + \frac{1}{4}\underline{\text{diag}}\,\{W\mathbb{1}\}\,\text{diag}\,\{\Sigma(t)\} + \frac{1}{2}W \odot \Sigma(t) + \\ &+ \frac{1}{4}\underline{\text{diag}}\,\{(W\text{diag}\,\{\Sigma(t)\})\mathbb{1}\} - \frac{1}{4}\text{diag}\,\{\Sigma(t)\}\,W \\ &- \frac{1}{4}W\text{diag}\,\{\Sigma(t)\} - \frac{1}{2}\text{diag}\,\{A(W \odot \Sigma(t))\} \end{aligned} \tag{4.26}$$

In order to prove the above Theorem we need the following technical Lemma.

Lemma 4.12 *Let $\mathcal{G} = (V, \mathcal{E})$ be a given graph satisfying Assumption* 4.9 *and let W be the matrix defined as in (4.22). Let Q any symmetric matrix. Then*

$$\mathbb{E}\left[E^{(i,j)}Q\right] = \underline{\text{diag}}\,\{W\mathbb{1}\}\,Q - WQ, \tag{4.27}$$

and

$$\begin{aligned}\mathbb{E}\left[E^{(i,j)}QE^{(i,j)}\right] = &\underline{\text{diag}}\,\{W\mathbb{1}\}\,\text{diag}\,\{Q\} + 2W \odot Q + \underline{\text{diag}}\,\{(W\text{diag}\,\{Q\})\mathbb{1}\} \\ &- [\text{diag}\,\{Q\}\,W + W\text{diag}\,\{Q\}] \\ &- 2\text{diag}\,\{A(W \odot Q)\} \end{aligned} \tag{4.28}$$

Proof: We have that

$$\begin{aligned}\mathbb{E}\left[E^{(i,j)}Q\right] &= \sum_{(i,j)\,\in\,\mathcal{E}} W_{ij}(e_i - e_j)(e_i - e_j)^*Q \\ &= \sum_{(i,j)\,\in\,\mathcal{E}} (e_ie_i^* + e_je_j^*)Q - \sum_{(i,j)\,\in\,\mathcal{E}} (e_ie_j^* + e_je_i^*)Q \\ &= \underline{\text{diag}}\,\{W\mathbb{1}\}\,Q - WQ. \end{aligned}$$

Then we have that

$$\begin{aligned}&\mathbb{E}\left[E^{(i,j)}Q\right] = \\ &= \sum_{(i,j)\,\in\,\mathcal{E}} W_{ij}(e_i - e_j)(e_i - e_j)^*Q(e_i - e_j)(e_i - e_j)^* \\ &= \sum_{(i,j)\,\in\,\mathcal{E}} W_{ij}(e_ie_i^* + e_je_j^*)Q(e_ie_i^* + e_je_j^*) \\ &\quad - \sum_{(i,j)\,\in\,\mathcal{E}} W_{ij}\left\{(e_ie_j^* + e_je_i^*)Q(e_ie_i^* + e_je_j^*) + (e_ie_i^* + e_je_j^*)Q(e_ie_j^* + e_je_i^*)\right\} \\ &\quad + \sum_{(i,j)\,\in\,\mathcal{E}} W_{ij}(e_ie_j^* + e_je_i^*)Q(e_ie_j^* + e_je_i^*) \end{aligned}$$

Consider now $\sum_{(i,j)\in\mathcal{E}} W_{ij}(e_ie_i^* + e_je_j^*)Q(e_ie_i^* + e_je_j^*)$. It follows that

$$\begin{aligned}
&\sum_{(i,j)\in\mathcal{E}} W_{ij}(e_ie_i^* + e_je_j^*)Q(e_ie_i^* + e_je_j^*) = \\
&\quad = \sum_{(i,j)\in\mathcal{E}} W_{ij}\left(e_ie_i^*Qe_ie_i^* + e_je_j^*Qe_je_j^*\right) + \sum_{(i,j)\in\mathcal{E}} W_{ij}\left(e_je_j^*Qe_ie_i^* + e_ie_i^*Qe_je_j^*\right) \\
&\quad = \sum_{(i,j)\in\mathcal{E}} \left(W_{ij}Q_{ii}e_ie_i^* + W_{ij}Q_{jj}e_je_j^*\right) + \sum_{(i,j)\in\mathcal{E}} \left(W_{ij}Q_{ji}e_je_i^* + W_{ij}Q_{ij}e_ie_j^*\right) \\
&\quad = \sum_{(i,j)\in\mathcal{E}} \left(W_{ij}Q_{ii}e_ie_i^* + W_{ij}Q_{jj}e_je_j^*\right) + \sum_{(i,j)\in\mathcal{E}} \left(W_{ji}Q_{ji}e_je_i^* + W_{ij}Q_{ij}e_ie_j^*\right) \\
&\quad = \underline{\text{diag}}\left\{W\mathbb{1}\right\}\text{diag}\left\{Q\right\} + W\odot Q
\end{aligned}$$

In a similar way it can be shown that

$$\begin{aligned}
&\sum_{(i,j)\in\mathcal{E}} W_{ij}\left\{(e_ie_j^* + e_je_i^*)Q(e_ie_i^* + e_je_j^*) + (e_ie_i^* + e_je_j^*)Q(e_ie_j^* + e_je_i^*)\right\} = \\
&\qquad = \left[\text{diag}\left\{Q\right\}W + W\text{diag}\left\{Q\right\}\right] + 2\text{diag}\left\{A(W\odot Q)\right\}
\end{aligned}$$

and that

$$\sum_{(i,j)\in\mathcal{E}} W_{ij}(e_ie_j^* + e_je_i^*)Q(e_ie_j^* + e_je_i^*) = W\odot Q + \underline{\text{diag}}\left\{(W\text{diag}\left\{Q\right\})\mathbb{1}\right\}.$$

Plugging together all the contributions we obtain (4.28). ■

We are able now to provide the proof of Theorem 4.11.
Proof: Consider the Equation (4.26). We have that

$$\begin{aligned}
\mathbb{E}\left[P(t)\Sigma(t)P(t)\right] &= \mathbb{E}\left[\left(I - \frac{1}{2}E_{ij}\right)\Sigma(t)\left(I - \frac{1}{2}E_{ij}\right)\right] \\
&= \Sigma_{yy} - \frac{1}{2}\mathbb{E}\left[E_{ij}\Sigma(t)\right] - \frac{1}{2}\mathbb{E}\left[\Sigma(t)E_{ij}\right] + \frac{1}{4}\mathbb{E}\left[E_{ij}\Sigma(t)E_{ij}\right]
\end{aligned}$$

Substituting in the above expectations the expressions given in Lemma 4.12 we obtain the recursive equation given in the statement of the Theorem. ■

Since the graph $\mathcal{G}$ satisfies Assumption 4.9 which guarantees, as previously observed, that the symmetric gossip algorithm (4.23) converges almost surely, we have that $\lim_{t\to\infty}\Sigma(t) = 0$ and hence also $\lim_{t\to\infty} d(t) = 0$. Unfortunately providing further theoretical analysis on the symmetric gossip

algorithms, for instance on the speed of convergence toward zero of $d(t)$, is quite hard in general. In [24] and in [58] the authors provide some bounds on the rate of convergence of $d(t)$ but these bound could be not tight.

However, further and interesting analytical developments in this direction can be made if we restrict the graph $\mathcal{G}$ to possess some symmetries. In the sequel we will work with the Cayley Graphs.

4.5.2 Mean-square Analysis for the Cayley Graphs

In this section we will show that when $\mathcal{G}$ is a Cayley graph, the analysis proposed above simplifies considerably. Let G be a finite Abelian group of order N and let S be a subset of G which does not contain zero. Let $\mathcal{G}(G,S)$ be the Cayley graph associated to the group G and the subset S as described in Section 3.3. Observe the fact that S does not contain zero implies that the graph $\mathcal{G}$ does not contain self-loops. In the following we will assume that $|S| = \nu$, namely that the degree of $\mathcal{G}$ is ν. Moreover we will assume that each edge of the edge set $\mathcal{E}$ of the Cayley graph $\mathcal{G}(G,S)$ can be selected with probability $1/|\mathcal{E}|$, where it is easy to see that $|\mathcal{E}| = \frac{2}{\nu N}$. Moreover observe that it follows that $W = \frac{2}{\nu N}A$. We start our analysis with the following simple result.

Lemma 4.13 *Consider (4.26). Assume that* $\Sigma(0) = \sigma^2 (I - 1/N \mathbb{1}\mathbb{1}^*)$. *Then* $\Sigma(t)$ *is a Cayley matrix for all* $t \geq 0$.

Proof: We prove the statement of the theorem by induction on t. Let $t = 0$. $\Sigma_{yy}(0)$ is Cayley since $(I - 1/N \mathbb{1}\mathbb{1}^*)$ is Cayley. Let now $\Sigma(t)$ be Cayley for a generic t. Consider $t+1$. Since A and W are Cayley it follows that all the matrices on the right-side of (4.26) are Cayley. Therefore also $\Sigma(t+1)$ is Cayley. ■

By using the fact that $W = \frac{2}{\nu N}A$ and the properties of the Cayley matrices we have (4.26) can be rewritten in a simpler way as

$$\Sigma(t+1) = \left(1 - \frac{2}{N}\right)\Sigma(t) + \frac{2}{\nu N}A\Sigma(t) + \frac{1}{N}\text{diag}\,\{\Sigma(t)\} + \frac{1}{\nu N}A \odot \Sigma(t) - \frac{1}{\nu N}\left[A\text{diag}\,\{\Sigma(t)\} + \text{diag}\,\{A\Sigma(t)\}\right] \quad (4.29)$$

Note now that a Cayley matrix is completely determined by the values that its generator π assume on the elements of the group G, equivalently by the

values of one of its row or column. In order to simplify further the problem we assume that the elements of G are ordered in some way, more precisely we assume that $G = \{g_0, g_1, \ldots, g_{N-1}\}$, where by convention $g_0 = 0$. Define the column vector in $\mathbb{R}^N$ as

$$\Pi(t) = \begin{bmatrix} \pi_{\Sigma(t)}(g_0) \\ \pi_{\Sigma(t)}(g_1) \\ \vdots \\ \pi_{\Sigma(t)}(g_{N-1}) \end{bmatrix} \tag{4.30}$$

In order to find a recursive relation for $\Pi(t)$, it is useful to introduce the $N \times N$ matrices B, C, D defined as follows

$$(B)_{ij} := \begin{cases} 1 & \text{if} \quad i = j \quad \text{and} \quad \pi_A(g_i) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.31}$$

$$(C)_{ij} := \begin{cases} 1 & \text{if} \quad i = j = 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.32}$$

and

$$(D)_{ij} := \begin{cases} 1 & \text{if} \quad i = 1 \quad \text{and} \quad \pi_A(g_j) = 1 \\ 1 & \text{if} \quad j = 1 \quad \text{and} \quad \pi_A(g_i) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.33}$$

Observe that $B = \underline{\text{diag}}\left\{e_1^* A\right\}, C = e_1 e_1^*$ and $D = e_1 e_1^* A + A e_1 e_1^*$ where $e_1 = [1, 0, \ldots, 0]^*$. We have the following result.

Theorem 4.14 *Let $\Pi(t)$ be as above defined. Then we have*

$$\Pi(t+1) = \left[\left(1 - \frac{2}{N}\right) I + \frac{2}{\nu N} A + \frac{1}{N} C + \frac{1}{\nu N} B - \frac{1}{\nu N} D\right] \Pi(t). \tag{4.34}$$

Proof: We start by observing that from the definition of A, B, C, D we have that

$$\begin{bmatrix} \pi_{A\Sigma(t)}(g_0) \\ \pi_{A\Sigma(t)}(g_1) \\ \vdots \\ \pi_{A\Sigma(t)}(g_{N-1}) \end{bmatrix} = A \begin{bmatrix} \pi_{\Sigma(t)}(g_0) \\ \pi_{\Sigma(t)}(g_1) \\ \vdots \\ \pi_{\Sigma(t)}(g_{N-1}) \end{bmatrix}, \tag{4.35}$$

$$\begin{bmatrix} \pi_{\text{diag}\{\Sigma(t)\}}(g_0) \\ \pi_{\text{diag}\{\Sigma(t)\}}(g_1) \\ \vdots \\ \pi_{\text{diag}\{\Sigma(t)\}}(g_{N-1}) \end{bmatrix} = C \begin{bmatrix} \pi_{\Sigma(t)}(g_0) \\ \pi_{\Sigma(t)}(g_1) \\ \vdots \\ \pi_{\Sigma(t)}(g_{N-1}) \end{bmatrix}, \tag{4.36}$$

$$\begin{bmatrix} \pi_{A\odot\Sigma(t)}(g_0) \\ \pi_{A\odot\Sigma(t)}(g_1) \\ \vdots \\ \pi_{A\odot\Sigma(t)}(g_{N-1}) \end{bmatrix} = B \begin{bmatrix} \pi_{\Sigma(t)}(g_0) \\ \pi_{\Sigma(t)}(g_1) \\ \vdots \\ \pi_{\Sigma(t)}(g_{N-1}) \end{bmatrix}, \tag{4.37}$$

and

$$\begin{bmatrix} \pi_{A\mathrm{diag}\{\Sigma(t)\}+\mathrm{diag}\{A\Sigma(t)\}}(g_0) \\ \pi_{A\mathrm{diag}\{\Sigma(t)\}+\mathrm{diag}\{A\Sigma(t)\}}(g_1) \\ \vdots \\ \pi_{A\mathrm{diag}\{\Sigma(t)\}+\mathrm{diag}\{A\Sigma(t)\}}(g_{N-1}) \end{bmatrix} = D \begin{bmatrix} \pi_{\Sigma(t)}(g_0) \\ \pi_{\Sigma(t)}(g_1) \\ \vdots \\ \pi_{\Sigma(t)}(g_{N-1}) \end{bmatrix}. \tag{4.38}$$

From (4.29) and from the above expressions we obtain (4.34). ■

For the sake of the notational convenience let

$$F := \left(1 - \frac{2}{N}\right) I + \frac{2}{\nu N} A + \frac{1}{N} C + \frac{1}{\nu N} B - \frac{1}{\nu N} D. \tag{4.39}$$

Notice that F is a doubly stochastic matrix and that $\mathbb{1}^* \Pi(t) = 0$ for all t. Clearly, since $\lim_{t\to\infty} \Sigma(t) = 0$ we have that also $\lim_{t\to\infty} \Pi(t) = 0$ and the speed of convergence of $\Pi(t)$ toward zero is given by the essential spectral radius of the matrix F that, accordingly to the definition provided in (2.12) we denote by $\rho_{ess}(F)$. Let now $\Pi_1(t)$ denote the first component of $\Pi(t)$. Observe that

$$d(t) = \Pi_1(t).$$

which implies that also the speed of convergence of $d(t)$ toward zero is determined by $\rho_{ess}(F)$. We have not been able to provide a theoretical characterization of $\rho_{ess}(F)$. However, supported by several numerical results, we conjecture the following.

Conjecture 4.15 *Let $\mathcal{G}$ be a graph satisfying Assumption 4.9, and let F be defined as in (4.39). Moreover let ν denote the degree of $\mathcal{G}$. Then*

$$\rho_{ess}(F) = 1 - \frac{C}{N^{\frac{4+\nu}{\nu}}} + o\left(\frac{1}{N^{\frac{4+\nu}{\nu}}}\right).$$

as $N \to \infty$, where C is a constant independent of the graph $\mathcal{G}$.

A possible way of proving the above Conjecture, could be by considering the matrix $\frac{1}{N}C + \frac{1}{\nu N}B - \frac{1}{\nu N}D$ as a perturbation of the matrix $\left(1 - \frac{2}{N}\right) I + \frac{2}{\nu N}A$. Indeed, we notice that the matrix $\frac{1}{N}C + \frac{1}{\nu N}B - \frac{1}{\nu N}D$ has only $3\nu + 1$ elements different from 0, which should become less and less significant as $N \to \infty$. Moreover note that, since $\left(1 - \frac{2}{N}\right) I + \frac{2}{\nu N}A$ is a Cayley doubly stochastic matrices, we could characterize precisely its essential spectral radius by the arguments illustrated in Section 3.3.1.

4.6 Conclusions

In this Chapter we have introduced the randomized consensus algorithms. We have reviewed the concept of probabilistic consensus and some algebraic conditions ensuring the achievement of this probabilistic consensus. We have then analyzed two particular random strategies, the time-varying strategy with bounded in-degree and the time-varying Cayley graphs strategy, proving that they allow to achieve better performance than deterministic strategies with comparable complexity. Finally we have considered the symmetric gossip algorithm, whose study is motivated by applications to sensors, peer-to-peer and *ad hoc* networks. In particular we have provided an interesting characterization of the symmetric gossip algorithm over the Cayley graphs.

Some important questions remain open. In particular, a detailed analysis of the convergence rate of the symmetric gossip algorithms needs to be carried out. We believe that quite rich and complete results can be obtained for the broad class of graphs possessing symmetries the Cayley graphs. Another interesting family of graphs to work on would be the random geometric graphs.

Chapter 5

Quantized Consensus: Time-Invariant case

5.1 Introduction

We have seen in the Introduction, in Chapter 2 and in Chapter 4 that different algorithms for average consensus have been proposed in the literature. They can be distinguished on the basis of the amount of communication and computation they require, of their scalability properties with respect to the number of nodes, of their adaptability to time-varying graphs, and, finally, on the basis of their deterministic or randomized operating protocol.

However there is a feature common to most of the literature on the consensus problem: the assumption that the communication channel between the nodes allows to transfer real numbers with no errors. In many practical applications this is not a realistic assumption: if we think, for instance, of sensor networks communicating in a wireless fashion, it is evident that energy and bandwidth limitation yield a finite capacity channel. This suggests that in many situations the communication channel should be rather considered as digital, accepting messages taking values in some finite alphabet. This clearly forces a quantization on the real numbers that agents have to transmit.

The effects of quantization in feedback control problems have been widely studied in the past [91], mainly in the stabilization problem. Moreover granularity effects different from quantization in the consensus problems have been tackled in few papers especially in the load balancing applications [54]. Our setting is however different: this chapter studies the consensus problem

under the assumption that transmissions are quantized, in particular we assume here that the communications network is constituted of ideal noiseless digital channels.

We review now the literature on the quantized consensus. The investigation of consensus under quantized communication started with [75]. In this paper the authors study systems having (and transmitting) integer-valued states and propose a class of gossip algorithms which preserve the average of states and are guaranteed to converge up to one quantization bin. Besides the fact there is not precise consensus, since the algorithm requires the use of a single link per iteration, the convergence is slow.
The authors in [150] analyzed the impact of the quantization noise through modification of the consensus algorithm proposed in [148], where the case of noisy communication links is addressed. Precisely, the authors in [148] consider the case in which the information transmitted by each node is noise-corrupted, being the noise additive and zero-mean. In [150] it is noted that the noise component can be considered as the quantization noise and by simulations, it is shown for small N that, if the increasing correlation among the node states is taken into account, the variance of the quantization noise diminishes and nodes converge to a consensus.
In [33] the authors propose a distributed algorithm that uses quantized values and preserves the average at each iteration. They showed favorable convergence properties using simulations on some static topologies, and provided performance bounds for the limit points of the generated iterates. In [92] the more general case of time-varying topologies is considered. A scheme reaching the consensus is proposed even if not at the initial average. However polynomial bounds on both the convergence time and the discrepancy from the initial average are provided in terms of the number of quantization levels.
Of note is that all the paper mentioned above considered quantized strategy based on deterministic uniform quantizers. Reference [8] proposes a simple distributed and iterative scheme to compute the average at each sensor node utilizing only quantized information communication. The authors in [8] adopt the probabilistic quantization (PQ) scheme (see [146]) to quantize the information before transmitting to the neighboring sensors. They show that the node states reach consensus to a quantized level; only in expectation do they converge to the desired average. Moreover if the quantization step size is large this approach will lead to large residual errors.

In this chapter and in the next two, the main goal is to provide an analysis identifying what are the effects due to the presence of the digital transmission and trying to overcome these effects. We will do this by following the analysis

presented in [61] which extends the algorithm appeared in [33, 34].

The first problem we will address is to understand what happens when in the equation (2.1), ruling the average consensus algorithm, in place of the exact values of the states of the systems we substitute a quantized version of these obtained through a deterministic uniform quantizer or a probabilistic quantizer (we will define rigorously these quantizers in Section (5.2)). Two questions mainly are in order:

(i) is it still possible to reach the average consensus, and

(ii) if not, how far is it the asymptotic behavior of the states of the systems from the average consensus?

Answering the above questions is quite challenging. We will try to provide some insights on the above issues in this and the next two chapters.

We postpone the explanation of the organization of this chapter inside the next Section, where we will formulate precisely the problem we are dealing with.

5.2 Problem formulation

In this chapter we consider the consensus algorithm (2.10) restricting to the case in which the consensus matrix P is doubly stochastic. For the sake of the clarity we briefly review it. Assume that we have a set V of systems and a graph $\mathcal{G}$ on V describing the feasible communications among the systems. For each system $i \in V$ let $x_i(t)$ denote the estimate of the average of agent i at time t. We assume that at every times t agent i updates its estimate according to

$$x_i(t+1) = \sum_{j=1}^{N} P_{ij} x_j(t). \tag{5.1}$$

where P_{ij} are the elements of a doubly stochastic $P \in \mathbb{R}^{N\times N}$ matrix compatible with $\mathcal{G}$. More compactly we can write

$$x(t+1) = Px(t), \tag{5.2}$$

where $x(t)$ is the column vector entries $x_i(t)$ represent the agents states.
We have seen in Section 2.2.1 that, if P is a doubly stochastic matrix with

positive diagonal entries and such that $\mathcal{G}_P$ is strongly connected, then the algorithm (5.2) solves asymptotically the *average consensus problem*, namely

$$\lim_{t\to+\infty} x(t) = \left(\frac{1}{N}\sum_{i=1}^{N} x_i(0)\right)\mathbb{1}, \tag{5.3}$$

where $\mathbb{1}$ is the column vector of all ones. From now on we will assume the following.

Assumption 5.1 *P is a doubly stochastic stochastic matrix such that $P_{ii} > 0$, for $i \in \{1,\dots,N\}$, and such that $\mathcal{G}_P$ is strongly connected.*

Note that the algorithm (5.2) relies upon a crucial assumption: each agent transmits to its neighboring agents the precise value of its state. This implies the exchange of perfect information through the communication network.
In what follows, we consider a more realistic case, i.e., we assume that the communication network is constituted only of rate-constrained digital links. This prevents the agents from having a precise knowledge about the state of the other agents. In fact, through a digital channel, the i-th agent can only send to its neighbors symbolic data in a finite alphabet; using only this data, the neighbors of the i-th agent can build at most an estimate of the i-th agent's state. In the following we will denote this estimate by $\hat{x}_i(t)$. Let

$$\hat{x}(t) = \begin{bmatrix} \hat{x}_1(t) \\ \vdots \\ \hat{x}_N(t) \end{bmatrix}$$

By remarking that (7.1) and (7.2) can be written respectively as

$$x_i(t+1) = x_i(t) + \sum_{j=1}^{N} P_{ij}\left(x_j(t) - x_i(t)\right)$$

and

$$x(t+1) = x(t) + (P - I)x(t),$$

in this chapter we consider the following updating rule

$$x_i(t+1) = x_i(t) + \sum_{j=1}^{N} P_{ij}\left(\hat{x}_j(t) - \hat{x}_i(t)\right) \tag{5.4}$$

or more compactly

$$x(t+1) = x(t) + (P - I)\hat{x}(t). \tag{5.5}$$

It is easy to see that this law has the remarkable property of maintaining the initial state average, whatever $\hat{x}(t)$ is. This fact is stated in the following proposition.

Proposition 5.2 *Consider (5.5). Let*

$$x_{ave}(t) = \frac{1}{N}\mathbb{1}^* x(t).$$

Then

$$x_{ave}(t) = x_{ave}(0)$$

for all $t \geq 0$.

Proof: We have that

$$\begin{aligned} \mathbb{1}^* x(t+1) &= \mathbb{1}^* x(t) + \mathbb{1}^*(P - I)\hat{x}(t) \\ &= \mathbb{1}^* x(t) \end{aligned}$$

where the last inequality follows from the fact, since P is doubly stochastic, it holds $\mathbb{1}^*(P - I) = 0$. ■

We proceed now by illustrating two types of quantizers which has been introduced in literature in order to transmit information throughout a digital channel. In [33], the authors analyze the case in which

$$\hat{x}_i(t) = q_d(x_i(t)), \tag{5.6}$$

where, given a real number z, $q_d : \mathbb{R} \to \mathbb{Z}$ is the mapping sending z to its nearest integer, namely,

$$q_d(z) = n \in \mathbb{Z} \quad \Leftrightarrow \quad \begin{array}{ll} z & \in [n - 1/2, n + 1/2[, \text{ if } z \geq 0 \\ z & \in]n - 1/2, n + 1/2], \text{ if } z < 0. \end{array} \tag{5.7}$$

We will refer to this quantizer as *deterministic quantizer*[1].
Instead in [8, 146], the so-called *probabilistic quantizer* is introduced. This quantizer woks as follows. Let $x \in \mathbb{R}$ and let $q_p(\cdot)$ denote the *probabilistic quantizer*[2]. Assume, as for the *deterministic quantizer* above described, that the set of quantization levels is composed by the integer numbers, then $q_p(x)$ is the binary random variable defined as follows

$$q_p(x) = \begin{cases} \lfloor x \rfloor \text{ with probability } \lceil x \rceil - x \\ \lceil x \rceil \text{ with probability } x - \lfloor x \rfloor \end{cases} \tag{5.8}$$

[1] the subscript d means *deterministic*
[2] The subscript p means *probabilistic*

The following lemma discusses two important properties of the *probabilistic quantizer* (for the proof see [146]).

Lemma 5.3 *Let $q_p(x)$ be a probabilistic quantization of $x \in \mathbb{R}$. Then $q_p(x)$ is an unbiased representation of x, i.e.,*

$$\mathbb{E}\left[q_p(x)\right] = x. \tag{5.9}$$

Moreover

$$\mathbb{E}\left[(x - q_p(x))^2\right] \leq \frac{1}{4}. \tag{5.10}$$

From now on, given a vector $x \in \mathbb{R}^N$, with a slight abuse of notation, we will use the notation $q_d(x) \in \mathbb{R}^N$ (respectively $q_p(x) \in \mathbb{R}^N$) to denote the vector such that $q_d(x) = [q_d(x_1), \ldots, q_d(x_N)]^*$ (respectively $q_p(x) = [q_p(x_1), \ldots, q_p(x_N)]^*$).
The main goal of this chapter will be to analyze the following systems

$$x(t+1) = x(t) + (P - I)q_d(x(t)) \tag{5.11}$$

and

$$x(t+1) = x(t) + (P - I)q_p(x(t)). \tag{5.12}$$

Some remarks are now in order.

Remark 5.4 In this paper we consider quantizers having quantization step equal to 1. More general quantizers, with quantization step a generic positive real number ϵ, can be obtained from q_d and q_p by defining $q_d^{(\epsilon)}(x) = \epsilon q_d(x/\epsilon)$ and $q_p^{(\epsilon)}(x) = \epsilon q_p(x/\epsilon)$. Namely, the general case can be simply recovered by a suitable scaling. For the sake of the completeness, it is worth noting that in this case

$$\mathbb{E}\left[(x - q_p^{(\epsilon)}(x))^2\right] \leq \frac{\epsilon^2}{4}.$$

Remark 5.5 We could define two different state update equations. By assuming that each node i uses the exact knowledge of its own state x_i instead of the estimate $\hat{x}_i$ we could adopt

$$\begin{aligned} x_i(t+1) &= x_i(t) + \sum_{j=1}^{N} P_{ij}\left(\hat{x}_j(t) - x_i(t)\right), \\ &= P_{ii}x_i(t) + \sum_{j \neq i} P_{ij}\hat{x}_j(t), \end{aligned} \tag{5.13}$$

instead of equation (5.4). Note that (7.9) can be written, in vector form, as

$$x(t+1) = \text{diag}\{P\}\, x(t) + (P - \text{diag}\{P\})\, \hat{x}(t),$$

where $\text{diag}\{P\}$ means a diagonal matrix with the same diagonal elements of the matrix P. Instead, by assuming that each node i uses only quantized information also related to its own state, we could adopt

$$\begin{aligned} x_i(t+1) &= \hat{x}_i(t) + \sum_{j=1}^{N} P_{ij}\left(\hat{x}_j(t) - \hat{x}_i(t)\right), \\ &= \sum_{j=1}^{N} P_{ij}\hat{x}_j(t), \end{aligned} \tag{5.14}$$

or in vector form

$$x(t+1) = P\hat{x}(t). \tag{5.15}$$

The drawback of these choices is that, differently from (5.5), they do not preserve the average of the initial condition. However, we will come back on these laws at the end of this chapter.

Remark 5.6 The authors in [8] analyze the rule (5.14), when the estimates are given by the *probabilistic quantizer*, namely the algorithm

$$x(t+1) = Pq_p(x(t)). \tag{5.16}$$

They prove that (5.16) reaches, almost surely, consensus on a quantization level. However the distance of the consensus point from the initial average depends heavily on the matrix P. We will review later the main features of this strategy.

It is worth noting that both the algorithm (5.11) and the algorithm (5.12), because of the quantization effects, are not expected to converge in the sense (5.3). What we can hope is for the agents to reach states which are close to each other and close to the average $x_a(0)$. To measure this asymptotic disagreement, we introduce the following quantity $\Delta_i(t) := x_i(t) - x_a(t) = x_i(t) - x_a(0)$ which represents the distance, at time t, of the i-th agent from the average of the initials conditions. Let now $Y = I - N^{-1}\mathbb{1}\mathbb{1}^*$ and $\Delta(t) = [\Delta_1(t), \ldots, \Delta_N(t)]^*$. Then, $\Delta(t) = Y\, x(t)$. Finally we define the performance index

$$d_\infty(P, x(0)) = \limsup_{t\to\infty} \frac{1}{\sqrt{N}} ||\Delta(t)||. \tag{5.17}$$

Note that the factor $\frac{1}{\sqrt{N}}$ is introduced in order to scale $d_\infty(P, x(0))$ with the dimension of the network. Moreover we can get rid of the initial condition by considering

$$d_\infty(P) = \sup_{x(0)} d(P, x(0)). \tag{5.18}$$

The problem we would like to address is, given a matrix P, to evaluate how big d_∞ is. In particular, we are interested in evaluating how, for sequences of graphs of increasing size, this quantity depends on the number of the nodes N.

In order to obtain a first insight in the problem, we provide immediately some simulative results regarding two particular communication topologies: the random geometric graph and the hypercube graph.

The random geometric graph is commonly used to model wireless networks [67]. It is constructed by randomly placing N nodes in the unit square, and joining them with edges whenever their distance is below a threshold $R = \Theta(\sqrt{\log N/N})$ for $N \to \infty$. Moreover we assume the weights of the matrix P are chosen by using the Metropolis weights, illustrated in Example 2.14.

The n-dimensional hypercube graph is the graph obtained drawing the edges of a n-dimensional hypercube. It has $N = 2^n$ nodes which can be identified with the binary words of length n. Two nodes are neighbors if the corresponding binary words differ in only one component. Thus every node exchanges information with other n nodes. In this case a matrix P can be constructed by setting equal every non-zero entry of P. We will be more precise in Section 5.3.1.

The behavior of $d_\infty(P)$, which is obtained as the average over several realizations on the initial conditions, for sequences of graphs of increasing size of both topologies is depicted in Figures 5.1 and 5.2 for the strategy using the deterministic quantizer, and Figures 5.3 and 5.5 for the strategy using the probabilistic quantizer. It is worth noting that remarkably in all the cases, $d_\infty(P)$ appears to be bounded on N, the size of the communication graph. Moreover the two different quantizers behave quite similarly for both the topologies.

We conclude this section with some considerations and explaining how the rest of the chapter is organized.
Obviously we expect that $d_\infty(P)$ depend on the particular communication topology we are considering and on the weights assigned to the matrix P. In general, assigned the communication topology, what one would like to do is

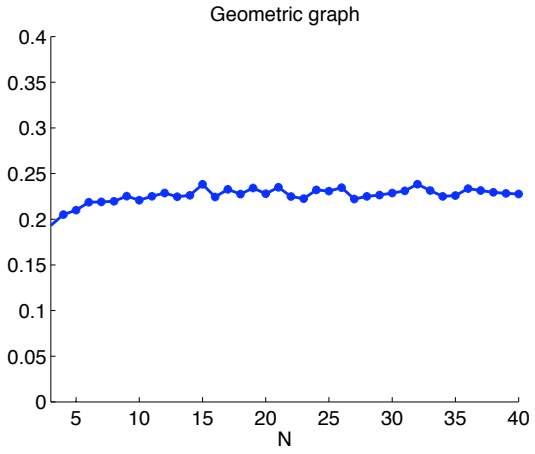

Figure 5.1: Performance of the random geometric graph when the deterministic quantizer is used. Since the graph itself is random, the plot of $d_\infty(P)$ comes from averaging over realizations of both the graph and the initial condition.

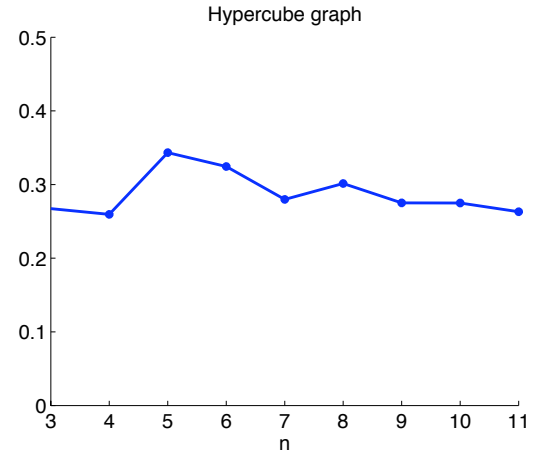

Figure 5.2: Performance of the n-dimensional hypercube graph (of order $N = 2^n$) when the deterministic quantizer is used.

finding the matrix of weights P minimizing $d_\infty(P)$. In order to do so, it is important to provide some tools which permit to estimate $d_\infty(P)$.

In spite of the fact that the proposed algorithms are still intrinsically linear, the quantization effects introduce nonlinearities which make the exact asymptotic analysis of the algorithm, quite hard. Indeed, we are able to carry

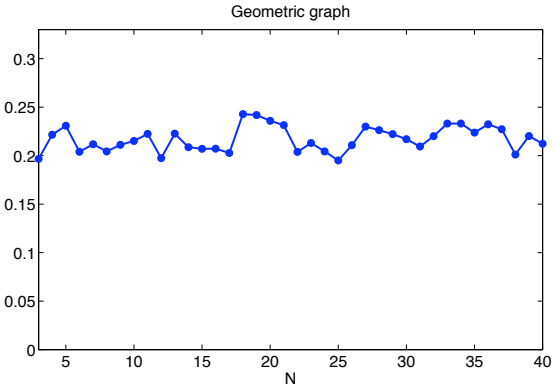

Figure 5.3: Performance of the random geometric graph when the probabilistic quantizer is used. Since the graph itself is random, the plot of $d_\infty(P)$ comes from averaging over realizations of both the graph and the initial condition.

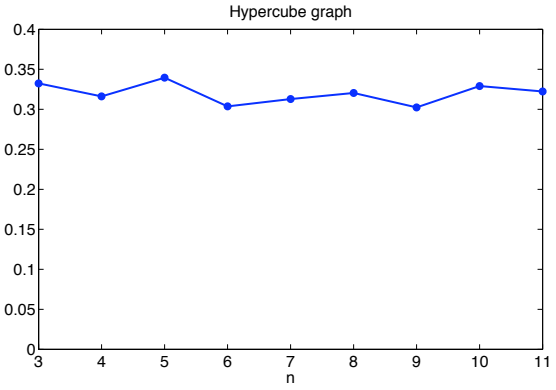

Figure 5.4: Performance of the n-dimensional hypercube graph when the probabilistic quantizer is used.

on such an analysis only in very specific examples.

In Section 5.3 we undertake a twofold analysis of the algorithm using deterministic quantizers: worst case and probabilistic. The basic idea is to study the system considering the communication errors induced by quantization as (unknown) bounded disturbances. The worst case analysis is obtained by introducing a bounded error model and maximizing $d_\infty(P)$ with respect to all possible realizations of the communication errors: of course in this way we obtain an upper bound to the performance of our algorithm. This

is done in in Section 5.3.1 where we prove in general the convergence to a neighborhood of the average, obtaining bounds on its size. These bounds are independent of the initial condition but depend on the diffusion matrix. However the worst-case analysis is intrinsically conservative. In fact we will show, for the hypercube example cited above, that the worst-case displacement grows logarithmically in N, behavior which is in disagreement with the experimental evidence displayed by Figure 5.2.

For this reason, in Section 5.3.2 we propose an alternative method and develop a probabilistic analysis, modeling the quantization error as additive random noise affecting the received data. A classical mean squared analysis for the asymptotic error can be carried on in this case. A similar analysis is done in [148]. It comes out that, under mild assumptions on the diffusion matrix, the expected behavior depends only on the assumed distribution of the errors and on the spectrum of the evolution matrix. Of course, this probabilistic analysis, in principle, does not offer any rigorous bound on our system. However, simulations clearly show that the probabilistic analysis is very close to the experimental evidence, contrarily to the worst case analysis which instead is quite conservative.

In spite to the fact that there is no theoretical reason explaining why the probabilistic model should be a suitable model for analyzing the algorithm (5.11), we will show in Section 5.4 that the probabilistic model, due to structure of the probabilistic quantizer, represent the correct way to analyze (5.12). In particular we will compare (5.12) with another scheme proposed in literature based on the use of probabilistic quantizers, and we will show how plugging together these two schemes it is possible to solve, almost surely, the consensus problem, very close to the average of the initial conditions.

5.3 Time Invariant case- Deterministic quantizers

Consider (5.11), i.e. the updating law using the deterministic quantizer. We start with a remark about the best achievable performance. It is clear that, when all states lie in the same quantization interval, namely $q(x_i(t)) = Q$ for all i, differences are not perceivable and states do not evolve. Therefore the best the algorithm can assure is that the system reaches such an equilibrium in which $q(x_i(t)) = Q$ for all i and for all $t \geq T$. In this case we would obtain that $|\Delta_i(t)| \leq 1$ for all $t \geq T$. This implies that the best we can obtain is $d_\infty \leq 1/2$. Unfortunately simulations show that in many cases the

error from the agreement can be bigger. However, it is worth noting that the states of the agents subject to (5.11) are bounded. In particular one can see, by convexity arguments, that for any node $i \in V$ and for all $t \geq 0$,

$$x_i(t) \in \left[\min_{j\in V}\{q(x_j(0))\} - \frac{1}{2},\ \max_{j\in V}\{q(x_j(0))\} + \frac{1}{2}\right]. \tag{5.19}$$

Of course (5.19) is a very weak result since we can hope that in general the disagreement will decrease as time goes on and that the asymptotic disagreement will not depend on the initial conditions, but just possibly on P. We consider now two examples in which the evolution of the system can be studied explicitly, the complete graph and the directed circuit graph.

Example 5.7 (Complete graph) If the communication graph is complete and the communication is exact, i.e. not quantized, the average consensus problem can be solved in one step taking $P = \frac{1}{N}\mathbb{1}\mathbb{1}^*$. We now compute the exact performance degradation due to quantization. The system is in this case

$$x(t+1) = x(t) - q(x(t)) + N^{-1}\mathbb{1}\mathbb{1}^* q(x(t)). \tag{5.20}$$

We have that, for $t \geq 1$,

$$\begin{aligned}
|\Delta_i(t)| &= \left| x_i(t) - \frac{1}{N}\sum_{j=1}^{N} x_j(0) \right| \\
&= \left| x_i(t) - \frac{1}{N}\sum_{j=1}^{N} x_j(t-1) \right| \\
&= \left| x_i(t-1) - q\left(x_i(t-1)\right) + \frac{1}{N}\sum_{j=1}^{N} q\left(x_j(t-1) - \frac{1}{N}\sum_{j=1}^{N} x_j(t-1)\right) \right| \\
&\leq |x_i(t-1) - q\left(x_i(t-1)\right)| + \frac{1}{N}\sum_{j=1}^{N} |q\left(x_j(t-1) - x_j(t-1)\right| \\
&= \frac{1}{2} + \frac{1}{N}\frac{N}{2} = 1.
\end{aligned}$$

This implies that

$$d_\infty(N^{-1}\mathbb{1}\mathbb{1}^*) \leq 1. \tag{5.21}$$

Example 5.8 (Directed circuit) Now we consider a more interesting example, the directed circuit graph, which is described by the Cayley graph

$\mathcal{G}(\mathbb{Z}_N, \{0,1\})$ (see Section 3.3). In this case each agent communicates with only one neighbor and evolves following

$$x_i(t+1) = x_i(t) + \frac{1}{2}[q(x_{i+1}(t)) - q(x_i(t))] \quad i = 1, \dots, N, \tag{5.22}$$

where summation of the indexes is to be intended mod N. The evolution of (5.22) can be studied exactly by means of a *symbolic dynamics* approach. This analysis is definitively not trivial, but permits us to characterize precisely the evolution of (5.22) and to obtain a strong result.
To start, we need the following technical lemma. Let $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling operators from $\mathbb{R}$ to $\mathbb{Z}$.

Lemma 5.9 *Given $\alpha, \beta \in \mathbb{N}$ and $x \in \mathbb{R}$, it holds*

$$\lfloor x \rfloor = \left\lfloor \frac{\lfloor \alpha x \rfloor}{\alpha} \right\rfloor \tag{5.23}$$

$$q(x) = \lfloor x + 1/2 \rfloor = \left\lceil \frac{1}{2} \left\lfloor \frac{\lfloor 2\beta x \rfloor}{\beta} \right\rfloor \right\rceil. \tag{5.24}$$

Proof: We first prove (5.23). Let $m = \lfloor x \rfloor$. So

$$m \leq x < m + 1$$
$$\alpha m \leq \alpha x < \alpha m + \alpha.$$

Hence, we can find $s \in \mathbb{N}, 0 \leq s \leq \alpha - 1$ such that $\alpha m + s \leq \alpha x < \alpha m + s + 1$. This yields $\lfloor \alpha x \rfloor = \alpha m + s$ and $\left\lfloor \frac{\lfloor \alpha x \rfloor}{\alpha} \right\rfloor = m$.

Then we prove equation (5.24). The equality $q(x) = \lfloor x + 1/2 \rfloor$ is clear from the definition of $q(x)$. To prove the second equality, let $h = \lfloor 2x \rfloor$. Then $h \leq 2x < h + 1$, from which follows that

$$\frac{h}{2} + \frac{1}{2} = \frac{h+1}{2} \leq x + 1/2 < \frac{h+2}{2} = \frac{h}{2} + 1.$$

From this inequality it follows that $\lfloor x + 1/2 \rfloor = \lceil \frac{h}{2} \rceil$. This, with (5.23), implies (5.24).

■

We define now $n_i(t) = \lfloor 2x_i(t) \rfloor$. Simple properties of floor and ceiling operators, together with the above lemma, allow us to remark that $q(x_i(t)) = \left\lceil \frac{n_i(t)}{2} \right\rceil$ and to derive from (5.22) that

$$\lfloor 2x_i(t+1) \rfloor = \lfloor 2x_i(t) \rfloor + q(x_{i+1}(t)) - q(x_i(t))$$

and hence

$$n_i(t+1) = n_i(t) + \left\lceil \frac{n_{i+1}(t)}{2} \right\rceil - \left\lceil \frac{n_i(t)}{2} \right\rceil$$
$$= \left\lfloor \frac{n_i(t)}{2} \right\rfloor + \left\lceil \frac{n_{i+1}(t)}{2} \right\rceil .$$

We have thus found an iterative system involving only the symbolic signals $n_i(t)$:

$$n_i(t+1) = g(n_i(t), n_{i+1}(t)) \tag{5.25}$$

where

$$g(h,k) = \left\lfloor \frac{h}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil . \tag{5.26}$$

The asymptotic analysis of (5.25) will then allow us to obtain information about the asymptotics of $x_i(t)$ (since $n_i(t) = \lfloor 2x_i(t) \rfloor$) up to quantization errors equal to 1.

We now start the analysis of system (5.25). Define the following quantities: $m(t) = \min_{1\le i\le N} n_i(t)$, $M(t) = \max_{1\le i\le N} n_i(t)$, $D(t) = M(t) - m(t)$. From the form of (5.25) one can easily remark that $m(t)$ can not decrease and $M(t)$ can not increase. Hence $D(t)$ is not increasing. A much stronger result about the monotonicity of $D(t)$ is the content of the following lemma.

Lemma 5.10 *If $D(t_0) \ge 2$, there exists $T \in \mathbb{N}$ such that $D(t_0+T) < D(t_0)$.*

Proof: Let $I_m(t) = \{j \in \mathbb{Z}_N \text{ s.t. } n_j(t) = m(t)\}$. The idea of the proof will be to show that the set $I_m(t)$ eventually decreases if we are in the range $D(t) \ge 2$.

Notice first that, for $h, k \in \mathbb{Z}$, $g(h+2, k+2) = g(h,k) + 2$. Hence, by an appropriate translation of the initial condition, we can always restrict ourselves to the case $m(0) \in \{0, 1\}$.

Case $m(t_0) = 0$. Notice that

$$g(h,k) > 0 \ \forall h \ge 0,\ k > 0, \quad g(h,0) > 0 \ \forall h \ge 2 . \tag{5.27}$$

This easily implies that $I_m(t)$ is not increasing. Now, since $D(t_0) \ge 2$, we can find $j_0 \in I_m(t_0)$ and two integers $U > 0$ and $W \ge 0$ such that

$$\begin{aligned} &n_{j_0-W-1}(t_0) > 1 \\ &n_{j_0-v}(t_0) = 1 \quad 0 < v \le W \\ &n_{j_0+u}(t_0) = 0 \quad 0 \le u < U \\ &n_{j_0+U}(t_0) > 0. \end{aligned}$$

After W instants step, we then obtain,

$$\begin{aligned} &n_{j_0-W-1}(t_0+W) > 1 \\ &n_{j_0-u}(t_0+W) = 0 \quad W-U+1 \le u \le W \\ &n_{j_0+W-U}(t_0+W) > 0. \end{aligned}$$

The time after one 0 will then disappear

$$\begin{aligned} &n_{j_0-W-1}(t_0+W+1) > 0 \\ &n_{j_0-u}(t_0+W+1) = 0 \quad W-U+2 \le u \le W \\ &n_{j_0+W-U-1}(t_0+W+1) > 0. \end{aligned}$$

This implies that $|I_m(t_0+W+1)| < |I_m(t_0)|$.

Case $m(t_0) = 1$. Notice that

$$g(h,k) > 1 \; \forall h \ge 2,\, k \ge 1, \quad g(1,k) > 1 \; \forall k \ge 3\,. \tag{5.28}$$

This easily implies that $I_m(t)$ is again not increasing. Now, since $D(t_0) \ge 2$, we can find $j_0 \in I_m(t_0)$ and an integer $W \ge 0$ such that

$$\begin{aligned} &n_{j_0}(t_0) = 1 \\ &n_{j_0+w}(t_0) = 2 \quad 1 \le w \le W \\ &n_{j_0+W+1}(t_0) > 2. \end{aligned}$$

The evolution of the above configuration yields, after W instant steps

$$\begin{aligned} &n_{j_0}(t_0+W) = 1 \\ &n_{j_0+1}(t_0+W) > 2. \end{aligned}$$

The next step, we obtain $n_{j_0}(t_0+W+1) > 1$. Therefore, $|I_m(t_0+W+1)| < |I_m(t_0)|$.

In both cases we have proven that $|I_m(t)|$ strictly decreases in finite number of steps. A straightforward induction principle then implies that a finite $T \in \mathbb{N}$ exists such that $m(t_0+T) > m(t_0)$. This proves the result. ■

The interesting consequence of this lemma is the following result which characterizes the asymptotic behavior of the variable n_i.

Theorem 5.11 *There exist* $T \in \mathbb{N}$ *and* $h \in \mathbb{Z}$ *such that, for all* $t > T$, $D(t) < 2$ *and, moreover, one of the following condition holds*

i) $n_i(t) = h$, $\forall i$;

ii) $\{n_i(t)\ i = 0, \dots N\} = \{h, h+1\}$ *and each* $n_i(t)$ *is constant in time;*

iii) $\{n_i(t)\ i = 0, \dots N\} = \{h, h+1\}$ *and each* $n_i(t)$ *is periodic in time of period* N.

Proof: From Lemma 5.10, it follows that a finite $T \in \mathbb{N}$ can be found, such that $D(t) < 2$ for all $t > T$. Once we reach his condition, there are two possibilities: either the $n_i(t)$ are all equal or they differ by 1. In the first case, the system remains constant (case i). In the second case, it follows from the way g is defined that if the lowest state is odd, the evolution is constant (case ii), while if the lowest one is even, the state evolution is a leftward shift (case iii). This is periodic of period N (and possibly also of some divisor of N). ■

We can now go back to the original system. The following result follows directly from Theorem 5.11.

Corollary 5.12 *For system (5.22), there exists* $T \in \mathbb{N}$ *such that*

$$|x_i(t) - x_j(t)| \leq 1 \quad \forall\, i, j \quad \forall\, t > T, \tag{5.29}$$

and hence

$$d_\infty(P) \leq 1/2.$$

Proof: Immediate consequence of Theorem 5.11, considering the relation $n_i(t) = \lfloor 2x_i(t) \rfloor$. ■

Unfortunately an exact analysis of the dynamics of system (5.11), as we did in the previous two examples, it is not feasible for general graphs. In the following subsections we will try to introduce some auxiliary models in order to provide some bounds on d_∞ and some further considerations.

5.3.1 Bounded error model-Worst case analysis

In this section we undertake a worst case analysis which can be applied general graphs. We start by observing that (5.11) can be rewritten in the following way

$$x(t+1) = Px(t) + (P - I)(q_d(x(t)) - x(t)), \tag{5.30}$$

where $q_d(x(t)) - x(t)$ is such that $\|q_d(x(t)) - x(t)\|_\infty \leq 1/2$. In order to carry out a worst-case analysis of (5.30), we introduce the following bounded error model

$$\begin{cases} x_w(t+1) = Px_w(t) + (I-P)e(t), & x_w(0) = x(0) \\ \Delta_w(t) = Yx_w(t) \end{cases} \tag{5.31}$$

where $e(t) \in \mathbb{R}^N$ is such that $\|e(t)\|_\infty \leq 1/2$ for all $t \geq 0$ and where we again $Y = I - \frac{1}{N}\mathbb{1}\mathbb{1}^*$. Notice that in this case $e(t)$ is no more a quantization error, but instead represents an unknown bounded disturbance. Clearly, when $e(t) = q(x(t)) - x(t)$ it turns out that $x_w(t) = x(t)$ and $\Delta_w(t) = \Delta(t)$ for all $t \geq 0$.

We define now a performance index for (5.31), considering the worst asymptotic disagreement, worst with respect to all the possible choices of the time sequence of the vectors $e(t)$. To be more precise, let us introduce $\mathcal{E}^\infty = \left\{\{e(\cdot)\}_{t=0}^\infty \,|\, \|e(t)\|_\infty \leq \frac{1}{2},\ \forall\, t \geq 0\right\}$, namely the set of all the sequences of N-dimensional vectors having sup norm less than $1/2$. Then, for the system (5.31), we define

$$d_\infty^w(P, x_w(0)) = \sup_{\mathcal{E}^\infty} \limsup_{t\to\infty} \frac{1}{\sqrt{N}} ||\Delta_w(t)||. \tag{5.32}$$

Note that $\lim_{t\to\infty} YP^t = 0$. This implies that the asymptotic behavior of $\Delta_w(t)$ is independent of the initial condition $x_w(0)$ and hence this is the case also for the quantity $d_\infty^w(P, x_w(0))$. Thus, from now on we will denote $d_\infty^w(P, x_w(0))$ simply by $d_\infty^w(P)$. As a preliminary remark, note that

$$d_\infty(P) \leq d_\infty^w(P). \tag{5.33}$$

We start our analysis of $d_\infty^w(P)$ by the following example.

Example 5.13 In this example we consider the hypercube graph. Precisely, we consider the group $\mathbb{Z}_2^n$ where $2^n = N$ and the Cayley graph $\mathcal{G}(\mathbb{Z}_2^n, S)$, where $S = \{e_0, e_1, \ldots, e_n\}$, with $e_0 = [0, \ldots, 0]^*$ and e_j the vector with all elements equal to 0 except a 1 in position j if $j \neq 0$. Clearly $|S| = n+1$. We assume that the matrix P has the following structure

$$P_{ij} = \begin{cases} \frac{1}{n+1} & \text{if } i - j = e_h \ \exists\ h\ :\ 0 \leq h \leq n, \\ 0 & \text{otherwise} \end{cases} \tag{5.34}$$

for all i and j belonging to $\mathbb{Z}_2^n$. In other words we have that $P = \frac{1}{n+1}A$ where A is the adjacency matrix of the Cayley graph $\mathcal{G}(\mathbb{Z}_2^n, S)$. For the sake of the

clarity we recall that, according to our convention, in the adjacency matrix we consider also the presence of self-loops (see Appendix B). We have the following result.

Theorem 5.14 *Let P be as above. Then*

$$d_\infty^w(P) = \frac{n}{2} = \frac{\log_2 N}{2}$$

where given a matrix $M \in \mathbb{R}^{N\times N}$, $\rho(M)$ denotes the spectral radius of M.

In order to prove the above Theorem we need of the following preliminary result.

Lemma 5.15 *Let P be as above. Then*

$$\sum_{s=0}^{\infty} \rho(P^s(I-P)) = n = \log_2 N, \tag{5.35}$$

where we recall that, according to definition (2.12), given a matrix $M \in \mathbb{R}^{N\times N}$ $\rho_{ess}(M)$ denotes the essential spectral radius of M.

Proof: The eigenvalues of P are $\lambda_k = 1-\frac{2k}{n+1} \quad k = 0\dots n$, with multiplicities $p_k = \binom{n}{k}$ (see [43]). Then,

$$\sum_{s=0}^{\infty} \rho(P^s(I-P)) = \sum_{s=0}^{\infty} \rho_{ess}(P^s)\rho(I-P) = \sum_{s=0}^{\infty}(1-\frac{2}{n+1})^s(2-\frac{2}{n+1}) = n,$$

■

We are able now to provide the proof of Theorem 5.14.

Proof: First we rewrite the expression of $d_\infty^w(P)$. Since our P is symmetric, P is diagonalizable by an orthogonal matrix. We can write that $P = \sum_{h=0}^{N-1} \lambda_h q_h q_h^*$ where q_h are orthonormal. These facts are true also for $P^s(I-P)$. Moreover we have that $\rho\left(P^s(I-P)\right) = ||P^s(I-P)||$.
Let

$$\Delta_w^{(f)}(t) := \sum_{s=0}^{t-1} P^s(I-P)e(t-s-1).$$

Then

$$\begin{aligned}\|\Delta_w^{(f)}(t)\|^2 &= \|\sum_{s=0}^{t-1} P^s(I-P)e(t-s-1)\|^2 \\ &= \|\sum_{s=0}^{t-1}\sum_{h=0}^{N-1} \lambda_h^s(1-\lambda_h)q_h q_h^* e(t-s-1)\|^2 \\ &= \|\sum_{h=0}^{N-1} q_h(1-\lambda_h)\sum_{s=0}^{t-1}\lambda_h^s q_h^* e(t-s-1)\|^2 \\ &= \sum_{h=0}^{N-1}\left[(1-\lambda_h)\sum_{s=0}^{t-1}\lambda_h^s q_h^* e(t-s-1)\right]^2.\end{aligned}$$

Hence, $[d_\infty^w(P)]^2 = \max_{\mathcal{E}^\infty} \limsup_{t\to\infty} \frac{1}{N}\|\Delta_w^{(f)}(t)\|^2$. Now we start using combinatorial tools. Indeed the vertices of the hypercube, as well as the eigenvalues and eigenvectors of P, can be indexed by the subsets of $\{1\ldots,n\}$ (see [69]). With this indexing, for each $I \subseteq \{1\ldots n\}$ the corresponding eigenvalue is $\lambda_I = 1 - \frac{2|I|}{n+1}$ and the eigenvector is the 2^n-dimesional vector $q^{(I)}$, such that its $J-$th component is equal to $q_J^{(I)} = 2^{-n/2}(-1)^{|I\cap J|}$. Let T be any positive integer and consider the sequence of vectors

$$e(0), e(1), \ldots, e(t), \ldots$$

such that J-th component of the vector $e(t)$ is equal to $\frac{1}{2}(-1)^{T-1-r}(-1)^{|J|}$, where r is the remainder in the euclidean division of t over T. Observe that $e(t+T) = e(t)$ for all $t \geq 0$. Observe, moreover, that $e(t)$ is an eigenvector of P corresponding to the eigenvalue $\frac{1-n}{1+n}$ for all $t \geq 0$. Hence we have that

$$\begin{aligned}&\frac{1}{N}\|\Delta_w^{(f)}(T)\|^2 = \\ &\quad= \frac{1}{N}\sum_{h=0}^{N-1}\left[(1-\lambda_h)\sum_{s=0}^{T-1}\lambda_h^s q_h^* e(T-s-1)\right]^2 \\ &\quad= \frac{1}{2^n}\left[(1-\frac{1-n}{n+1})\sum_{s=0}^{T-1}\left(\frac{1-n}{n+1}\right)^s 2^{-\frac{n}{2}}\sum_{J\subseteq\{1,\ldots,n\}}(-1)^{|J|}\frac{1}{2}(-1)^s(-1)^{|J|}\right]^2\end{aligned}$$

$$= \frac{1}{4^n}\left[\frac{n}{n+1}\sum_{s=0}^{T-1}\left(\frac{n-1}{n+1}\right)^s 2^n\right]^2$$
$$= \frac{n^2}{(n+1)^2}\left[\frac{1-\left(\frac{n-1}{n+1}\right)^T}{1-\frac{n-1}{n+1}}\right]^2$$
$$= \frac{n^2}{4}\left[1-\left(\frac{n-1}{n+1}\right)^T\right]^2.$$

Assume now that T is an even positive integer. By recalling that $e(t+T) = e(t)$ for all $t \geq 0$, for $t = kT$ where $k \in \mathbb{N}$ it turns out that

$$\frac{1}{N}\|\Delta_w^{(f)}(kT)\|^2 = \frac{n^2}{4}\left[1-\left(\frac{n-1}{n+1}\right)^T\right]^2\sum_{u=0}^{k-1}\left(\frac{1-n}{n+1}\right)^{uT}$$
$$= \frac{n^2}{4}\left[1-\left(\frac{n-1}{n+1}\right)^T\right]^2\left[\frac{1-\left(\frac{n-1}{n+1}\right)^{kT}}{1-\left(\frac{n-1}{n+1}\right)^T}\right]^2$$
$$== \frac{n^2}{4}\left[1-\left(\frac{n-1}{n+1}\right)^{kT}\right]^2.$$

Letting $k \to \infty$ we obtain that, for the particular sequence considered

$$\lim_{k\to\infty}\frac{1}{N}\|\Delta_w^{(f)}(kT)\|^2 = \frac{n^2}{4} \tag{5.36}$$

Therefore we have proved that

$$\limsup_{t\to\infty}\frac{1}{N}\|\Delta_w^{(f)}(kT)\|^2 \geq \frac{n^2}{4}$$

and hence $[d_\infty^w(P)]^2 \geq \frac{n^2}{4}$. Now, Lemma 5.15 implies that $[d_\infty^w(P)]^2 \leq \frac{n^2}{4}$, and then the claim follows. ■

The above example shows immediately that there exists a discrepancy between the simulative evidence on d_∞ and the estimate d_∞^w. Indeed, while d_∞ seems to be uniformly bounded on N (see Figure 5.2), $d_\infty^w \to \infty$ as $N \to \infty$.
Nevertheless, in the literature of the quantized control, the bounded error model is the only model which permits us to infer some theoretical analysis

on (5.11) and thus to provide some bound on d_∞. There is an another model, the probabilistic model that we will propose in the next section, that seems to be more in accordance with the experimental results. But the fact that the probabilistic model seems to capture the main features of (5.11) comes only from the experimental evidence: there is no theoretical justification motivating this agreement. For this reason we proceed in the analysis of d_∞^w. We start from the following result that provides a general bound for d_∞^w.

Proposition 5.16 *Let P be a matrix satisfying Assumption 5.1. Then*

$$\|PY\| < 1 \tag{5.37}$$

and

$$d_\infty^w(P) \leq \frac{1}{1-\|PY\|}. \tag{5.38}$$

Proof: We have that

$$\|PY\| = \sqrt{\rho\left((PY)^*PY\right)}.$$

Since $PY = YP$ and $Y^2 = Y$ we can write that $(PY)^*PY = P^*PY$. Notice that the fact that P satisfies Assumption 5.1 implies both that $(P^*P)_{ii} > 0$ and $\mathcal{G}_{P^*P}$ is strongly connected. Therefore we can write that $\sigma(P^*P) = \{1, \lambda_1, \ldots, \lambda_{N-1}\}$, where $|\lambda_i| < 1$, $1 \leq i \leq N-1$. Observe that $\sigma(P^*PY) = \{\sigma(P^*P) - \{1\}\} \cup \{0\}$. Hence $\|PY\| < 1$.
Consider now $\Delta_w(t)$. From standard algebraic tools we have that

$$\begin{aligned}\Delta_w(t) &= YP^tx(0) + Y\sum_{s=0}^{t-1} P^s(I-P)e(t-s-1)\\ &= (PY)^t\Delta(0) + \sum_{s=0}^{t-1}(PY)^s(I-P)e(t-s-1)\end{aligned}$$

where in the last equality we have used again the facts that $PY = YP$ and that $Y^k = Y$ for all $k > 0$. Now we have that

$$\begin{aligned}\|\Delta_w(t)\| &= \|(PY)^t\Delta_w(0) + \sum_{s=0}^{t-1}(PY)^s(I-P)e(t-s-1)\|\\ &\leq \|(PY)^t\|\,\|\Delta_w(0)\| + \|I-P\|\sum_{s=0}^{t-1}\|(PY)\|^s\|e(t-s-1)\|\\ &= \|(PY)^t\|\,\|\Delta_w(0)\| + \sqrt{N}\frac{1-\|PY\|^t}{1-\|PY\|}\end{aligned}$$

where in the last inequality we used the facts that $\|I-P\| \leq 2$ and $\|e(t)\| \leq \sqrt{N}/2$ for all $t \geq 0$. By letting $t \to \infty$ we obtain (5.38). ■

Note that, if P is normal we have that $\|PY\| = \rho_{ess}(P)$ and hence (5.38) becomes

$$d_\infty^w(P) \leq \frac{1}{1-\rho_{ess}(P)}.$$

However, when P is a normal matrix the bound on $d_\infty^w(P)$ can be improved as stated in the next proposition.

Proposition 5.17 *If P is normal, then*

$$d_\infty^w(P) \leq \frac{1}{2}\sum_{s=0}^{\infty}\rho(P^s(I-P)). \tag{5.39}$$

Proof: Starting from the expression of $\Delta_w(t)$ provided along the proof of Proposition 5.16 we can write that

$$\begin{aligned}\|\Delta_w(t)\| &\leq \|(PY)^t\Delta_w(0)\| + \|\sum_{s=0}^{t-1}(PY)^s(I-P)e(t-s-1)\| \\ &\leq \|(PY)^t\Delta_w(0)\| + \frac{\sqrt{N}}{2}\|(PY)^s(I-P)\|\end{aligned}$$

Since P is normal we have that $\|(PY)^s(I-P)\| = \rho((PY)^s(I-P)) = \rho(P^s(I-P))$. By letting $t \to \infty$ in the last inequality, we obtain (5.39). ■

Remark 5.18 It is worth noting that, from the sub-multiplicative inequality $\|(PY)^s(I-P)\| \leq \|PY\|^s \|I-P\|$, it follows immediately that $\frac{1}{2}\sum_{s=0}^{\infty}\rho(P^s(I-P)) \leq \frac{1}{1-\rho_{ess}(P)}$ which shows that the bound (5.38) is indeed an improvement of the bound (5.39).

Example 5.19 (Complete graph) We recall that, in this case, $P = \frac{1}{N}\mathbb{1}\mathbb{1}^*$. Hence $PY = 0$. Thus we have that $\frac{1}{1-\|PY\|} = 1$. This is an alternative way to prove (5.21). However, since $\left(\frac{1}{N}\mathbb{1}\mathbb{1}^*\right)^k = \frac{1}{N}\mathbb{1}\mathbb{1}^*$ for all $k > 0$, it follows immediately that $\frac{1}{2}\sum_{s=0}^{\infty}\rho(P^s(I-P)) = \frac{1}{2}$ and this represents a refinement of (5.21).

In general it is quite hard to evaluate (5.39). We provide two results which permit us to approximate (5.39) under some mild assumptions. First a notational definition. Given $c \in \mathbb{C}$ and $r \in \mathbb{R}$ such that $r \geq 0$, we denote

$$B_{c,r} := \{z \in \mathbb{C} \,|\, \|z - c\| \leq r\},$$

namely the closed ball of complex numbers of radius r and centered in c.

Proposition 5.20 *Let P be a normal matrix satisfying the Assumption 5.1. Let R be such that $0 < R < 1$ and $\sigma(P) \subseteq B_{1-R,R}$ and let $\bar{\rho} = \rho_{ess}(P)$ denote the essential spectral radius of P. Then*

$$\sum_{s=0}^{\infty} \rho(P^s(I-P)) \leq \frac{1}{1-R} + \sqrt{\frac{8R}{(1-R)(1-\bar{\rho})}}. \tag{5.40}$$

Proof: We want to upper bound $\rho(P^s(I-P)) = \max_{k=1}^{N-1} |\lambda_k^s(1-\lambda_k)|$. In order to do so we consider the function $f : \mathbb{C} \to \mathbb{R}$ defined as $f(z) = z^s(1-z)$. Let us consider the closed balls $B_{1-R,R}$ and $B_{0,\bar{\rho}}$. By Gershgorin's Theorem we have that $\sigma(P) \subseteq B_{1-R,R}$. By the definition of essential spectral radius it holds that $\sigma(P) \setminus \{1\} \subseteq B_{0,\bar{\rho}}$. Hence $\sigma(P) \setminus \{1\} \subseteq B_{0,\bar{\rho}} \cap B_{1-R,R}$. Let $A := B_{1-R,R} \cap B_{0,\bar{\rho}}$. Clearly

$$\max_{k=1}^{N-1} |\lambda_k^s(1-\lambda_k)| \leq \max_{z \in A} |f(z)|.$$

Since f is an analytic function and A is a compact set, from the Maximum Modulus Principle it follows that

$$\max_{k=1}^{N-1} |\lambda_k^s(1-\lambda_k)| \leq \max_{z \in \partial A} |f(z)|,$$

where ∂A denotes the boundary of A.
Consider now the curves $\gamma : [0, 2\pi] \to \mathbb{C}$,

$$\gamma(t) = 1 - R + Re^{jt},$$

and $\theta : [0, 2\pi] \to \mathbb{C}$,

$$\theta(t) = \bar{\rho}e^{jt},$$

which represent, respectively, the boundaries of $B_{1-R,R}$ and of $B_{0,\bar{\rho}}$. In the following, since $|f(z)| = |f(z^*)|$, we will consider γ and θ only on the interval $[0, \pi]$.

By calculating the intersection between γ and θ one can see that $\partial A = \tilde{\gamma} \cup \tilde{\theta}$ where

$$\tilde{\gamma} := \left\{ z = z_x + iz_y \in \gamma \,:\, z_x \leq \frac{1 - 2R + \bar{\rho}^2}{2(1-R)} \right\}$$

and

$$\tilde{\theta} := \left\{ z = z_x + iz_y \in \theta \,:\, z_x \geq \frac{1 - 2R + \bar{\rho}^2}{2(1-R)} \right\}$$

We consider now $|f(z)|$ on $\tilde{\gamma}$. By straightforward calculations one can show that

$$|f(\gamma(t))|^2 = 2R^2(1-\cos t)\left[1 - 2R(1-R)(1-\cos t)\right]^s .$$

Now let $x = R\cos t + 1 - R$. In order to analyze the behavior of $|f(\gamma(t))|^2$ we introduce the following auxiliary function

$$F(x) = 2R(1-x)\left[1 - 2(1-R)(1-x)\right]^s .$$

A straightforward calculation shows that studying $|f(z)|^2$ on $\tilde{\gamma}$ is equivalent to study F on $\left[1-2R, \frac{1-2R+\bar{\rho}^2}{2(1-R)}\right]$. By taking the first derivative of F we obtain

$$\frac{\partial F}{\partial x} = 2R\left[1 - 2(1-R)(1-x)\right]^{s-1}\left[-1 + 2(1-R)(s+1)(1-x)\right]$$

We have that $\frac{\partial F}{\partial x} = 0$ for $x := x_1 = 1 - \frac{1}{2(1-R)}$ and $x := x_2 = 1 - \frac{1}{2(1-R)(s+1)}$. Note that $1 - \frac{1}{2(1-R)} \leq 1 - 2R$ for all $R > 0$. Moreover note that F is monotone increasing in $[x_1, x_2]$ and monotone decreasing for $[x_2, +\infty)$. Hence F reached its maximum value inside the interval $\left[1-2R, \frac{1-2R+\bar{\rho}^2}{2(1-R)}\right]$ on $1-2R$ if $x_2 \leq 1-2R$, on x_2 if $1-2R \leq x_2 \leq \frac{1-2R+\bar{\rho}^2}{2(1-R)}$, on $\frac{1-\bar{\rho}^2}{2R(1-R)}$ if $x_2 \geq \frac{1-\bar{\rho}^2}{2R(1-R)}$. We have that $x_2 \leq 1-2R \Leftrightarrow s \leq \frac{(1-2R)^2}{4R(1-R)}$, $1-2R \leq x_2 \leq \frac{1-2R+\bar{\rho}^2}{2(1-R)} \Leftrightarrow \frac{(1-2R)^2}{4R(1-R)} < s < \frac{\bar{\rho}^2}{1-\bar{\rho}^2}$, $x_2 \geq \frac{1-\bar{\rho}^2}{2R(1-R)} \Leftrightarrow s \geq \frac{\bar{\rho}^2}{1-\bar{\rho}^2}$. Let $\bar{s} = \lfloor \frac{(1-2R)^2}{4R(1-R)} \rfloor$ and $s^* = \lfloor \frac{\bar{\rho}^2}{1-\bar{\rho}^2} \rfloor$. Therefore

$$\max_{1-2R \leq x \leq \frac{1-2R+\bar{\rho}^2}{2(1-R)}} F(x) = \begin{cases} 4R^2\left(1-2R\right)^{2s} & \text{if} \quad s \leq \bar{s} \\ \frac{R}{1-R}\frac{s^s}{(s+1)^{s+1}} & \text{if } \bar{s}+1 \leq s \leq s^* \\ \frac{R}{1-R}\bar{\rho}^{2s}\left(1-\bar{\rho}^2\right) & \text{if} \quad s \geq s^*+1 \end{cases}$$

Consider now $|f(z)|^2$ on $\tilde{\theta}$. By simple algebraic manipulations one can see that

$$|f(\theta(t))|^2 = \bar{\rho}^{2s}\left(1 + \bar{\rho}^2 - 2\bar{\rho}\cos t\right)$$

Note that $|f(\theta(t))|^2$ is monotone increasing for $t \in [0, \pi]$ and hence it reaches its maximum on $\tilde{\theta}$ when $\cos t = \frac{1-2R+\bar{\rho}^2}{2(1-R)}$ Therefore we can conclude that

$$\max_{k=1}^{N-1} |\lambda_k^s(1-\lambda_k)| \leq \begin{cases} 2R\,|1-2R|^s & \text{if} \qquad s \leq \bar{s} \\ \sqrt{\frac{R}{1-R}\frac{s^s}{(s+1)^{s+1}}} & \text{if } \bar{s}+1 \leq s \leq s^* \\ \sqrt{\frac{R}{1-R}\bar{\rho}^{2s}\,(1-\bar{\rho}^2)} & \text{if} \qquad s \geq s^*+1. \end{cases}$$

Hence we can write

$$\sum_{s=0}^{t-1} \rho(P^s(I-P)) \leq \sum_{s=0}^{\bar{s}} 2R\,|1-2R|^s + \sum_{s=\bar{s}+1}^{s^*} \sqrt{\frac{R}{1-R}\frac{s^s}{(s+1)^{s+1}}} + \\ + \sum_{s=s^*+1}^{t-1} \sqrt{\frac{R}{1-R}\bar{\rho}^{2s}\,(1-\bar{\rho}^2)}.$$

Notice now that

$$\sum_{s=0}^{\bar{s}} 2R\,|1-2R|^s \leq \frac{2R}{1-|1-2R|} \\ \leq \frac{1}{1-R}.$$

and that

$$\sum_{s=s^*+1}^{t-1} \sqrt{\frac{R}{1-R}\bar{\rho}^{2s}\,(1-\bar{\rho}^2)} \leq \sqrt{\frac{R}{1-R}}\sqrt{1-\bar{\rho}^2}\sum_{s=0}^{\infty}\bar{\rho}^s \\ = \sqrt{\frac{R(1-\bar{\rho}^2)}{(1-R)(1-\bar{\rho})^2}} \\ \leq \sqrt{\frac{2R}{(1-R)(1-\bar{\rho})}}.$$

Notice finally that, since $\sum_{i=1}^{m}\sqrt{\frac{1}{i+1}} \leq 2\sqrt{m+1}$, we can argue that

$$\sum_{s=\bar{s}+1}^{s^*} \sqrt{\frac{R}{1-R}\frac{s^s}{(s+1)^{s+1}}} \leq \sqrt{\frac{R}{1-R}}\sum_{s=1}^{s^*}\sqrt{\frac{1}{2}}\sqrt{\frac{1}{1+s}} \\ \leq \sqrt{\frac{4R}{2(1-R)(1-\bar{\rho}^2)}} \\ \leq \sqrt{\frac{2R}{(1-R)(1-\bar{\rho})}}.$$

Putting together these three inequalities we obtain (5.40). ■

Example 5.21 (Direct circuit) Consider the direct circuit graph introduced in Example 5.8 and consider the evolution law given by (5.22). Note that in this case P is the circulant matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

We have theoretically proved (see Corollary 5.12) that $d_\infty \leq \frac{1}{2}$. Instead we could not evaluate d_∞^w in this case. Consider the bound introduced in Proposition 5.16. Since $\rho_{ess} = 1 - \frac{\pi^2}{2}\frac{1}{N^2} + o\left(\frac{1}{N^2}\right)$ (see [35]) and since each circulant matrix is a normal matrix we have that $\frac{1}{1-\|PY\|} = \frac{1}{1-\rho_{ess}(P)} = \Theta\left(N^2\right)$. Observe now that all the eigenvalues of P are inside the ball $B_{\frac{1}{2},\frac{1}{2}}$. Hence we obtain that $\frac{1}{1-R} + \sqrt{\frac{8R}{(1-R)(1-\bar{\rho})}} = \Theta(N)$. This means that the bound (5.40) improves the bound proposed in (5.38). Moreover, by numerical experiments one can see that $\frac{1}{2}\sum_{s=0}^{\infty} \rho(P^s(I-P)) = \Theta(N)$ meaning, that for $N \to \infty$, (5.40) behaves as (5.39).

If P is symmetric we can provide a stronger result.

Proposition 5.22 *Let P be a symmetric stochastic matrix satisfying Assumption 5.1. Let R be such that $0 < R < 1$ and $\sigma(P) \subseteq B_{1-R,R}$ and let $\bar{\rho} = \rho_{ess}(P)$ denote the essential spectral radius of P. Then,*

$$\sum_{s=0}^{+\infty} \rho\left(P^s(I-P)\right) \leq \frac{3}{2} + \frac{1}{1-R} + \frac{1}{2}\log\left(\frac{1}{1-\bar{\rho}}\right). \tag{5.41}$$

Proof: Assume that $\sigma(P) = \{\lambda_0 = 1, \lambda_1, \ldots, \lambda_{N-1}\}$. Note that $\sigma(P)\setminus\{1\} \subseteq [1-2R, \bar{\rho}]$. We want to upper bound $\rho(P^s(I-P)) = \max_{k=1}^{N-1} |\lambda_k^s(1-\lambda_k)|$. To do this, consider the function $f(x) = |x^s(1-x)|$. It is continuous in $[-1,1]$, positive and decreasing in $[-1,0]$, it vanishes in $x=0$ and in $x=1$ and has a local maximum in $x = x_M = \frac{s}{1+s}$, with $f(x_M) = (\frac{s}{1+s})^s(\frac{1}{1+s})$. We need to evaluate $\max_{x\in[1-2R,\bar{\rho}]} f(x)$.

First observe that there exists $\bar{s}$, only depending on the value $1-2R$, such that for all $s > \bar{s}$ we have $f(1-2R) < (\frac{s}{1+s})^s(\frac{1}{1+s})$ and then the global

maximum of $f(x)$ is assumed at $x = x_M$ if $x_M \leq \bar{\rho}$. Since x_M tends to 1 as s goes to infinity, it happens that, when $s > \frac{\bar{\rho}}{1-\bar{\rho}}$, $x_M(s) > \bar{\rho}$. In conclusion we have that

$$\max_{x\in[1-2R,\bar{\rho}]} f(x) = \begin{cases} 2R|1-2R|^s, \text{ if } 0 \leq s \leq \bar{s}; \\ (\frac{s}{1+s})^s(\frac{1}{1+s}), \text{ if } \bar{s} < s \leq s^*; \\ \bar{\rho}^s(1-\bar{\rho}), \quad \text{if } s^* < s < \infty. \end{cases}$$

where $s^* := \lfloor \frac{\bar{\rho}}{1-\bar{\rho}} \rfloor$. Hence we can write

$$\sum_{s=0}^{t-1} \rho(P^s(I-P)) \leq \sum_{s=0}^{\bar{s}} 2R|1-2R|^s + \sum_{s=\bar{s}+1}^{s^*} \left(\frac{s}{1+s}\right)^s \left(\frac{1}{1+s}\right) + \\ + \sum_{s=s^*+1}^{t-1} \bar{\rho}^s(1-\bar{\rho}).$$

Notice now that

$$\sum_{s=0}^{\bar{s}} 2R|1-2R|^s \leq \frac{1}{1-R},$$

and that

$$\sum_{s=s^*+1}^{t-1} \bar{\rho}^s(1-\bar{\rho}) = \bar{\rho}^{s^*+1} - \bar{\rho}^t \leq 1.$$

Notice finally that, since $\sum_{i=1}^{m} 1/i \leq 1 + \ln m$ we can argue that

$$\begin{aligned} \sum_{s=\bar{s}+1}^{s^*} \left(\frac{s}{1+s}\right)^s \left(\frac{1}{1+s}\right) &\leq \sum_{s=0}^{s^*} \left(\frac{1}{2}\right)\left(\frac{1}{1+s}\right) \\ &\leq \frac{1}{2}(1+\log(s^*+1)) \\ &\leq \frac{1}{2} + \frac{1}{2}\log\left(\frac{\bar{\rho}}{1-\bar{\rho}} + 1\right) \\ &= \frac{1}{2} + \frac{1}{2}\log\left(\frac{1}{1-\bar{\rho}}\right). \end{aligned}$$

Putting together these three inequalities we obtain (5.41). ■

Example 5.23 (Undirect circuit) In this example we consider the circulant matrix

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \frac{1}{3} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

By simulations one can see that d_∞ seems to be uniformly bounded on N. Also in this case we are not able to evaluate d_∞^w. By considering the bound (5.38), since $\rho_{ess}(P) \cong 1 - \frac{4}{3}\frac{\pi^2}{N^2}$ (see [35]) we obtain that $\frac{1}{1-\|PY\|} = \frac{1}{1-\rho_{ess}} = \Theta(N^2)$. Observe that all the eigenvalues of P are greater than $-\frac{1}{3}$. Hence it results, letting $R = 2/3$, that $\frac{3}{2} + \frac{1}{1-R} + \frac{1}{2}\log\left(\frac{1}{1-\bar{\rho}}\right) = \Theta(\log N)$. Moreover, numerically it is possible to observe that also $\frac{1}{2}\sum_{s=0}^{\infty}\rho(P^s(I-P))$ grows logarithmically, meaning that asymptotically in N, we have that (5.41) behaves as (5.39).

Remark 5.24 It is worth noting that this improvement is more general. Consider a sequence of symmetric Cayley matrices P, having the elements on the diagonal uniformly lower bounded on N and supported on Cayley graphs having in-degree uniformly upper bounded on N. Then the above arguments can be applied to argue that, for this class of graphs, we can not obtain from (5.39) a bound for $d^w(P)$ stronger than a logarithmic dependence on N. In the next example we will show that a logarithmic bound can be proved to be tight, that is $d^w(P) = \Theta(\log N)$.

Example 5.25 (Hypercube) Consider the hypercube graph and the matrix P compatible with the hypercube graph as defined in Example 5.13. We have already seen, that, by simulations, $d_\infty(P)$ seems to be a uniformly bounded quantity on N, while we have analytically proved that $d_\infty^w = \frac{\log_2 N}{2}$ (see Example 5.13). It is possible to see that also $\frac{1}{2}\sum_{s=0}^{\infty}\rho(P^s(I-P)) = \frac{\log_2 N}{2}$. This fact is stated in Lemma 5.15. Moreover, since the eigenvalues of P are $1 - \frac{2k}{n+1}$ for $k = 0, \ldots, n$ we have that all the eigenvalues of P different from 1 are inside the interval $\left[\frac{1-n}{n+1}, \frac{n-1}{n+1}\right]$. This implies that $\frac{3}{2} + \frac{1}{1-R} + \frac{1}{2}\log\left(\frac{1}{1-\bar{\rho}}\right) = \Theta(\log_2(N))$.

5.3.2 Probabilistic model

In the previous section we have shown that the bounded error model does not seem to really capture the behavior of the quantized model. In particular, the upper bound to the performance we have found, seems to be quite conservative. In this section we undertake a probabilistic approach, modeling the quantization error as a random variable. We will carry on a classical mean square analysis and we will show that it gets quite close to simulations of the real quantized model. This suggests that the probabilistic approach is more appropriate to describe quantization errors even though we have no theoretical evidence of this fact. Therefore we have to point out that the probabilistic model, differently from the worst case analysis, does not provide any bound to the performance.

For $i \in V$ and $t \in \mathbb{N}$, let $n_i(t)$ be random variables of zero mean and variance σ^2, which have their supports inside $[-1/2, 1/2]$ and are uncorrelated and identically distributed in both i and t, i.e., $\mathbb{E}[n_i(t)n_j(\tau)] = 0$ if $i \neq j$ or $t \neq \tau$. Define $n(t)$ as the random vector whose components are $n_i(t)$ and consider the stochastic model

$$\begin{cases} x_r(t+1) = Px_r(t) + (I-P)n(t), & x_r(0) = x(0) \\ \Delta_r(t) = Yx_r(t) \end{cases} \tag{5.42}$$

where $Y = I - \frac{1}{N}\mathbb{1}\mathbb{1}^*$. We define

$$d^r_\infty(P, x_r(0)) = \limsup_{t\to\infty} \sqrt{\frac{1}{N}\mathbb{E}[\|\Delta_r(t)\|^2]}.$$

Since $\lim_{t\to\infty} YP^t = 0$ also in this case, we have that $d^r_\infty(P, x_r(0))$ is independent of the initial condition $x_r(0)$. Hence, in the sequel we will denote $d^r_\infty(P, x_r(0))$ with the symbol $d^r_\infty(P)$. It is worth to point out that this index, as the previously defined $d^w(P)$, captures the asymptotic error induced by quantization, but no information about diverse issues like speed of convergence or finite time behavior.
We start our analysis of the probabilistic model with the following result.

Theorem 5.26 *Let P be a matrix satisfying Assumption 5.1. Then*

$$[d^r_\infty(P)]^2 = \frac{\sigma^2}{N}\mathrm{tr}\left[(I-P)(I-\tilde{P}\tilde{P}^*)^{-1}(I-P)^*\right]. \tag{5.43}$$

where $\tilde{P} = PY$. In particular, if P is normal, and $\sigma(P) = \{1, \lambda_1, \ldots, \lambda_{N-1}\}$ denotes the spectrum of P, we have that

$$[d^r_\infty(P)]^2 = \frac{\sigma^2}{N}\sum_{i=1}^{N-1}\frac{|1-\lambda_i|^2}{1-|\lambda_i|^2}. \tag{5.44}$$

Proof: Define $Q(t) = \mathbb{E}[\Delta_r(t)\Delta_r(t)^*]$, and remark that $\frac{1}{N}\mathbb{E}[||\Delta_r(t)||^2] = \frac{1}{N}\mathrm{tr}\, Q(t)$. Using the facts that $Y^k = Y$ for all positive integer k and $Y(P - I) = P - I$, it is easy to see that Δ_r satisfies the following recursive equation

$$\Delta_r(t+1) = \tilde{P}\Delta_r(t) + (P - I)n(t).$$

Now, thanks to the hypotheses on $n_i(t)$,

$$\begin{aligned} Q(t+1) &= \mathbb{E}[\Delta_r(t+1)\Delta_r(t+1)^*] \\ &= \mathbb{E}[\tilde{P}\Delta_r(t)\Delta_r(t)^*\tilde{P}^*] + (I-P)\mathbb{E}[n(t)n(t)^*](I-P)^* \\ &= \tilde{P}\, Q(t)\, \tilde{P}^* + \sigma^2(I-P)(I-P)^*, \end{aligned}$$

and by a simple recursion

$$Q(t) = \tilde{P}^t\, Q(0)\, (\tilde{P}^*)^t + \sigma^2 \sum_{s=0}^{t-1} \tilde{P}^s (I-P)(I-P)^* (\tilde{P}^*)^s.$$

Recall now from the proof of Proposition 5.16 that, since P satisfies Assumption 1, then $\rho_{ess}(P^*P) < 1$. Moreover we have that $\rho(\tilde{P}) = \rho_{ess}(P) < 1$ and $\rho(\tilde{P}^*\tilde{P}) = \rho_{ess}(P^*P) < 1$. Using the linearity and properties of the trace,

$$\begin{aligned} \mathrm{tr}\, Q(t) &= \mathrm{tr}\, \left[\tilde{P}^t\, Q(0)\, (\tilde{P}^*)^t\right] + \mathrm{tr}\, \left[\sigma^2 \sum_{s=0}^{t-1} (PP^* - (P+P^*) + I)(\tilde{P}\tilde{P}^*)^s\right] \\ &= \mathrm{tr}\, \left[\tilde{P}^t\, Q(0)\, (\tilde{P}^*)^t\right] + \\ &\quad + \sigma^2 \mathrm{tr}\, \left[(PP^* - (P+P^*) + I)(I - (\tilde{P}\tilde{P}^*)^t)(I - \tilde{P}\tilde{P}^*)^{-1}\right] \end{aligned}$$

and hence

$$\lim_{t\to\infty} \mathrm{tr}\, Q(t) = \sigma^2 \mathrm{tr}\, \left[(PP^* - (P+P^*) + I)(I - \tilde{P}\tilde{P}^*)^{-1}\right].$$

If moreover P is normal, we can find a unitary matrix O of eigenvectors and a diagonal matrix of eigenvalues Λ, such that $P = O\Lambda O^*$. This implies

$$\mathrm{tr}\, \left[(PP^* - (P+P^*) + I)(1 - \tilde{P}\tilde{P}^*)^{-1}\right] = \sum_{i=1}^{N-1} \frac{|1-\lambda_i|^2}{1-|\lambda_i|^2}.$$

■

From now on we restrict to the case in which P is normal. Note that the expression for the mean square error of formula (5.44) is the product of two terms, $[d^r_\infty(P)]^2 = \sigma^2\Phi(P)$ where

$$\Phi(P) := \frac{1}{N}\sum_{i=1}^{N-1}\frac{|1-\lambda_i|^2}{1-|\lambda_i|^2},$$

is a functional[3] of the matrix P, depending only on its spectral structure.

As in the previous section, we are mainly interested in sequences of matrices of increasing size. We will see that the above functional scales well with N in the following examples.

Example 5.27 (Complete graph) In this case it is easy to compute $\Phi(P)$. We have that $\Phi(P) = \frac{N-1}{N}$.

Example 5.28 (Direct graph) Consider the matrix P defines in Example 5.21. In this case $\Phi(P)$ can be exactly computed. We have

$$\begin{aligned}\Phi(P_N) &= \frac{1}{N}\sum_{h=1}^{N-1}\frac{(1-\lambda_h)(1-\lambda_h^*)}{1-\lambda_h\lambda_h^*}\\ &= \frac{1}{N}\sum_{h=1}^{N-1}\frac{(1-(1/2+1/2\,e^{i\frac{2\pi}{N}h}))(1-(1/2+1/2\,e^{-i\frac{2\pi}{N}h}))}{1-(1/2+1/2\,e^{i\frac{2\pi}{N}h})(1/2+1/2\,e^{-i\frac{2\pi}{N}h})}\\ &= \frac{1}{N}\sum_{h=1}^{N-1}\frac{1/2(1-\cos(\frac{2\pi}{N}h))}{1/2(1-\cos(\frac{2\pi}{N}h))} = \frac{N-1}{N}.\end{aligned}$$

Example 5.29 (Undirected graph) Consider the undirected undirected circuit graph and the matrix P introduced in Example 5.23. The eigenvalues of P are

$$\lambda_h = \frac{1}{3}+\frac{2}{3}\cos\,(\frac{2\pi}{N}h)\quad h=0,\dots,N-1,$$

and we have

$$\Phi(P_N) = \frac{1}{N}\sum_{h=1}^{N-1}\frac{1-\lambda_h}{1+\lambda_h} = \frac{1}{N}\sum_{h=1}^{N-1}\frac{(1-\cos\,(\frac{2\pi}{N}h))}{2+\cos\,(\frac{2\pi}{N}h)}.$$

[3]Remarkably, the functional $\Phi(P)$ also arises, with a rather different meaning, in [47], as a cost functional describing the transient of the diffusion methods for average consensus over graphs with ideal communication.

In this case it is difficult to work out the computation explicitly. However, it is possible to compute the limit for $N \to \infty$, since the summation can be interpreted as Riemann sum relative to the function $f(x) = \dfrac{1-\cos(x)}{2+\cos(x)}$. We thus obtain

$$\lim_{N\to\infty} \Phi(P_N) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-\cos(x)}{2+\cos(x)} dx = \sqrt{3} - 1. \tag{5.45}$$

Example 5.30 (Hypercube) Consider the hypercube graph and the matrix P defined in (5.34). We have that

$$\begin{aligned}
\Phi(P) &= \frac{1}{N} \sum_{i=1}^{N-1} \frac{|1-\lambda_i|^2}{1-|\lambda_i|^2} \\
&= \frac{1}{2^n} \sum_{k=1}^{n} \frac{\left(\frac{2k}{n+1}\right)^2}{1-\left(\frac{n+1-2k}{n+1}\right)^2} \binom{n}{k} \\
&= \frac{1}{2^n} \sum_{k=1}^{n} \frac{k}{n+1-k} \binom{n}{k} \\
&= \frac{1}{2^n} \sum_{k=1}^{n} \binom{n}{k-1} \\
&= \frac{2^n - 1}{2^n} \\
&= \frac{N-1}{N}.
\end{aligned}$$

Then

$$\Phi(P) = \frac{N-1}{N}.$$

While in the previous section the hypercube provided the negative example for the worst case behavior, the probabilistic analysis is in agreement with the evidence showed in the simulations. This highlights the differences between the bounded error model and the probabilistic model: with the same assumptions on P the two worst-case analysis and the mean-square analysis give different results.

Example 5.31 (Random geometric graph) For the random geometric graph, as defined in the introduction, we have no explicit formula for the eigenvalues. However, d_∞^r can be numerically evaluated, and compared with d_∞. The results are shown in Figure 5.5 where d_∞^r seems to describe well the behavior of d_∞.

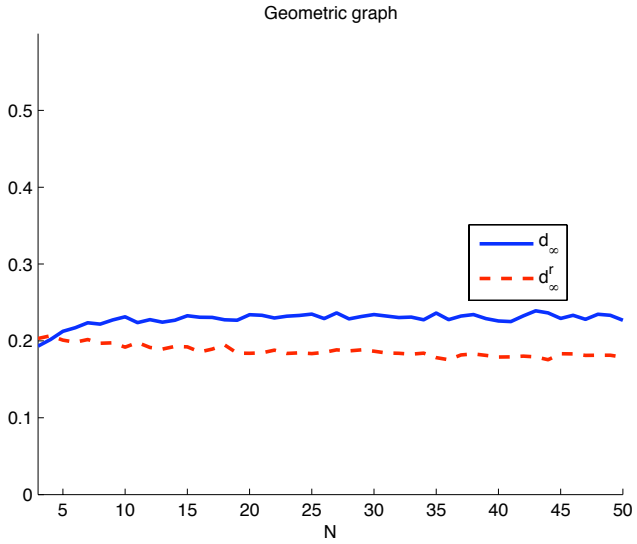

Figure 5.5: Comparison of d^r_∞ and d_∞ for random geometric graphs, averaged over 25 realizations both of the initial condition and of the graphs, for each N. The plot assumes $\sigma^2 = 1/12$.

In all the above examples we have that $\Phi(P)$ scales well with N. We provide now some general conditions ensuring that the functional P for a given sequence of matrices of increasing size is uniformly bounded on N.
We start by observing that, if $\rho_{ess}(P) \leq B < 1$ then $\Phi(P) \leq \frac{N-1}{N}\frac{4}{1-B^2}$. This implies that, given a sequence of matrices of increasing size, if the essential spectral radius of the sequence is uniformly bounded away from 1 then the functional cost Φ is uniformly bounded in N. This fact is true also for d^w_∞ as can be easily seen by recalling the expression of the bound provided by Proposition 5.16. The interesting fact is that the performance index d^r_∞ can exhibit the same behavior even when the essential spectral radius is not bounded away from 1 as the following proposition shows.

Proposition 5.32 *Let $B_{c,r} \subset \mathbb{C}$ denote the closed ball of complex numbers with center the point c and radius r. If there exists $0 < R < 1$ such that $\sigma(P) \subseteq B_{1-R,R}$ then*

$$\Phi(P) \leq \frac{R}{1-R}. \tag{5.46}$$

Proof: The inequality $0 < R < 1$ is clear from Assumption 5.1. It means that the spectrum is contained in a disc of radius R internally tangent in 1

to the unit disc of the complex plane. Then, we need to prove (5.46). For all i, the eigenvalue $\lambda_i \in B_{1-R,R}$, so

$$\lambda_i = (1-r) + r\, e^{i\theta}$$

with $\theta \in [0, 2\pi[$ and $0 \leq r \leq R$. Moreover, if $i \geq 1$, then $\theta > 0$. Hence,

$$\begin{aligned}
\frac{|1-\lambda_i|^2}{1-|\lambda_i|^2} &= \frac{|r - re^{i\theta}|^2}{1-|1-r+r\,e^{i\theta}|^2} \\
&= \frac{r^2|1-e^{i\theta}|^2}{1-(1-r)^2-2r(1-r)\cos\theta - r^2} \\
&= \frac{r^2 2(1-\cos\theta)}{2r(1-r)(1-\cos\theta)} \\
&= \frac{r}{1-r} \\
&\leq \frac{R}{1-R} \quad \forall\, i.
\end{aligned}$$

This yields the result. ∎

Note that the above bound depends only on R and does not depend on the essential spectral radius of the matrix P, while the worst case bounds provided in Proposition 5.40 and in Proposition 5.22 did. This is why (5.46) is bounded in all the cases we considered.

Many of them are covered by the following corollary.

Corollary 5.33 *Let $p = \min_i P_{ii}$ and R as above. Then, $R \leq 1-p$, and*

$$\Phi(P) \leq \frac{1-p}{p}. \tag{5.47}$$

Proof: By Gershgorin theorem, $\sigma(P) \subseteq \bigcup_i B_{P_{ii},1-P_{ii}} \subseteq B_{p,1-p}$. ∎

The above result has the following interpretation. If in a family of matrices P_N we have that $\min_i (P_N)_{ii}$ is lower bounded uniformly in N, then (5.47) gives a finite bound, uniform in N, on the asymptotic displacement. This is a useful hint to construct sequences of matrices whose performance scales well with N. It would be enough to prescribe that the agents assign a minimum weight to their own values.

5.4 Time Invariant case- Probabilistic quantizers

Consider (5.12), i.e., the updating rule using the probabilistic quantizer. In this section we will show that a probabilistic model, similar to the one used in Section 5.3.2 for the deterministic quantizer-model, is a theoretically based way to analyze (5.12).

We start our analysis from the following assumption on the initial condition $x(0)$.

Assumption 5.34 *The initial condition $x(0)$ is a random variable such that $\mathbb{E}[x(0)] = 0$ and $\mathbb{E}\left[x(0)x^*(0)\right] = \sigma_0^2 I$ for some $\sigma_0^2 > 0$.*

Observe now that (5.12) can be rewritten in the following way

$$x(t+1) = Px(t) + (P - I)\left(q_p(x(t)) - x(t)\right). \tag{5.48}$$

We introduce the variables

$$y(t) := \left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right)x(t) = x(t) - \left(\frac{1}{N}\mathbb{1}^*x(t)\right)\mathbb{1} = x(t) - \left(\frac{1}{N}\mathbb{1}^*x(0)\right)\mathbb{1}$$

and

$$w(t) = q(x(t)) - x(t).$$

Note that the variable $y(t) = [y_1(t), \ldots, y_N(t)]^*$ represents the distance from the initial average and the variable $w(t) = [w_1(t), \ldots, w_N(t)]^*$ represents the quantization error. Moreover note that $\mathbb{1}^*y(t) = 0$ for all $t \geq 0$. From (5.48) have that

$$\left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right)x(t+1) =$$
$$= \left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right)Px(t) + \left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right)(P - I)\left(q_p(x(t)) - x(t)\right).$$

Since $\left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right)P = P\left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right)$ and $\left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right)(P - I) = P - I$ we obtain the following recursive equation

$$y(t+1) = Py(t) + (P - I)w(t). \tag{5.49}$$

In order to perform an asymptotic analysis of (5.49) we introduce the following matrices. Let

$$\Sigma_{yy}(t) := \mathbb{E}\left[y(t)y^*(t)\right],$$

$$\Sigma_{ww}(t) := \mathbb{E}\left[w(t)w^*(t)\right],$$

and

$$\Sigma_{yw}(t) := \mathbb{E}\left[y(t)w^*(t)\right].$$

Equation (5.49) leads to the following recursive equation in terms of the above matrices

$$\begin{aligned}\Sigma_{yy}(t+1) = P\Sigma_{yy}(t)P^* + P\Sigma_{yw}(t)(P-I)^* + (P-I)\Sigma_{yw}^*P^* + \\ + (P-I)\,\Sigma_{ww}(t)\,(P-I)^*.\end{aligned} \tag{5.50}$$

Notice that, since $x(0)$ is a random variable satisfying Assumption 5.34, one can show that

$$\Sigma_{yy}(0) = \sigma_0^2\,(I - 1/N\,\mathbb{1}\mathbb{1}^*). \tag{5.51}$$

The following proposition states some remarkable properties of the variables y and w.

Proposition 5.35 *Consider the variables $y(t)$ and $w(t)$ above defined. Then*

$$\mathbb{E}\left[w(t)\right] = 0 \qquad \textit{and} \qquad \mathbb{E}\left[w(t)w^*(t)\right] = \operatorname{diag}\left\{\sigma_1^2(t), \ldots, \sigma_N^2(t)\right\} \tag{5.52}$$

where $\sigma_i^2(t) := \mathbb{E}\left[w_i^2(t)\right]$ is such that $\sigma_i^2(t) \leq 1/4$ for all $1 \leq i \leq N$ and for all $t \geq 0$.
Moreover

$$\Sigma_{yw}(t) = 0, \tag{5.53}$$

for all $t \geq 0$.

Proof: Observe that

$$\begin{aligned}\mathbb{E}\left[w_i(t)\right] &= \mathbb{E}\left[\,\mathbb{E}\left[q_p(x_i(t)) - x_i(t)|x_i(t)\right]\right] \\ &= \mathbb{E}\left[\,\mathbb{E}\left[q_p(x_i(t))|x_i(t)\right] - x_i(t)\right] \\ &= \mathbb{E}\left[x_i(t) - x_i(t)\right] \\ &= 0,\end{aligned} \tag{5.54}$$

and that, for $i \neq j$,

$$\begin{aligned}\mathbb{E}\left[w_i(t)w_j(t)\right] &= \mathbb{E}\left[w_i(t)(q(x_j(t)) - x_j(t))\right] \\ &= \mathbb{E}\left[\,w_i(t)\mathbb{E}\left[q(x_j(t)) - x_j(t)|x_i(t), x_j(t)\right]\right] \\ &= \mathbb{E}\left[\,w_i(t)\mathbb{E}\left[q(x_j(t)) - x_j(t)|x_j(t)\right]\right] \\ &= \mathbb{E}\left[\,w_i(t)\left(\mathbb{E}\left[q(x_j(t))|x_j(t)\right] - x_j(t)\right)\right] \\ &= \mathbb{E}\left[\,w_i(t)\left(x_j(t) - x_j(t)\right)\right] \\ &= 0\end{aligned} \tag{5.55}$$

where, both in (5.54) and in (5.55), we have used the fact that by Lemma 5.3, $\mathbb{E}\left[q(x_j(t))|x_j(t)\right] = x_j(t)$. If $i = j$ we have that

$$\begin{aligned}\mathbb{E}\left[w_i^2(t)\right] &= \mathbb{E}\left[(q(x_i(t)) - x_i(t))^2\right] \\ &= \mathbb{E}\left[\,\mathbb{E}\left[(q(x_i(t)) - x_i(t))^2\,|x_i(t)\right]\right] \\ &\leq \mathbb{E}\left[\frac{1}{4}\right] \\ &= \frac{1}{4} \end{aligned} \tag{5.56}$$

where again by Lemma 5.3, we used the fact $\mathbb{E}\left[(q(x_i(t)) - x_i(t))^2\,|x_i(t)\right] \leq 1/4$. (5.54), (5.55) and (5.56) establish the validity of (5.52).
Observe now that, given any pair of indexes i, j it holds also that

$$\begin{aligned}\mathbb{E}\left[x_i(t)w_j(t)\right] &= \mathbb{E}\left[x_i(t)(q(x_j(t)) - x_j(t))\right] \\ &= \mathbb{E}\left[\,x_i(t)\mathbb{E}\left[q(x_j(t)) - x_j(t)|x_i(t), x_j(t)\right]\right] \\ &= \mathbb{E}\left[\,x_i(t)\mathbb{E}\left[q(x_j(t)) - x_j(t)|x_j(t)\right]\right] \\ &= \mathbb{E}\left[\,x_i(t)\left(\mathbb{E}\left[q(x_j(t))|x_j(t)\right] - x_j(t)\right)\right] \\ &= \mathbb{E}\left[\,x_i(t)\left(x_j(t) - x_j(t)\right)\right] \\ &= 0 \end{aligned}$$

It is easy to see that the fact that $\mathbb{E}\left[x_i(t)w_j(t)\right] = 0$ implies also that $\mathbb{E}\left[y_i(t)w_j(t)\right] = 0$ and hence that $\Sigma_{yw}(t) = 0$ for all $t \geq 0$. ■

From (5.53), it follows that (5.50) can be rewritten as

$$\Sigma_{yy}(t+1) = P\Sigma_{yy}(t)P^* + (P - I)\,\Sigma_{ww}(t)\,(P - I)^*\,. \tag{5.57}$$

In order to evaluate the asymptotic distance from the initial average, a suitable functional cost is

$$d_\infty^p(P) := \limsup_{t\to\infty} \sqrt{\frac{1}{N}\text{tr}\,\{\Sigma_{yy}(t)\}}. \tag{5.58}$$

Note that $d_\infty^p(P)$ has a similar meaning to $d_\infty^r(P)$ defined in the previous Section, when we introduced the probabilistic model to study the evolution of (5.11). By straightforward calculations one can show that,

$$\begin{aligned} d_\infty^p(P) &= \lim_{t\to\infty}\left(P^t\Sigma_{yy}(0)\,(P^*)^t + \sum_{i=0}^{t-1} P^{t-1-i}\Sigma_{ww}(i)\,(P^*)^{t-1-i}\right) \\ &= \lim_{t\to\infty}\left(\sum_{i=0}^{t-1} P^{t-1-i}\Sigma_{ww}(i)\,(P^*)^{t-1-i}\right), \end{aligned} \tag{5.59}$$

where the last equality derives from the fact that

$$\begin{aligned}\lim_{t\to\infty}\left(P^t\Sigma_{yy}(0)\left(P^*\right)^t\right) &= \frac{1}{N}\mathbb{1}\mathbb{1}^*\Sigma_{yy}(0)\mathbb{1}\mathbb{1}^* \\ &= \frac{1}{N}\mathbb{1}\mathbb{1}^*\left(I-\frac{1}{N}\mathbb{1}\mathbb{1}^*\right)\mathbb{1}\mathbb{1}^* \\ &= 0.\end{aligned}$$

In general, inferring some theoretical analysis on (5.59) is quite hard. We try now to simplify the problem. Let us introduce the matrix

$$\bar{W} = \frac{1}{4}I,$$

and the auxiliary system

$$\begin{aligned}\bar{\Sigma}_{yy}(t+1) &= P\bar{\Sigma}_{yy}(t)P^* + (P-I)\bar{W}(P-I)^* \\ &= P\bar{\Sigma}_{yy}(t)P^* + \frac{1}{4}(P-I)(P-I)^*. \end{aligned} \tag{5.60}$$

where $\bar{\Sigma}_{yy}(0) = \Sigma_{yy}(0)$. Moreover let

$$\bar{d}^p_\infty(P) := \limsup_{t\to\infty}\sqrt{\frac{1}{N}\text{tr}\,\left\{\bar{\Sigma}_{yy}(t)\right\}}.$$

We have the following proposition.

Proposition 5.36 *Let $d^p_\infty(P)$ and $\bar{d}^p_\infty(P)$ be as previously defined. Then*

$$d^p_\infty(P) \le \bar{d}^p_\infty(P).$$

Proof: In order to prove the statement of the Theorem we prove that

$$\bar{\Sigma}(t) \ge \Sigma(t) \tag{5.61}$$

for all $t \ge 0$, where the inequality is meant in the matricial sense, namely $\bar{\Sigma}(t) - \Sigma(t) \ge 0$. We prove (5.61) by induction on t.
Let $t = 0$, then (5.61) is trivially true since $\bar{\Sigma}_{yy}(0) = \Sigma_{yy}(0)$. Assume now that (5.61) is verified for a generic t and consider $t+1$. We have that

$$\begin{aligned}&\bar{\Sigma}(t+1) - \Sigma(t+1) = \\ &= P\bar{\Sigma}(t)P^* + (P-I)\bar{W}(P-I)^* - (P\Sigma(t)P^* + (P-I)W(t)(P-I)^*) \\ &= P\left(\bar{\Sigma}(t) - \Sigma(t)\right)P^* + (P-I)\left(\bar{W} - W(t)\right)(P-I)^* \end{aligned} \tag{5.62}$$

Since, by inductive hypothesis we have that $\bar{\Sigma}(t) - \Sigma(t) \geq 0$ and since $\bar{W} - W(t) \geq 0$ for all t, it follows from (5.62) that also $\bar{\Sigma}(t+1) - \Sigma(t+1) \geq 0$ holds true. ■

Now note that, from (5.60), by simple algebraic tools one can show that

$$\begin{aligned}\left[\bar{d}^p_\infty(P)\right]^2 &= \frac{1}{4N}\mathrm{tr}\left\{\lim_{t\to\infty}\sum_{i=0}^{t-1} P^i(P-I)(P-I)^*(P^*)^i\right\} \\ &= \frac{1}{4N}\mathrm{tr}\left\{(P-I)(I-\tilde{P}\tilde{P})^{(-1)}(P-I)^*\right\}\end{aligned}$$

where $\tilde{P} = PY$ with $Y = I - \frac{1}{N}\mathbb{1}\mathbb{1}^*$. Note that, if in (5.43) $\sigma^2 = \frac{1}{4}$ then $\bar{d}^p_\infty(P) = d^r_\infty(P)$. Moreover, if P is normal, also $\bar{d}^p_\infty$ can be expressed by (5.44). These observations imply that all the considerations and the results stated in Example 5.28, in Example 5.29, in Example 5.30, in Example 5.31, in Lemma 5.32 and in Corollary 5.33 hold true also for the Equation (5.60). In particular this shows that $\bar{d}^p_\infty$ is uniformly bounded on N for many interesting families of graphs of increasing size and that asymptotically reaches a value less than $\frac{1}{4}$.

Remark 5.37 As pointed out in Remark 5.6, the authors in [8] proposed the following updating scheme

$$x(t+1) = Pq_p(x(t)). \tag{5.63}$$

They proved the following theorem

Theorem 5.38 *Consider (5.63). Then, almost surely, it holds that*

$$\lim_{t\to\infty} x(t) = c\mathbb{1}$$

for some integer c.

Note that the scheme (5.63) in general do not preserve the initial average. In order to quantify the displacement from the initial average, the authors in [9] introduced the following variable $\tilde{y}(t) := q_p(x(t)) - (1/N\mathbb{1}^*x(0))\,\mathbb{1}$ and the following functional cost

$$\tilde{J} := \limsup_{t\to\infty}\sqrt{\frac{1}{N}\mathrm{tr}\,\{\mathbb{E}\,[\tilde{y}(t)\tilde{y}(t)^*]\}}.$$

Note that the meaning of the variables $\tilde{y}(t)$ and $\tilde{J}$ is very close to the meaning of the variable $y(t)$ and J, respectively. The following result, stated in [9], characterizes $\tilde{J}$.

Theorem 5.39 *Let $\tilde{y}(t)$ and $\tilde{J}$ as above defined. Then*

$$\lim_{N\to\infty} \tilde{J} \leq \frac{1}{2}\frac{1}{1-\rho_{ess}(P)}. \tag{5.64}$$

Consider now the sequence of circulant matrices $P_N \in \mathbb{R}^{N\times N}$ defined by

$$P_N = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \frac{1}{3} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}. \tag{5.65}$$

For this sequence of symmetric stochastic matrices we have that (see Section 3.3.1)

$$\rho_{ess}(P_N) = \frac{1}{3} + \frac{2}{3}\cos\left(\frac{2\pi}{N}\right).$$

Hence, for $N \to \infty$ we have that

$$\rho_{ess}(P_N) = 1 - \frac{4}{3}\frac{\pi^2}{N^2} + o\left(\frac{1}{N^2}\right)$$

implying that

$$\lim_{N\to\infty} \frac{1}{2}\frac{1}{1-\rho_{ess}(P_N)} = +\infty.$$

Hence, even if this strategy ensures, that, almost surely, the consensus is reached, the distance from the average of the initial conditions could increase drastically for those consensus matrices P, whose essential spectral radius tends to 1.

Remark 5.40 For the sake of the completeness we briefly consider again the laws

$$x(t+1) = Pq_d(x(t)) \tag{5.66}$$

and

$$x(t+1) = \operatorname{diag}\{P\}\, x(t) + (P - \operatorname{diag}\{P\})\, \hat{x}(t) \tag{5.67}$$

introduced in Remark 5.5. Besides the fact that these strategies do not preserve the average of the state, the authors in [33], supported by several simulations, stated that (5.66) and (5.67) exhibit also poor performance in terms of reaching a consensus. In particular for matrices possessing symmetries, as (5.65), they asymptotically lead the overall system to periodic orbits whose amplitude seems to be increasing as $N \to \infty$.

5.4.1 An alternative solution to the consensus problem

Consider (5.12), i.e.,

$$x(t+1) = x(t) + (P - I)q_p(x(t)), \tag{5.68}$$

and (5.63), i.e.,

$$x(t+1) = Pq_p(x(t)). \tag{5.69}$$

Since (5.69) uses only quantized information we refer to it as *globally quantized*, while since in (5.68) the systems use also perfect information related their own state we refer to it as *partially quantized*.

We have seen in Remark 5.37 that the distance from the average of the initial condition of the consensus point toward which the globally quantized strategy leads the systems could be not negligible. On the other hand the partially quantized strategy does not assure a consensus in a strict sense. If, depending on the application, one can not relax the convergence requirement we could suggest the following heuristic solution to the consensus problem, which combines the positive features of both strategies

$$x(t+1) = Pq_p(x(t)) + \epsilon(t)(x(t) - q_p(x(t))),$$

where $\epsilon(t),\ t \geq 0$, is a nonnegative sequence such that $\epsilon(t) \leq 1,\ \forall\, t \geq 0$ and $\lim_{t\to\infty} \epsilon(t) = 0$.

An example, could be the case in which, given $T \in \mathbb{N}$, $\epsilon(t) = 1$ for $t \leq T$ and $\epsilon(t) = 0$ for $t > T$. In other words, the overall system uses as updating rule (5.68), i.e., the *partially quantized* strategy, for $t \leq T$, and (5.69), i.e., the *globally quantized* strategy, for $t > T$. It is clear that this combined strategy reaches almost surely the consensus, and the deviation from the initial average will be smaller than using only (5.69).

5.5 Conclusions

In this chapter we studied the effects of a uniform quantization (both deterministic and probabilistic) on the average consensus problem, and, starting from the standard average consensus algorithm, we proposed a simple and effective adaptation which is able to preserve the average of states and to drive the system reasonably near to the consensus.

An exact analysis of this adaptation is, in general, very hard and we have been able to perform it only in one special case. The main features of this

new law have been investigated by a worst case analysis and by a probabilistic analysis. A special attention has been given to the scalability in N of the performance, which is a crucial issue for applications in which the number of agents is huge. In this direction we obtained several favorable results, which we applied to sequences of graphs with symmetries.

A potential future development is be the extension to random geometric graphs of the analysis we performed on Cayley examples. Moreover an interesting question remain open: why and in which sense the probabilistic analysis seems to be closer to experimental results also when the information is quantized by means of deterministic quantizers.

Chapter 6

Quantized Consensus: Gossip Algorithms

6.1 Introduction

In the previous chapter we have studied the average consensus algorithm, when the consensus matrix P is time-invariant, by assuming that the communication network is constituted of only digital links and hence the information exchanged between the systems is quantized. Also in this chapter, we assume that the systems exchange information through digital channels but, differently from the previous chapter, we consider the case in which the consensus matrix is time-varying. Precisely, the main goal of this chapter is to analyze the effects of the quantization on the symmetric gossip algorithm described in Section 4.5.

Similarly to the previous chapter, we will introduce two particular strategies, the *partially quantized* strategy and the *globally quantized* strategy, depending on whether the systems use exact information regarding their own state, or not, to update their states. We will analyze these strategies both via the deterministic quantizer and via the probabilistic quantizer. We will show that the *globally quantized* strategy, via both the deterministic quantizer and the probabilistic quantizer, ensures that, almost surely, the consensus is reached. The drawback of this strategy is that it does not preserve the average of the initial conditions. On the other hand, the *partially quantized* strategy maintains the initial average at each iteration of the algorithm, but does not guarantee that the consensus is reached in general. However, we will see that the *partially quantized* drives asymptotically all the states very

close to the initial average (we will quantify how close).

The chapter is organized as follows. In Section 6.2 we formulate the problem. In particular we introduce the *partially* quantized strategy and the *globally* quantized strategy. In section 6.3 and in Section 6.4, we analyze these two strategies assuming, respectively, that the systems quantize the information by means of deterministic quantizers and by means of probabilistic quantizers. Finally in Section 6.5 we gather out our conclusions.

6.2 Quantized Symmetric Gossip Algorithms

For the sake of the clarity, we start by briefly reviewing the symmetric gossip consensus algorithm as described in Section 4.5, where the systems communicate each other the exact value of their states. Assume we are given an undirected graph $\mathcal{G} = (V, \mathcal{E})$. At each time step, one edge (i, j) is randomly selected in $\mathcal{E}$ with probability $W^{(i,j)}$ such that $\sum_{(i,j) \in \mathcal{E}} W^{(i,j)} = 1$. The two agents connected by that edge average their states according to

$$\begin{aligned} x_i(t+1) &= \frac{1}{2}x_i(t) + \frac{1}{2}x_j(t) \\ x_j(t+1) &= \frac{1}{2}x_j(t) + \frac{1}{2}x_i(t) \end{aligned} \tag{6.1}$$

and

$$x_h(t+1) = x_h(t) \tag{6.2}$$

if $h \neq i, j$. Let

$$E_{ij} = (e_i - e_j)(e_i - e_j)^*$$

and

$$P(t) = I - \frac{1}{2}E_{ij}$$

where $e_i = [0, \dots, 0, 1, 0, \dots, 0]^*$ is a $N \times 1$ unit vector with the i-th component equal to 1, then (6.1) and (6.2) can be written in a vector form as

$$x(t+1) = P(t)x(t) \tag{6.3}$$

where $x(t) = [x_1(t), \dots, x_N(t)]^*$ denotes the state of the overall system. Note that $P(t)$ is a doubly stochastic matrix. We have seen in in Section 4.5 that, if the graph $\mathcal{G}$ is connected and each edge $(i, j) \in \mathcal{E}$ can be selected with

a probability $W^{(i,j)}$ strictly positive, then (6.3) reaches, almost surely, the *average consensus*, namely

$$\lim_{t\to\infty} x(t) = x_{ave}\mathbb{1},$$

where $x_{ave} = \frac{1}{N}\mathbb{1}^* x(0)$. In the sequel, as in Section 4.5, we will make the following assumption.

Assumption 6.1 *The graph $\mathcal{G} = (V, \mathcal{E})$ is a undirected connected graph and, at every time instant $t \in \mathbb{N}$, each edge $(i, j) \in \mathcal{E}$ can be selected with a strictly positive probability $W^{(i,j)}$.*

As in the previous chapter, here we assume that the comunication network is constituted only of digital channels through which the systems can exchange only quantized information.We have already emphasized that this implies that the systems can not have access to the exact value of the state x of the systems with which they are communicating, but only to an estimate $\hat{x}$ of it. In this chapter we will introduce two updating rules of the state which represent the gossip-version of the two updating rules (5.5) and (5.15), considered in the previous chapter. Precisely, if (i, j) is the edge selected at the t-th iteration, in the first strategy we assume that i and j, in order to update its state, use only the estimates of their states, i.e.,

$$\begin{aligned} x_i(t+1) &= \frac{1}{2}\hat{x}_i(t) + \frac{1}{2}\hat{x}_j(t) \\ x_j(t+1) &= \frac{1}{2}\hat{x}_j(t) + \frac{1}{2}\hat{x}_i(t), \end{aligned} \tag{6.4}$$

or, equivalently in vector form, by recalling the definition of $P(t)$,

$$x(t+1) = P(t)\hat{x}(t), \tag{6.5}$$

where $\hat{x}(t) = [\hat{x}_1(t), \ldots, \hat{x}_N(t)]$. In the second strategy, by remarking that (6.1) can be written as

$$\begin{aligned} x_i(t+1) &= x_i(t) - \frac{1}{2}x_i(t) + \frac{1}{2}x_j(t) \\ x_j(t+1) &= x_j(t) - \frac{1}{2}x_j(t) + \frac{1}{2}x_i(t) \end{aligned} \tag{6.6}$$

we propose the following updating rule, where the systems use also perfect information regarding their own state,

$$\begin{aligned} x_i(t+1) &= x_i(t) - \frac{1}{2}\hat{x}_i(t) + \frac{1}{2}\hat{x}_j(t) \\ x_j(t+1) &= x_j(t) - \frac{1}{2}\hat{x}_j(t) + \frac{1}{2}\hat{x}_i(t), \end{aligned} \tag{6.7}$$

or, equivalently in vector form,

$$x(t+1) = x(t) + (P(t) - I)\hat{x}(t). \tag{6.8}$$

Accordingly to Section 5.4.1 of the previous chapter, we call the law (6.4) *globally quantized* and the law (6.7) *partially quantized*. It is easy to see that the *partially quantized* law (6.7), as the law (6.1) and the time-invariant partially quantized rule (5.5), maintains the initial state average. This fact is stated in the following proposition.

Proposition 6.2 *Consider (6.7). Let*

$$x_{ave}(t) = \frac{1}{N}\mathbb{1}^* x(t).$$

Then

$$x_{ave}(t) = x_{ave}(0)$$

for all $t \geq 0$.

Proof: We have that

$$\begin{aligned} \mathbb{1}^* x(t+1) &= \mathbb{1}^* x(t) + \mathbb{1}^*(P(t) - I)\hat{x}(t) \\ &= \mathbb{1}^* x(t) \end{aligned}$$

where the last inequality follows from the fact that, since $P(t)$ is doubly stochastic for all $t \geq 0$, it holds $\mathbb{1}^*(P(t) - I) = 0$ for all $t \geq 0$. ■

We proceed our analysis of these two rules by assuming first the $\hat{x}_i(t) = q_d(x_i(t))$, i.e., the information transmitted is quantized by means of deterministic quantizer, and then by assuming that $\hat{x}_i(t) = q_p(x_i(t))$, i.e., the information transmitted is quantized by means of probabilistic quantizer, introduced in Section 5.2.

Remark 6.3 As in the previous chapter, we assume here that the set of quantization levels is constituted of only integers numbers. As underlined in Remark 5.4, the more general case can be treated by a suitable scaling.

6.3 Quantized symmetric gossip algorithms via deterministic quantizers

In this section we assume that the information exchanged between the systems is quantized by means of the deterministic quantizer q_d described in

(5.7), namely $\hat{x}_i(t) = q_d(x_i(t))$. Hence (6.4) and (6.7) become

$$\begin{aligned} x_i(t+1) &= \frac{1}{2}q_d[x_i(t)] + \frac{1}{2}q_d[x_j(t)] \\ x_j(t+1) &= \frac{1}{2}q_d[x_j(t)] + \frac{1}{2}q_d[x_i(t)], \end{aligned} \tag{6.9}$$

and

$$\begin{aligned} x_i(t+1) &= x_i(t) - \frac{1}{2}q_d[x_i(t)] + \frac{1}{2}q_d[x_j(t)] \\ x_j(t+1) &= x_j(t) - \frac{1}{2}q_d[x_j(t)] + \frac{1}{2}q_d[x_i(t)], \end{aligned} \tag{6.10}$$

In the following we will analyze the two strategies separately, starting from the latter.

6.3.1 Partially quantized strategy

Consider the partially quantized strategy (6.10). Let us define

$$d(t) = \frac{1}{\sqrt{N}}\|y(t)\|_2, \tag{6.11}$$

where

$$y(t) = \left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right)x(t). \tag{6.12}$$

Such quantity represents the distance of the state $x(t)$ from the current average of the state. Since the *partially quantized strategy* is an average-preserving law, $y(t)$ coincides with the distance of the state from the initial average.

As an example we report in Figure 6.1 the result of simulations relative to a connected random geometric graph. Such graph has been drawn placing $N = 50$ nodes uniformly at random inside the unit square and connecting two nodes whenever the distance between them is less that $R = 0.3$. The initial condition $x_i(0)$ is randomly chosen inside the interval $[-100, 100]$ for all $1 \leq i \leq N$. Note that $d(t)$ does not converge to 0, meaning that the average consensus is not reached, but the values get very close to the initial average.

We now prove this fact in general, quantifying how close asymptotically the systems get to the initial average of their states. To this aim, we will take advantage again of the *symbolic dynamics* which lies under the real

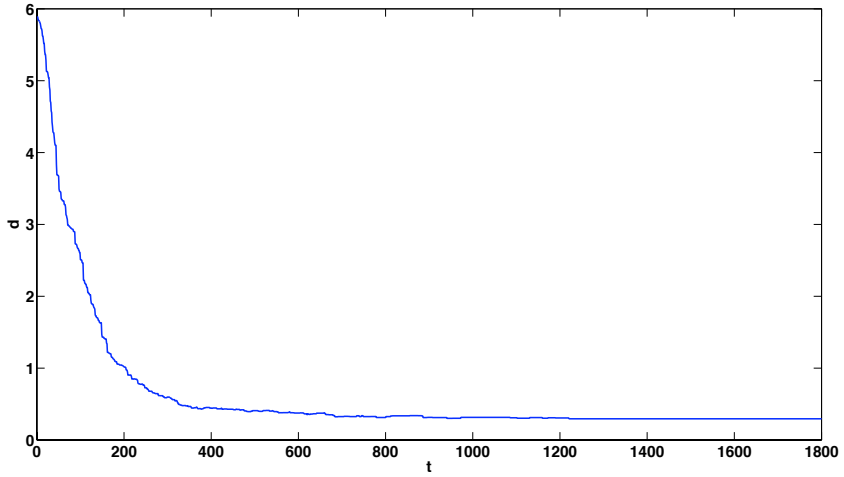

Figure 6.1: Behavior of d for a connected random graph with $N = 50$ in case of deterministic quantizers and of partially quantized strategy.

states dynamics and that we have introduced in Example 5.8. To the idea of this construction we will adapt the results presented in [75].

Let us start our analysis. We define $n_i(t) = \lfloor 2x_i(t) \rfloor$ for all $i \in V$. Simple properties of floor and ceiling operators, together with the Lemma 5.9 stated in Example 5.8, allow us to remark that $q[x_i(t)] = \left\lceil \frac{n_i(t)}{2} \right\rceil$ and that

$$\begin{aligned} x_i(t+1) &= x_i(t) - \frac{1}{2} q[x_i(t)] + \frac{1}{2} q[x_j(t)] \\ \lfloor 2x_i(t+1) \rfloor &= \lfloor 2x_i(t) \rfloor - q[x_i(t)] + q[x_j(t)], \end{aligned}$$

from which we can obtain that

$$\begin{aligned} n_i(t+1) &= n_i(t) - \left\lceil \frac{n_i(t)}{2} \right\rceil + \left\lceil \frac{n_j(t)}{2} \right\rceil \\ &= \left\lfloor \frac{n_i(t)}{2} \right\rfloor + \left\lceil \frac{n_j(t)}{2} \right\rceil. \end{aligned}$$

We have thus found an iterative system involving only the symbolic signals $n_i(t)$. When the edge (i, j) is selected, i and j adjourn their states following the pair dynamics

$$(n_i(t+1), n_j(t+1)) = g(n_i(t), n_j(t)) \tag{6.13}$$

where

$$g(h,k) = \left(\left\lfloor \frac{h}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil , \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{h}{2} \right\rceil \right).$$

It is clear that g is symmetric in the arguments, in the sense that if $g(h,k) = (\eta, \chi)$, then $g(k,h) = (\chi, \eta)$.

The analysis of the evolution of (6.13) will then allow us to obtain information about the asymptotics of $x_i(t)$, since $n_i(t) = \lfloor 2x_i(t) \rfloor$.

Before stating the main result regarding the convergence properties of (6.13), we define the following quantities

$$m(t) = \min_{1 \le i \le N} n_i(t) \tag{6.14}$$

and

$$M(t) = \max_{1 \le i \le N} n_i(t), \tag{6.15}$$

and, finally,

$$D(t) = M(t) - m(t).$$

Moreover we provide the following two notational definitions. Let

$$\mathcal{R} = \left\{ r \in \mathbb{Z}^N : r - \alpha \mathbb{1} \in \{0,1\}^N , \exists\, \alpha \in \mathbb{Z} \right\}. \tag{6.16}$$

and let $n(t) = [n_1(t), \ldots, n_N(t)]^*$.

We have the following result.

Theorem 6.4 *Almost surely there exists $T_{con} \in \mathbb{N}$ such that $n(t) \in \mathcal{R}$ for all $t \ge T_{con}$.*

Proof: The proof is based on verifying the following three facts:

(i) the evolution of $n(t)$ is a Markov process with a finite number of states;

(ii) the set $\mathcal{R}$, defined in (6.16), is an invariant subset for the evolution described by (6.13);

(iii) there is a positive probability of reaching a state belonging to the invariant subset $\mathcal{R}$, starting from any initial condition.

Standard results in the Markov chains literature (see [93]), ensure that, if the above three facts yield true, then any trajectory starting from a state $\bar{n} \notin \mathcal{R}$, will reach with probability 1 in a finite time, a state belonging to $\mathcal{R}$.

Let us now check them in order.

(i) We start by observing that the Markov property of the process follows from Assumption 6.1. We prove now that the states are finite. To this aim let

$$(h', k') = g(h, k).$$

By the structure of g, it is easy to see that

$$\max\{h', k'\} \leq \max\{h, k\}$$

and

$$\min\{h, k\} \leq \min\{h', k'\}.$$

Therefore we have that $m(n(t)) \geq m(n(0))$ and $M(n(t)) \leq M(n(0))$ and hence that the cardinality of the set of the states is upper bounded by $(M(n(0)) - m(n(0)) + 1)^N$.

(ii) Let $h \in \mathbb{Z}$. Observe that

$$g(h, h+1) = \begin{cases} (h+1, h) & \text{if } h \text{ is even} \\ (h, h+1) & \text{if } h \text{ is odd} \end{cases}$$

This implies that $\mathcal{R}$ is an invariant subset for the dynamics described by (6.13).

(iii) The proof of this fact is based on the following strong result about the monotonicity of $D(t)$: if $D(t) \geq 2$, then there exists $\tau \in \mathbb{N}$ such that

$$\mathbb{P}[D(t+\tau) < D(t)] > 0. \tag{6.17}$$

Now we prove (6.17).

Let $\mathcal{I}(t) = \{j \in \mathcal{V} \text{ s.t. } n_j(t) = m(t)\}$. We start by proving that $|\mathcal{I}(t)|$, i.e., the cardinality of $\mathcal{I}(t)$, does not increase and that, if $D(t) \geq 2$, then there is a positive probability that it decreases within a finite number of time steps. Notice first that, for $h, k \in \mathbb{Z}$, $g(h+2, k+2) = g(h, k) + 2$. Hence, by an appropriate translation of the initial condition, we can always restrict ourselves to the case $m(t) \in \{0, 1\}$, which of course is easier to handle.

Case $m(t) = 0$. In this case it is possible for a nonzero state to decrease to 0, but only in the case of a swap between 0 and 1. This assures that $|\mathcal{I}(t)|$ is nonincreasing. Let $\mathcal{S}(t)$ denote the set of nodes which have value $m(t) + 2$ or more. Since $D(t) \geq 2$ then $\mathcal{S}(t)$ is non empty at time t. Now let $(v_1, v_2, \ldots, v_{p-1}, v_p)$ be a shortest path between $\mathcal{I}(t)$ and $\mathcal{S}(t)$. Such a path exists since $\mathcal{G}$ is connected. Note that $v_1 \in \mathcal{I}(t)$ and $v_p \in \mathcal{S}(t)$ and that $\{v_2, \ldots, v_{p-1}\}$ could be an empty set; in this case

a shortest path between $\mathcal{I}(t)$ and $\mathcal{S}(t)$ has length 1. Moreover note also that all the nodes in the path except v_1 and v_p have value 1 at time t, otherwise $(v_1, v_2, \ldots, v_{p-1}, v_p)$ is not a shortest path. Since each edge of the communication graph has a positive probability of being selected in any time, there is also a positive probability that in the $p-1$ time units following t the edges of this path are selected sequentially, starting with the edge (v_1, v_2). At the last step of this sequence we have that the values of v_{p-1} and v_p are updated. By observing again, that the pair of value $(0, 1)$ is transformed by (6.13) into the pair $(1, 0)$ we have that the value of v_{p-1}, when the edge (v_{p-1}, v_p) is selected, is equal to 0. This update, for the form of (6.13), will cause the value of both nodes to be strictly greater than 0. Therefore, this proves that $|\mathcal{I}(t+p-1)| < |\mathcal{I}|$ with positive probability. Clearly, if $|\mathcal{I}(t)| = 1$ then we have also that $D(t+p-1) < D(t)$ with positive probability.
Case $m(t) = 1$. In this case no state can decrease to 1, and then $|\mathcal{I}(t)|$ is not increasing. Let $\mathcal{I}(t)$, $\mathcal{S}(t)$ and $(v_1, v_2, \ldots, v_{p-1}, v_p)$ be defined as in the previous case. Obviously in this case all the nodes $v_2, \ldots, v_{p-1}$ in the path have value equal to 2. Moreover observe that also the sequence of edges (v_{p-1}, v_p), (v_{p-2}, v_{p-1}),$\ldots$,(v_2, v_3), (v_1, v_2) has positive probability of being selected in the $p-1$ time units following t. At the last step of this sequence of edges, the values of v_1 and v_2 are updated. Clearly the value of v_1 is equal to 1. Since the value of v_p at time t is greater or equal to 3, and since the pair $(2, 3)$ is transformed by (6.13) into $(3, 2)$, we have that the value of v_2 when the edge (v_1, v_2) is selected, is greater or equal to 3. This update, for (6.13), will cause the value of both nodes to be strictly greater than 1. Hence $|\mathcal{I}(t+p-1)| < |\mathcal{I}|$ with positive probability. Again, if $|\mathcal{I}(t)| = 1$ then we have also that $D(t+p-1) < D(t)$ with positive probability.
Consider now the following sequence of times $t_0 = t, t_1, t_2, \ldots$. For each $i \geq 0$, if $|\mathcal{I}(t)| > 1$, then we let t_{i+1} to be the first time for which there is a positive probability that $|\mathcal{I}(t_{i+1})| < |\mathcal{I}(t_i)|$. Let now $k \in \mathbb{N}$ be such that $|\mathcal{I}(t_k)| = 1$. Then we have that $D(t_{k+1}) < D(t_k)$. This ensures the validity of (6.17).

The proof of the fact (iii) follows directly from (6.17). Indeed, let $\bar{n} \notin \mathcal{R}$, then, from a repeated application of (6.17) it follows that, there exists a path connecting $\bar{n}$ to a state $\bar{n}' = [\bar{n}'_1, \ldots, \bar{n}'_N]$, such that

$$\max\{\bar{n}'_1, \ldots, \bar{n}'_N\} - \min\{\bar{n}'_1, \ldots, \bar{n}'_N\} < 2,$$

i.e, $\bar{n}' \in \mathcal{R}$.

This proves the thesis. ■

We can go back to the original system, and prove the following result.

Corollary 6.5 *Consider the algorithm (6.10). Then, almost surely, there exists* $T_{con} \in \mathbb{N}$ *such that*

$$|x_i(t) - x_j(t)| \leq 1 \quad \forall\, i, j \quad \forall\, t \geq T_{con}, \tag{6.18}$$

and hence

$$\|x(t) - x_{ave}\mathbb{1}\|_\infty \leq 1$$

where $x_{ave} = \frac{1}{N}\mathbb{1}^* x(0)$.

Proof: The proof is an immediate consequence of Theorem 6.4 and of the relation $n_i(t) = \lfloor 2x_i(t) \rfloor$. ■

Remark 6.6 It is worth noting that Theorem 6.4 is an extension of Lemma 3 and Theorem 1 in [75]. In [75] the authors introduced a class of *quantized gossip algorithms*, satisfying the following assumptions. Assume that (i, j) is the edge selected at time t and that $n_i(t)$ and $n_j(t)$ are respectively the values of node i and of node j at time t. If $n_i(t) = n_j(t)$ then $n_i(t+1) = n_i(t)$ and $n_j(t+1) = n_j(t)$. Otherwise, defined $D_{ij} = |n_i(t) - n_j(t)|$, the method used to update the values has to satisfy the following three properties:

(P1) $n_i(t+1) + n_j(t+1) = n_i(t) + n_j(t)$,

(P2) if $D_{ij}(t) > 1$ then $D_{ij}(t+1) < D_{ij}(t)$, and

(P3) if $D_{ij}(t) = 1$ and (without loss of generality) $n_i(t) < n_j(t)$, then $n_i(t+1) = n_j(t)$ and $n_j(t+1) = n_i(t)$. This update is called *s*wap.

Now we substitute the property $(P3)$ either with the property

(P3') if $D_{ij}(t) = 1$ and (without loss of generality) $n_i(t) < n_j(t)$, then, if $n_i(t)$ is odd, then $n_i(t+1) = n_j(t)$ and $n_j(t+1) = n_i(t)$, otherwise if $n_i(t)$ is even then $n_i(t+1) = n_i(t)$ and $n_j(t+1) = n_j(t)$

or with the property

(P3") if $D_{ij}(t) = 1$ and (without loss of generality) $n_i(t) < n_j(t)$, then, if $n_i(t)$ is even then $n_i(t+1) = n_j(t)$ and $n_j(t+1) = n_i(t)$, otherwise if $n_i(t)$ is odd then $n_i(t+1) = n_i(t)$ and $n_j(t+1) = n_j(t)$.

If we consider the class of algorithms satisfying (P1), (P2), (P3') or satisfying (P1), (P2), (P3"), it is possible to prove that Lemma 3 and Theorem 1 stated in [75] hold true also for this class. The proofs are analogous to that of Lemma **??** and Theorem 6.4 provided in this paper. Moreover it is easy to see that the algorithm (6.13) satisfies the properties (P1), (P2), (P3'). This represents an alternative way to prove Theorem 6.4.

6.3.2 Globally quantized strategy

In this subsection we consider the globally quantized strategy (6.9). We underline immediately that the fact that (6.9) uses only quantized information and not perfect information combined with quantized information as in (6.10) makes the analysis of (6.10) slightly easier than the analysis of (6.9).
Remarkably, we will show in this subsection that the law (6.10) drives, almost surely, the systems to exact consensus at an integer value. Unfortunately, the initial average of states is not preserved in general. Again, the analysis of this algorithm can be performed efficiently by means of the *symbolic dynamics* previously introduced.
Let again $n_i(t) = \lfloor 2x_i(t) \rfloor$ for all $i \in V$. By recalling that $q_d(x_i(t)) = \left\lceil \frac{n_i(t)}{2} \right\rceil$ and by observing that

$$\lfloor 2x_i(t+1) \rfloor = q_d(x_i(t)) + q_d(x_j(t)),$$

we obtain

$$n_i(t+1) = \left\lceil \frac{n_i(t)}{2} \right\rceil + \left\lceil \frac{n_j(t)}{2} \right\rceil . \tag{6.19}$$

Hence also for (6.9), we have found an iterative system involving only the symbolic signals $n_i(t)$. When the edge (i, j) is selected, i and j adjourn their states following the pair dynamics

$$(n_i(t+1), n_j(t+1)) = \left(\left\lceil \frac{n_i(t)}{2} \right\rceil + \left\lceil \frac{n_j(t)}{2} \right\rceil , \left\lceil \frac{n_i(t)}{2} \right\rceil + \left\lceil \frac{n_j(t)}{2} \right\rceil \right) . \tag{6.20}$$

Let now $n(t) = [n_1(t), \dots, n_N(t)]^*$ and let $g_1 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined as

$$g_1(h, k) := \left\lceil \frac{h}{2} \right\rceil + \left\lceil \frac{k}{2} \right\rceil .$$

Note that

$$(n_i(t+1), n_j(t+1)) = (g_1(n_i(t), n_j(t)), g_1(n_i(t), n_j(t))) . \tag{6.21}$$

Then we have the following result.

Theorem 6.7 *Let $n(t)$ be as defined above and consider* (6.21)*. Then almost surely there exists $T_{con} \in \mathbb{N}$ and $\alpha \in \mathbb{Z}$ such that $n(t) = 2\alpha\mathbb{1}$ for all $t \geq T_{con}$.*

Proof: The proof is based on verifying the following three facts:

(i) the evolution process is a Markov process with a finite number of states;

(ii) the process (6.21) has absorbing states;

(iii) there is a positive probability of reaching an absorbing state in a finite time, starting for any initial state.

These facts, similarly to Theorem 6.4, ensures the validity of the thesis.

Let us now check them in order. Given $n(t)$ let $m(t)$ and $M(t)$ be defined as in (6.14) and (6.15).

(i) The Markov property of the process follows from Assumption 6.1. We show now that the states are finite. Let $p, q \in \mathbb{Z}$, where, without loss of generality, we assume that $p \leq q$. Then, from the structure of g_1 we have that

- $p \leq g_1(p, q)$;
- if q is even, $g_1(p, q) \leq q$;
- if q is odd, $g_1(p, q) \leq q + r_q$ where r_q denotes the remainder in the euclidean division of q over 2.

It follows that $m(0) \leq n_i(t) \leq M(0) + r_{M(0)}$ for all $i \in \mathcal{V}$ and for all $t \geq 0$, where $r_{M(0)}$ denotes the remainder in the euclidean division of $M(0)$ over 2. Hence the cardinality of the set of the states is upper bounded by

$$(M(0) + r_{M(0)} + 1 - m(0))^N.$$

(ii) We denote as $\mathcal{A}$ the set of absorbing states for (6.21). The form of g_1 implies that if $n_i(t) \neq n_j(t)$, then $n_i(t+1) = n_j(t+1)$, and if $n_i(t) = n_j(t) = a$ then $n_i(t+1) = n_j(t+1) = a$ if and only if a is even. Thus $\mathcal{A} = \{y \in \mathbb{Z}^N : \exists\, \alpha \in \mathbb{Z} \text{ such that } y = 2\alpha\mathbb{1}\}$. Thus the absorbing states are consensus states.

(iii) Let us fix $t = t_0$, and assume that $n(t_0) \notin \mathcal{A}$. We will prove that there exists $\tau \in \mathbb{N}$ such that $\mathbb{P}[n(t_0 + \tau) \in \mathcal{A}] > 0$. We start by observing that, from the assumption of having a connected graph, there exists $(i, j) \in \mathcal{E}$ such that $n_i(t_0) = m(t_0)$, $n_j(t_0) = q$ and $g(m(t_0), q) > m(t_0)$. Indeed, two cases are given when $n(t_0) \notin \mathcal{A}$.

- If $m(t_0) < M(t_0)$, then it suffices to consider there an edge (i, j) such that $n_i(t_0) = m(t_0)$ and $n_j(t_0) = q > m(t_0)$ which gives $g_1(m(t_0), p) > m(t_0)$. Note that this edge there exists from the hypothesis of having a connected graph;
- if $m(t_0) = M(t_0)$, necessarily we have that $m(t_0)$ and $M(t_0)$ are odd; then $g(m(t_0), m(t_0)) > m(t_0)$.

We define now $\mathcal{I}_a(t) = \{i \in \mathcal{V} : n_i(t) = a\}$. The above discussion implies that $|\mathcal{I}_{m(t_0)}(t_0 + 1)| < |\mathcal{I}_{m(t_0)}(t_0)|$ with the positive probability of choosing the edge (i, j) and hence that there is also a positive probability that at some finite time $t' > t_0$, $|\mathcal{I}_{m(t_0)}(t_0)| = 0$, that is $m(t') > m(t_0)$. Iterating this argument and, recalling that $M(t) \leq M(t_0) + r_{M(t_0)}$ for all $t \geq t_0$, show that there exists $\tau \in \mathbb{N}$ such that $\mathbb{P}[n(t_0 + \tau) \in \mathcal{A}] > 0$.

This proves the thesis. ■

We can go back to the original system. The following corollary follows immediately form the definition of $n(t)$.

Corollary 6.8 *Let $x(t)$ evolve according to (6.9). Then almost surely there exists $T_{con} \in \mathbb{N}$ and $\alpha \in \mathbb{Z}$ such that $x_i(t) = \alpha$ for all $i \in \mathcal{V}$ and for all $t \geq T_{con}$.*

Proof: It follows directly from the definition of $n_i(t) = \lfloor 2x_i(t) \rfloor$. ■

We have already underlined the fact that this strategy does not preserve the initial average, in general. Providing some probabilistic estimation of the distance of the consensus point from the initial average is a challenging problem. Unfortunately we have not been able to obtain any theoretical result so far. We limits our analysis to the following simulation. In Figure 6.2 we plotted the variable z that is defined as follows. In the globally quantized strategy we have that, almost surely $\lim_{t\to\infty} = \alpha\mathbb{1}$ for some random integer α. Let $z = |\alpha - 1/N\mathbb{1}^* x(0)|$. In other words z represents the distance from the consensus point to which the globally quantized strategy leads the systems and the average of the initial condition. We have depicted the value of z for a family of random geometric graphs of increasing size from $N = 10$ up to $N = 80$. The initial condition $x_i(0)$ is chosen randomly inside the interval $[-100, 100]$ for all $1 \leq i \leq N$. Moreover for each N, z is computed as the mean of 100 trials. We can see that the value of z is increasing in N and

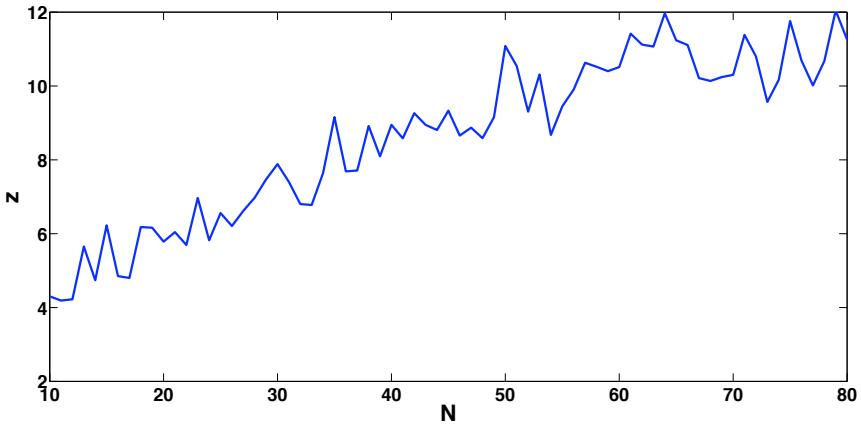

Figure 6.2: Behavior of z for a family of random geometric graphs in case of deterministic quantizers and of globally quantized strategy.

assumes values that are not negligible with respect to the quantization step size.

6.3.3 Some considerations on the speed of convergence of the globally quantized strategy and of the partially quantized strategy

Providing some insights on the speed of convergence of (6.10) and of (6.9) is quite hard in general. In Figure 6.3 and Figure 6.4 we report, respectively, a comparison between the partially quantized strategy (6.10) and the gossip algorithm with exchange of perfect information (6.1) and between the globally quantized strategy (6.9) and again the gossip algorithm with exchange of perfect information (6.1). The simulations are made on the same random geometric graph considered in Figure 6.1.

For all strategies we plotted the behavior of the variable $d(t)$ defined in (6.11). For both the simulations the initial conditions are randomly chosen inside the interval $[-100, 100]$. It is worth noting that the variable $y(t)$, introduced in (6.12) and whose normalized two-norm defines $d(t)$, represents the distance of the state $x(t)$ from its current average, that, only for (6.10) and (6.1), coincides also with the distance of the state $x(t)$.

From the Figure 6.3 and Figure 6.4 we can infer that the speed of convergence toward the steady state of the quantized strategies (6.9) and (6.10) is

similar to the one of the gossip algorithm with perfect exchange of information. Unfortunately we have not been able to prove this numerical evidence so far.

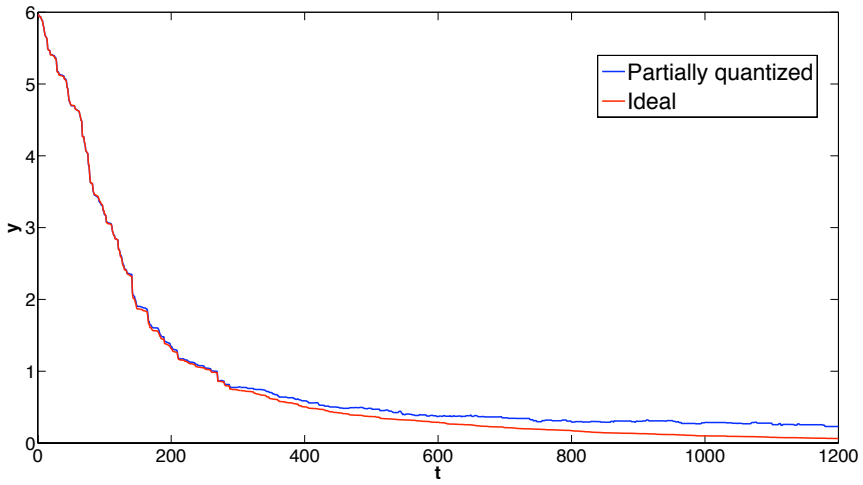

Figure 6.3: Behavior of d, when using the *partially quantized* strategy, for a connected random geometric graph with $N = 50$. Note that since the *partially quantized* strategy does not converge to a consensus, $d(t)$ does not go to 0.

Remark 6.9 We have observed in Figure 6.2, that the distance from the average of the initial state of the consensus point toward which the globally quantized strategy leads the systems is not negligible. On the other hand, the partially quantized strategy does not assure a consensus in a strict sense.

If, depending on the application, one can not relax the convergence requirement, inspired by the last subsection of the previous chapter, we could suggest the following heuristic solution to the consensus problem, which combines the positive features of both strategies,

$$x(t+1) = Pq_d(x(t)) + \epsilon(t)(x(t) - q_d(x(t))),$$

where $\epsilon(t),\ t \geq 0$, is a nonnegative sequence such that $\epsilon(t) \leq 1,\ \forall\, t \geq 0$ and $\lim_{t\to\infty} \epsilon(t) = 0$.

Remark 6.10 Some more insights on the speed of convergence of the quantized strategies and on the combined strategy presented in the above remark,

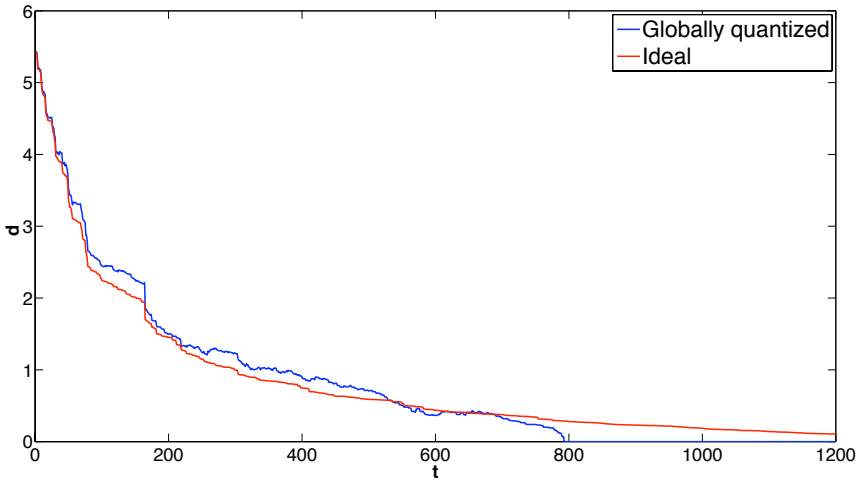

Figure 6.4: Behavior of d, when using the *globally quantized* strategy, for a connected random geometric graph with $N = 50$. In this case, accordingly to the theoretical result stated in Corollary 6.8, $d(t)$ tends to 0.

could be provided by introducing a probabilistic model in which the quantization error is regarded as a white noise uncorrelated with the state of the system. As we have widely underlined in the previous chapter, there is no theoretical reason justifying the validity of this model when using deterministic quantizers. Instead the probabilistic model is a suitable model to explain the quantized algorithms via probabilistic quantizers: we will see this in the next Section.

6.4 Quantized symmetric gossip algorithms via probabilistic quantizers

In this section we assume that the information exchanged between the systems is quantized by means of the probabilistic quantizer q_p described in (5.8), namely $\hat{x}_i(t) = q_p(x_i(t))$. Hence (6.4) and (6.7) become

$$\begin{aligned} x_i(t+1) &= \frac{1}{2} q_p[x_i(t)] + \frac{1}{2} q_p[x_j(t)] \\ x_j(t+1) &= \frac{1}{2} q_p[x_j(t)] + \frac{1}{2} q_p[x_i(t)], \end{aligned} \tag{6.22}$$

and

$$x_i(t+1) = x_i(t) - \frac{1}{2}q_p[x_i(t)] + \frac{1}{2}q_p[x_j(t)]$$
$$x_j(t+1) = x_j(t) - \frac{1}{2}q_p[x_j(t)] + \frac{1}{2}q_p[x_i(t)], \tag{6.23}$$

From now on we will assume that the initial condition $x(0)$ satisfes the following.

Assumption 6.11 *The initial condition $x(0)$ is a random variable such that $\mathbb{E}[x(0)] = 0$ and $\mathbb{E}[x(0)x^*(0)] = \sigma_0^2 I$ for some $\sigma_0^2 > 0$*

In the remainder of the section we will analyze the two strategies separately starting from the latter.

6.4.1 Partially quantized strategy

Consider the partially quantized strategy (6.23). Similarly to the partially quantized strategy via deterministic quantizers (6.10), also (6.23) does not reach the consensus in general. Again we report a simulation showing this fact. In Figure 6.5 the behavior of the quantity $d(t)$, defined in (6.11), is depicted for the same connected random geometric graph considered in Figure 6.1. Note that the quantity $d(t)$ does not converge a 0, meaning that the average consensus is not reached.

We observe immediately that the analysis of (6.23) is more complicate that the corresponding law (6.10). This is mainly due to the lack of some convexity arguments which we used in the analysis of (6.10). We will be more explicit in the following example.

Example 6.12 Consider (6.10) and assume that the edge (i, j) has been selected at time t. Without loss of generality assume that $x_i(t) \leq x_j(t)$. Then, by convexity arguments, we have that $\lfloor x_i(t) \rfloor \leq x_i(t+1), x_j(t+1) \leq \lceil x_j(t) \rceil$. This is not true for (6.23) anymore. As a numerical example assume that $x_i(t) = 3.4$ and $x_j(t) = 3.6$ and that $q_p(x_i(t)) = 4$ and $q_p(x_j(t)) = 3$. Then by (6.23) we have that $x_i(t+1) = 2.9$ and that $x_j(t+1) = 4.1$, namely $x_i(t+1), x_j(t+1)$ do not belong to the interval $[\lfloor x_i(t) \rfloor, \lceil x_j(t) \rceil]$.

By simulations we can see that (6.23) does not drive the states of the systems inside the same bin of quantization, as the corresponding strategy (6.10)

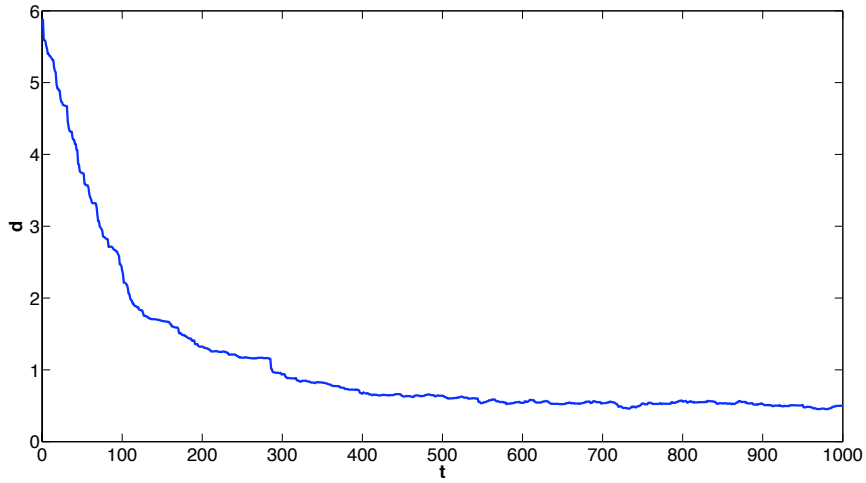

Figure 6.5: Behavior of d for a connected random geometric graph with $N = 50$.

using deterministic quantizers. In Figure 6.6, we depict the behavior of the quantity

$$s(t) = \max_{1 \leq i,j \leq N} |x_i(t) - x_j(t)|.$$

for the same random geometric graph considered in Figure 6.5. In this simulation we assume that the initial condition $x_i(0)$ is randomly chosen inside the interval $[-10, 10]$. Note that s asymptotically oscillates around 2. This

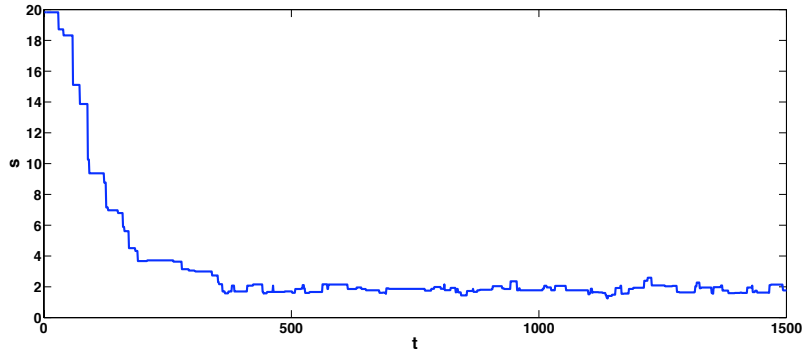

Figure 6.6: Behavior of s for a connected random geometric graph with $N = 50$.

suggests that (6.23) seems to work a little bit worse than (6.10). However, of note, is that also (6.23) behaves quite well.

Interesting results on (6.23), in terms of both the asymptotic distance from the initial average and the speed of convergence, can be provided by a mean-square analysis that we will carry out in the sequel of this subsection. We start by observing that (6.23) can be rewritten as

$$x(t+1) := P(t)x(t) + (P(t) - I)\,(q(x(t)) - x(t)) \tag{6.24}$$

We consider again the variable

$$y(t) := \left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right) x(t) = x(t) - \left(\frac{1}{N}\mathbb{1}^* x(t)\right)\mathbb{1} = x(t) - \left(\frac{1}{N}\mathbb{1}^* x(0)\right)\mathbb{1}$$

where the last equality follows from the fact that the partially quantized strategy is an average-preserving law and we introduce the new variable

$$e(t) = q(x(t)) - x(t).$$

which represents the quantization error. Note that $\mathbb{1}^* y(t) = 0$ for all $t \geq 0$.

Now, from (6.24), we can write

$$\left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right) x(t+1) = \left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right) P(t)x(t) + \\ + \left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right)(P(t) - I)\,(q(x(t)) - x(t))\,.$$

Since $\left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right) P(t) = P(t)\left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right)$ and $\left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right)(P(t) - I) = P(t) - I$, from the above expression we obtain the following recursive equation in terms of the variables y, e,

$$y(t+1) = P(t)y(t) + (P(t) - I)e(t). \tag{6.25}$$

In order to perform an asymptotic analysis of (6.25) it is convenient to introduce the following matrices. Let

$$\Sigma_{yy}(t) := \mathbb{E}\left[y(t)y^*(t)\right],$$

$$\Sigma_{ee}(t) := \mathbb{E}\left[e(t)e(t)^*\right],$$

and

$$\Sigma_{ye}(t) := \mathbb{E}\left[y(t)e(t)^*\right].$$

Equation (6.25) leads to the following recursive equation in terms of the above matrices

$$\Sigma_{yy}(t+1) = \mathbb{E}\left[P(t)\Sigma_{yy}(t)P(t)\right] + \mathbb{E}\left[P(t)\Sigma_{ye}(t)(P(t)-I)\right] + \\ + \mathbb{E}\left[(P(t)-I)\Sigma_{ye}^{*}P(t)\right] + (P(t)-I)\,\Sigma_{ee}(t)\,(P(t)-I)\,. \tag{6.26}$$

Notice that, since $x(0)$ is a random variable satisfying Assumption 6.11, one can show that

$$\Sigma_{yy}(0) = \sigma_0^2\,(I - 1/N\,\mathbb{1}\mathbb{1}^*)\,. \tag{6.27}$$

We recall, from Proposition 5.35 stated in Section 5.4, that the variables y and e satisfy the following remarkable properties

(i) $\mathbb{E}\left[e(t)\right] = 0$ and $\mathbb{E}\left[e(t)e^*(t)\right] = \text{diag}\,\{\sigma_1^2(t),\ldots,\sigma_N^2(t)\}$ where $\sigma_i^2(t) := \mathbb{E}\left[e_i^2(t)\right]$ is such that $\sigma_i^2(t) \leq 1/4$ for all $1 \leq i \leq N$ and for all $t \geq 0$.

(ii) $\Sigma_{ye}(t) = 0$, for all $t \geq 0$.

From the above properties we have that (6.26) can be rewritten as

$$\Sigma_{yy}(t+1) = \mathbb{E}\left[P(t)\Sigma_{yy}(t)P(t)\right] + \mathbb{E}\left[(P(t)-I)\,\Sigma_{ee}(t)\,(P(t)-I)\right]. \tag{6.28}$$

To estimate the asymptotic distance from the initial average, we introduce the following cost function

$$J(W) := \limsup_{t\to\infty} \sqrt{\frac{1}{N}\text{tr}\,\{\Sigma_{yy}(t)\}}. \tag{6.29}$$

We can rewrite the above evolution law as

$$\Sigma_{yy}(t+1) = \mathcal{A}(\Sigma_{yy}(t)) + \mathcal{B}(\Sigma_{ee}(t)),$$

where $\mathcal{A}$ and $\mathcal{B}$ are linear operators from $R^{N\times N}$ to itself. Namely, given a matrix M, $\mathcal{A}(M) = \mathbb{E}\left[P(t)MP(t)\right]$ and $\mathcal{B}(M) = \mathbb{E}\left[(P(t)-I)\,M\,(P(t)-I)\right]$. It is immediate to remark that $\mathcal{A}$ induces the natural evolution of the dynamical system, in the absence of quantization error, while $\mathcal{B}$ can be regarded as a disturbance due to the quantization error.

From [58], we know that in the case of no quantization the systems converge almost surely to consensus. This implies that $\mathcal{A}$ is an asymptotically stable operator when restricted to the subspace $\mathcal{S} = \{M \in \mathbb{R}^{N\times N} : \mathbb{1}^*M\mathbb{1} = 0\}$. Since $\mathbb{1}^*\mathcal{B}(M)\mathbb{1} = 0$ for any matrix M, we have that $\Sigma_{yy}(t) \in \mathcal{S}$ for all $t \geq 0$.

It is interesting to remark that these observations show how, during the transient, the symmetric gossip algorithm, with exchange of information quantized by means of probabilistic quantizers, behaves like the symmetric gossip algorithm with exchange of exact information.

Providing an expression for $J(W)$ is quite hard in general. We try now to simplify the problem by introducing the following auxiliary system

$$\bar{\Sigma}(t+1) = \mathbb{E}\left[P(t)\bar{\Sigma}(t)P(t)\right] + \frac{1}{4}\mathbb{E}\left[(P(t)-I)^2\right], \tag{6.30}$$

where $\bar{\Sigma}(0) = \Sigma_{yy}(0)$, and the following cost function

$$\bar{J}(W) := \limsup_{t\to\infty} \sqrt{\frac{1}{N}\mathrm{tr}\left\{\bar{\Sigma}(t)\right\}}.$$

We have the following comparison result.

Proposition 6.13 *Consider the functional costs $J(W)$ and $\bar{J}(W)$. We have that*

$$J(W) \leq \bar{J}(W).$$

Proof: To prove the statement of the Proposition we will show, by induction on t, that $\bar{\Sigma}(t) \geq \Sigma_{yy}(t)$ for all $t \geq 0$, where the inequality is meant in matricial sense, that is, $\bar{\Sigma}(t) - \Sigma_{yy}(t)$ is a semidefinite positive matrix.

Since $\bar{\Sigma}(0) = \Sigma_{yy}(0)$ the assertion holds true for $t = 0$. Assume now that $\bar{\Sigma}(t) \geq \Sigma_{yy}(t)$ holds true for a generic t. We have that

$$\begin{aligned}
&\bar{\Sigma}(t+1) - \Sigma_{yy}(t+1) = \\
&\quad = \mathbb{E}\left[P(t)\bar{\Sigma}(t)P(t)\right] + \frac{1}{4}\mathbb{E}\left[(P(t)-I)^2\right] \\
&\qquad - \left(\mathbb{E}\left[P(t)\Sigma_{yy}(t)P(t)\right] + \mathbb{E}\left[(P(t)-I)\,\Sigma_{ee}(t)\,(P(t)-I)\right]\right) \\
&\quad = \mathbb{E}\left[P(t)(\bar{\Sigma}(t) - \Sigma_{yy}(t))P(t)\right] \\
&\qquad + \mathbb{E}\left[(P(t)-I)\left(\frac{1}{4}I - \Sigma_{ee}(t)\right)(P(t)-I)\right].
\end{aligned}$$

Since by inductive hypothesis $\bar{\Sigma}(t) \geq \Sigma_{yy}(t)$ and since by Proposition 5.35 we know that $\Sigma_{ee}(t) \leq \frac{1}{4}I$ for all $t \geq 0$, we have that $\bar{\Sigma}(t+1) - \Sigma_{yy}(t+1) \geq 0$. ■

Observe now that, since $P(t)^2 = P(t)$ we obtain that $\mathbb{E}[(I - P(t))^2] = I - \mathbb{E}[P(t)]$. From this fact we obtain the following result.

Proposition 6.14 *Given the above definitions,*

$$\lim_{t\to\infty} \bar{\Sigma}(t) = \frac{1}{4}\left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right).$$

Proof: Since $\bar{\Sigma}_{yy}(0) \in S$, and $\mathcal{A}$ is asymptotically stable if restricted to the subspace $\mathcal{S}$, then

$$\lim_{t\to\infty} \bar{\Sigma}_{yy}(t) = \sum_{t=0}^{+\infty} \mathcal{A}^{(t)}(\bar{\mathcal{B}}),$$

where $\bar{\mathcal{B}} := \mathbb{E}[(I - P(t))^2]$. This is the only fixed point of the iteration law (6.30). Thus we are left to prove that $\Sigma^* = \frac{1}{4}\left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right)$ is a fixed point, that is $\Sigma^* = \mathcal{A}(\Sigma^*) + \bar{\mathcal{B}}$. Indeed observe that

$$\begin{aligned}\mathcal{A}(\Sigma^*) + \bar{\mathcal{B}} &= \frac{1}{4}\mathbb{E}\left[P(t)\left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right)P(t)\right] + \frac{1}{4}(I - \mathbb{E}[P(t)]) \\ &= \frac{1}{4}\left\{\mathbb{E}\left[P(t)^2\right] - \frac{1}{N}\mathbb{1}\mathbb{1}^* + I - \mathbb{E}[P(t)]\right\} = \frac{1}{4}\left\{I - \frac{1}{N}\mathbb{1}\mathbb{1}^*\right\}\end{aligned}$$

■

Corollary 6.15 *For all probability matrix W we have that $J(W) \leq \frac{1}{2}$.*

Proof: From the above theorem we can argue that $\bar{J}(W) = \frac{1}{2}\sqrt{\frac{N-1}{N}}$, and since $J(W) \leq \bar{J}(W)$, we can conclude. ■

From these theorems we draw a strong conclusion about the convergence of the algorithm. In spite of missing a consensus in the strict sense, the asymptotical mean squared error of the algorithm is smaller than the size of the quantization bin, and has a bound which does not depend on the number of the agents nor on the topology of the graph.

6.4.2 Globally quantized strategy

In this subsection we consider the globally quantized strategy (6.22). We will prove that the law (6.22), as the law (6.9), drives almost surely the systems to exact consensus at an integer value. Moreover, we will show by simulations, that the consensus point, even if (6.22) does not preserve the average of the

state, is rather close to the average of the initial condition. This represents a significant improvement with respect to the strategy (6.9), that, as seen in Figure 6.2, leads to a consensus point whose distance from the average of the initial condition, is not negligible in general.

We observe immediately that, with the globally quantized strategy (6.22), we have to deal with two sorts of randomness, since the interacting pair is randomly selected, and the quantization map is itself random. This makes the analysis of (6.22) more complicate than the analysis of (6.9). However, again, we prove the convergence by a *symbolic dynamics*.

Let again $n_i(t) = \lfloor 2x_i(t) \rfloor$ for all $i \in V$ and let $n(t) = [n_1(t), \dots, n_N(t)]^*$. Before finding a recursive equation for $n(t)$, we need to introduce the following random variable. Let

$$T_{all} = \inf \{t \, : \, \text{at time t every edge in } \mathcal{E} \text{ has been selected at least once}\}$$

T_{all} is an integer random variable which, since the edges are selected with positive probability, is almost surely finite. Note that, from (6.22), $x_i(t) \in \{a, a + 1/2\}$ for some integer number a, for all $t \geq T_{all}$. This allows us to disregard the evolution before T_{all} and to analyze, for $t > T_{all}$, the symbolic dynamics as follows. For $t \geq T_{all}$, by recalling how the probabilistic quantizer works, we have that

$$q_p[x(t)] = \begin{cases} \frac{n_i(t)}{2} & \text{if } n_i(t) \text{ is even} \\ \lceil \frac{n_i(t)}{2} \rceil \text{ with probability } 1/2 & \text{if } n_i(t) \text{ is odd} \\ \lfloor \frac{n_i(t)}{2} \rfloor \text{ with probability } 1/2 & \end{cases}$$

Let ξ_1 and ξ_2 be two independent Bernoulli random variables with parameter $1/2$ and assume that, at time instant t, the edge (i, j) is selected. Then

$$n_i(t+1) = n_j(t+1) = \left\lceil \frac{n_i(t)}{2} \right\rceil + \left\lceil \frac{n_j(t)}{2} \right\rceil - \xi_1 r_{n_i(t)} - \xi_2 r_{n_j(t)} \qquad (6.31)$$

where $r_{n_i(t)}$ and $r_{n_j(t)}$ denote, respectively, the remainders of the euclidean division of $n_i(t)$ over 2 and of $n_j(t)$ over 2. Let us define $g_2 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ such that

$$g_2(h, k) = \left\lceil \frac{h}{2} \right\rceil + \left\lceil \frac{k}{2} \right\rceil - \xi_1 r_h - \xi_2 r_k,$$

then we have that

$$(n_i(t+1), n_j(t+1)) = (g_2(n_i(t), n_j(t)), g_2(n_i(t), n_j(t))) \, . \qquad (6.32)$$

The following result characterizes the convergence properties of (6.32).

Theorem 6.16 *Let $n(t)$ be as above. Consider* (6.32)*. Then almost surely there exists $T_{con} \in \mathbb{N}$ and $\alpha \in \mathbb{Z}$ such that $n_i(t) = 2\alpha$ for all $i \in V$ and for all $t \geq T_{con}$.*

Proof: The proof is similar to the proof of Theorem 6.7 and Theorem 6.4, and it is based again on proving the following three facts:

(i) The evolution process is a Markov process with a finite number of states;

(ii) The process (6.32) has absorbing states;

(iii) There is a positive probability of reaching an absorbing state in a finite time, starting for any initial state.

Let us now check them in order. Given $n(t)$ let $m(t)$ and $M(t)$ be defined as in (6.14) and (6.15).

(i) The Markov property of the process follows from Assumption 6.1 and from (6.32). We show now that the states are finite. Let $h \in \mathbb{Z}$. Then, from the structure of g_2 we have that

- $g_2(h,h) = h$ if h is even;
- $h - 1 \leq g_2(h,h) \leq h + 1$ if h is odd.

The above two properties imply that $m(0) - r_{m(0)} \leq n_i(t) \leq M(0) + r_{M(0)}$ for all $i \in \mathcal{V}$ and for all $t \geq 0$, where $r_{M(0)}$ and $r_{m(0)}$ denote, respectively, the remainders in the euclidean division of $M(0)$ over 2 and of $m(0)$ over 2. Hence the cardinality of the set of the states is upper bounded by

$$(M(0) + r_{M(0)} + 1 - m(0) + r_{m(0)})^N.$$

(ii) We denote as $\mathcal{A}$ the set of absorbing states for (6.32). The form of g_2 implies that if $n_i(t) \neq n_j(t)$, then $n_i(t+1) = n_j(t+1)$, and if $n_i(t) = n_j(t) = a$ then $n_i(t+1) = n_j(t+1) = a$ with probability 1, if and only if a is even. Thus, as for (6.21), we have that $\mathcal{A} = \{y \in \mathbb{Z}^N : \exists\, \alpha \in \mathbb{Z} \text{ such that } y = 2\alpha \mathbb{1}\ \forall\, i\}$.

(iii) Observe that

$$g_2(h,k) = g_1(h,k) - \xi_1 r_h - \xi_2 r_k,$$

where g_1 is the map defining the evolution of (6.21). Hence

$$\mathbb{P}\left[g_2(h,k) = g_1(h,k)\right] \geq \frac{1}{4}.$$

This fact, combined with the fact (iii) proved along the proof of Theorem 6.7, ensures that, also for (6.32), there is a positive probability of reaching an absorbing state in a finite time, starting from any initial state.

This proves the thesis.

■

The above theorem and the previous remarks about T_{all} lead to the following claim about the original system.

Corollary 6.17 *Let $x(t)$ evolve following (6.22). Then almost surely there exists $T_{con} \in \mathbb{N}$ and $\alpha \in \mathbb{Z}$ such that $x_i(t) = \alpha$ for all $i \in \mathcal{V}$ and for all $t \geq T_{con}$.*

Proof: It follows directly from the definition of $n_i(t) = \lfloor 2x_i(t) \rfloor$. ■

As for (6.9) we have not been able so far to provide a theoretical estimation of the consensus point to which (6.22) leads the systems from the average of the initial condition.

We limit our analysis to the following simulations. In Figure 6.7 we plot the variable z as previously defined for the *globally quantized* strategy using deterministic quantizers, i.e., $z = |c - 1/N \mathbb{1}^* x(0)|$ where c is such that $\lim_{t\to\infty} x(t) = c\mathbb{1}$. z represents the distance from the consensus point to which the globally quantized strategy leads the systems and the average of the initial condition. We plot the value of z for a family of random geometric graphs of increasing size from $N = 10$ up to $N = 80$. The initial condition $x_i(0)$ is chosen randomly inside the interval $[-100, 100]$ for all $1 \leq i \leq N$. Moreover for each N, z is calculated as the mean of 100 trials. In Figure 6.8 we provide a comparison between (6.9) and (6.22). Surprisingly, the globally quantized strategy using probabilistic quantizers, differently from the globally quantized strategy using deterministic quantizers, seems to reach the consensus very close to the average of the initial condition.

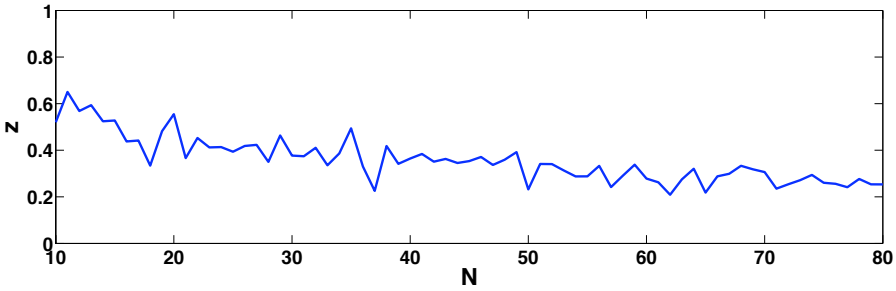

Figure 6.7: Behavior of z for a family of random geometric graphs when considering the globally quantized strategy using probabilistic quantizers.

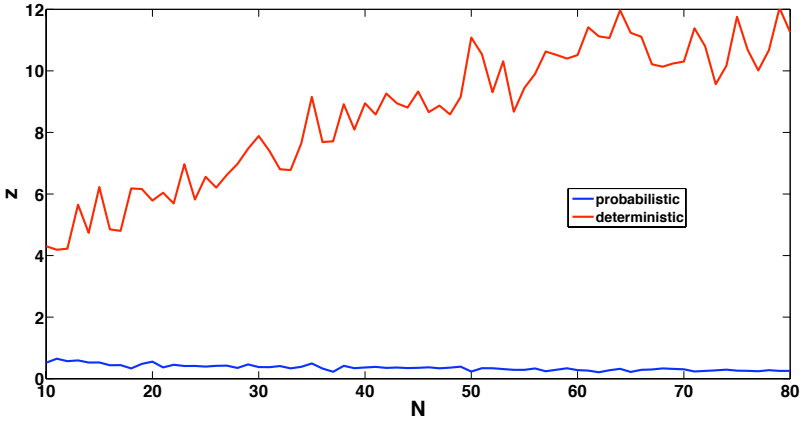

Figure 6.8: Comparison in terms of z between the "deterministic" and the "probabilistic" strategy, for a family of random geometric graphs.

6.5 Conclusions

In this chapter we addressed the quantized consensus problem for the symmetric gossip algorithm. In order to face the effects due to the quantization (both deterministic and probabilistic) we proposed here two updating rules: the globally quantized strategy and the partially quantized strategy. In the former the nodes use only quantized information in order to update their state. In the latter they have access also to exact information regarding their own state.

We have seen that the partially quantized strategy, with both the quantiz-

ers, deterministic and probabilistic, does not reach the consensus in general, but maintain the average of the state at each iteration and drive all the states very close to the average of the initial condition.

On the other hand, we have shown that the globally strategy leads almost surely to a consensus which, however, does not coincide with the average of the initial condition. We have provided some simulations characterizing the distance between the consensus point and the initial average. While using the deterministic quantizer this distance turns out to be not negligible, with the probabilistic quantizer the consensus is reached surprisingly very close to the average of the initial condition.

Providing some theoretical insights on this fact will be the object of future research. Moreover, an another interesting issue to address will be the evaluation of the speed of convergence of the strategies considered in this chapter.

Chapter 7

Quantized average consensus via dynamic coding/decoding schemes

7.1 Introduction

We have seen in Chapter 2 that the main features of the *ideal*[1] average consensus algorithm, both for time-varying version and the time-invariant version, are

- preservation of the average of the state at each iteration
- asymptotic convergence to an agreement that, since the algorithm preserves the average of the state at each iteration, coincides with the initial average.

In a more realistic case, in which the systems can communicate only through digital channels, i.e., they can exchange only symbolic data, the above conditions are in general violated. In the previous two chapters, in presence of quantized communications, we have elaborated strategies that:

- maintain the average of the state at each iteration, but do not converge asymptotically to the average consensus, even if they drive the states of all the systems quite close to it;

[1] with *ideal* average consensus algorithm is meant the algorithm in which the agents can exchange perfect information between them

- do not maintain the average of the state at each iteration, but reach asymptotically a consensus which is, in general, different from the average of the initial condition.

The main contribution of this chapter is to introduce a novel quantized strategy that permits both to maintain the initial average and to reach it asymptotically. More precisely, we adapt coding/decoding strategies, that were proposed for the centralized quantized control problems, to the distributed consensus problem. In particular, we present two coding/decoding strategies, one based on the exchange of logarithmically quantized information, the other on a zoom in - zoom out strategy (this latter involves the use of uniform quantizers). We provide analytical and simulative results illustrating the convergence properties of these strategies. In particular we show that the convergence factors depend smoothly on the accuracy parameter of the quantizers used and that, remarkably, the critical quantizer accuracy sufficient to guarantee convergence is independent from the network dimension.

The paper is organized as follows. Section 7.2 briefly reviews the standard average consensus algorithm. In Section 7.3 we present two strategies of coding/decoding of the data throughout reliable digital channels: one based on logarithmic quantizers, the other on uniform quantizers. We analyze the former from a theoretical point in Section 7.4 and Section 7.5. We provide simulations results for the latter in Section 7.6. Finally, we gather our conclusions in Section 7.7.

7.2 Problem Formulation

For the sake of the clarity, we start this section by briefly describing the standard discrete-time consensus algorithm. Assume that we have a set of agents V and a graph $\mathcal{G}$ on V describing the feasible communications among the agents. For each agent $i \in V$ we denote by $x_i(t)$ the estimate of the average of agent i at time t. We have seen in Chapter 2 that standard average consensus algorithms are constructed by choosing a doubly stochastic matrix $P \in \mathbb{R}^{N\times N}$ compatible with $\mathcal{G}$ and assuming that at every times t agent i updates its estimate according to

$$x_i(t+1) = \sum_{j=1}^{N} P_{ij} x_j(t). \tag{7.1}$$

More compactly we can write

$$x(t+1) = Px(t), \tag{7.2}$$

where $x(t)$ is the column vector whose entries $x_i(t)$ represent the agents states.

In our treatment we will restrict to the case in which P is symmetric, i.e., $P^* = P$. Note that a stochastic symmetric matrix P is automatically doubly stochastic.
We have seen in Chapter 2, that, if P is a symmetric stochastic matrix with positive diagonal entries and such that $\mathcal{G}_P$ is connected, then the algorithm (7.2) solves the *average consensus problem*, namely

$$\lim_{t \to +\infty} x(t) = x_{ave}\mathbb{1},$$

where $x_{ave} = 1/N\mathbb{1}^* x(0)$. From now on we will make the following assumption.

Assumption 7.1 *P is a symmetric stochastic matrix such that $P_{ii} > 0$, for $i \in \{1, \ldots, N\}$, and $\mathcal{G}_P$ is connected.*

Before proceeding we provide the following notational definition. Given the symmetric stochastic matrix P, let $\sigma(P)$ denote the set of eigenvalues of P. We will assume that

$$\sigma(P) = \{1, \lambda_1(P), \ldots, \lambda_{N-1}(P)\},$$

where $1, \lambda_1(P), \ldots, \lambda_{N-1}(P)$ denote the eigenvalues of P and are such that $\lambda_1(P) \geq \lambda_2(P) \geq \ldots \geq \lambda_{N-1}(P)$. We define

$$\lambda_{\max}(P) = \lambda_1(P),$$

and

$$\lambda_{\min}(P) = \lambda_{N-1}(P).$$

Note that $\max\{|\lambda_1(P)|, |\lambda_{N-1}(P)|\}$ is the essential spectral radius of the matrix P.

Note that the algorithm (7.2) relies upon a crucial assumption: each agent transmits to its neighboring agents the precise value of its state. This implies the exchange of perfect information through the communication network. As in the previous two Chapters , we consider a more realistic case, i.e., we assume that the communication network is constituted only of rate-constrained digital links. As already emphasized, the presence of a rate constraint prevents the agents from having a precise knowledge about the state of the other agents. In fact, through a digital channel, the i-th agent can only send to

the j-th agent symbolic data in a finite or countable alphabet; using only this data, the j-th agent can build at most an estimate of the i-th agent's state.The main objectives of this Chapter are to understand if there exists some smart way of coding/ decoding information through the digital channels that permits to overcome the forced quantization effects due to the digital channel and to modify the *ideal* standard average consensus algorithm into a scheme which both preserves the average of the state at each iteration and reaches asymptotically the average consensus.

To tackle this problem we take a two step approach. First, we introduce a coding/decoding scheme; each agent uses this scheme to estimate the positions of its neighbors. Second, we consider the standard consensus algorithm where, in place of the exact knowledge of the states of the systems, we substitute estimates calculated according to the proposed coding/decoding scheme.

7.3 Coder/decoder pairs for digital channels

In this section we discuss a general and two specific coder/decoder models for reliable digital channels; we follow the treatment in the survey [91]. We will later adopt this coder/decoder structure to define communication protocols in the robotic network.

Suppose a source wants to communicate to a receiver some time-varying data $x : \mathbb{N} \to \mathbb{R}$ via repeated transmissions at time instants in $\mathbb{N}$. Each transmission takes place through a digital channel, i.e., messages can only be symbols in a finite or countable set (to be designed). The channel is assumed to be reliable, that is, each transmitted symbol is received without error. A coder/decoder pair for a digital channel is defined by the sets:

(i) a set Ξ, serving as *state space* for the coder/decoder; a fixed $\xi_0 \in \Xi$ is the *initial coder/decoder state*;

(ii) a finite or countable set $\mathcal{A}$, serving as *transmission alphabet*; elements $\alpha \in \mathcal{A}$ are called message;

and by the maps:

(i) a map $F : \Xi \times \mathcal{A} \to \Xi$, called the *coder/decoder dynamics*;

(ii) a map $Q : \Xi \times \mathbb{R} \to \mathcal{A}$, being the *quantizer function*;

(iii) a map $H : \Xi \times \mathcal{A} \to \mathbb{R}$, called the *decoder function*.

The coder computes the symbols to be transmitted according to, for $t \in \mathbb{N}$,

$$\xi(t+1) = F(\xi(t), \alpha(t)), \quad \alpha(t) = Q(\xi(t), x(t)).$$

Correspondingly, the decoder implements, for $t \in \mathbb{N}$,

$$\xi(t+1) = F(\xi(t), \alpha(t)), \quad \hat{x}(t) = H(\xi(t), \alpha(t)).$$

Coder and decoder are jointly initialized at $\xi(0) = \xi_0$. Note that an equivalent representation for the coder is $\xi(t+1) = F(\xi(t), Q(\xi(t), x(t)))$, and $\alpha(t) = Q(\xi(t), x(t))$. In summary, the coder/decoder dynamics is given by

$$\begin{aligned} \xi(t+1) &= F(\xi(t), \alpha(t)), \\ \alpha(t) &= Q(\xi(t), x(t)), \\ \hat{x}(t) &= H(\xi(t), \alpha(t)). \end{aligned} \tag{7.3}$$

In what follows we present two interesting coder/decoder pairs: the logarithmic quantizer strategy and the "zoom in - zoom out" uniform quantizer strategy.

7.3.1 Zoom in - zoom out uniform coder

In this strategy the information transmitted from source to receiver is quantized by a scalar uniform quantizer which can be described as follows. For $L \in \mathbb{N}$, define the *uniform set of quantization levels*

$$S_L = \Big\{ -1 + \frac{2\ell - 1}{L} \;\Big|\; \ell \in \{1, \ldots, L\} \Big\} \cup \{-1\} \cup \{1\}$$

and the corresponding the *uniform quantizer* (see Figure 7.1) $\mathrm{unq}_L : \mathbb{R} \to S_L$ by

$$\mathrm{unq}_L(x) = -1 + \frac{2\ell - 1}{L}$$

if $\ell \in \{1, \ldots, L\}$ satisfies $-1 + \frac{2(\ell-1)}{L} \leq x \leq -1 + \frac{2\ell}{L}$, otherwise $\mathrm{unq}_L(x) = 1$ if $x > 1$ or $\mathrm{unq}_L(x) = -1$ if $x < -1$.
Note that larger values of the parameter L correspond to more accurate uniform quantizers unq_L. Moreover note that, if we define m to be the number of quantization levels we have that $m = L + 2$.

For $L \in \mathbb{N}$, $k_{in} \in]0,1[$, and $k_{out} \in]1,+\infty[$, the *zoom in - zoom out uniform coder/decoder* has the state space $\Xi = \mathbb{R} \times \mathbb{R}_{>0}$, the initial state $\xi_0 = (0,1)$, and the alphabet $\mathcal{A} = S_L$. The coder/decoder state is written as $\xi = (\hat{x}_{-1}, f)$ and the coder/decoder dynamics are

$$\hat{x}_{-1}(t+1) = \hat{x}_{-1}(t) + f(t)\alpha(t),$$
$$f(t+1) = \begin{cases} k_{\text{in}}\, f(t), & \text{if } |\alpha(t)| < 1, \\ k_{\text{out}}\, f(t), & \text{if } |\alpha(t)| = 1. \end{cases}$$

The quantizer and decoder functions are, respectively,

$$\alpha(t) = \text{unq}_L\Big(\frac{x(t) - \hat{x}_{-1}(t)}{f(t)}\Big),$$
$$\hat{x}(t) = \hat{x}_{-1}(t) + f(t)\alpha(t).$$

The coder/decoder pair is analyzed as follows. One can observe that $\hat{x}_{-1}(t+1) = \hat{x}(t)$ for $t \in \mathbb{Z}_{\geq 0}$, that is, the first component of the coder/decoder state contains the estimate of the data x. The transmitted messages contain a quantized version of the estimate error $x - \hat{x}_{-1}$ scaled by factor f. Accordingly, the second component of the coder/decoder state f is referred to as the *scaling factor*: it grows when $|x - \hat{x}_{-1}| \geq f$ ("zoom out step") and it decreases when $|x - \hat{x}_{-1}| < f$ ("zoom in step").

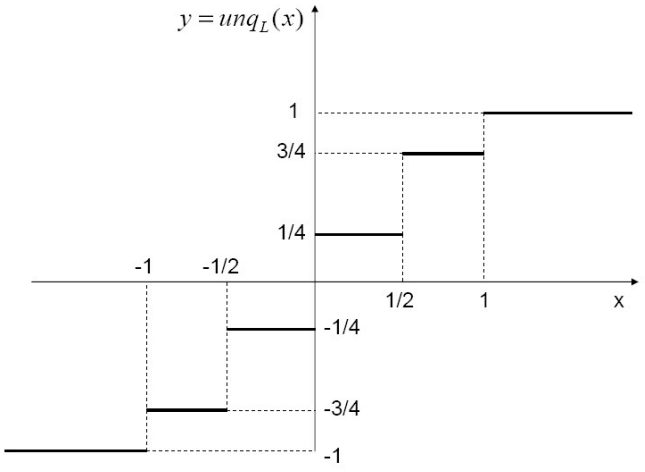

Figure 7.1: The uniform quantizer ($m = 6$).

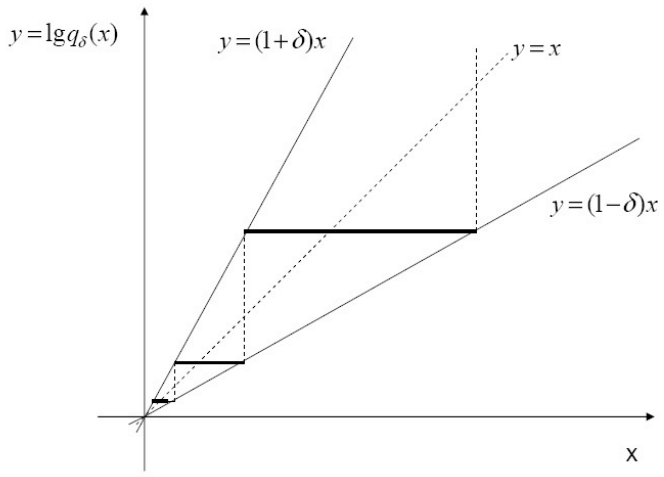

Figure 7.2: The logarithmic quantizer.

7.3.2 Logarithmic coder

This strategy is presented for example in [53]. Given an *accuracy parameter* $\delta \in]0,1[$, define the *logarithmic set of quantization levels*

$$S_\delta = \Big\{ \Big(\frac{1+\delta}{1-\delta}\Big)^\ell \Big\}_{\ell\in\mathbb{Z}} \cup \{0\} \cup \Big\{ -\Big(\frac{1+\delta}{1-\delta}\Big)^\ell \Big\}_{\ell\in\mathbb{Z}}, \tag{7.4}$$

and the corresponding *logarithmic quantizer* (see Figure 7.2) $\mathrm{lgq}_\delta : \mathbb{R} \to S_\delta$ by

$$\mathrm{lgq}_\delta(x) = \Big(\frac{1+\delta}{1-\delta}\Big)^\ell,$$

if $\ell \in \mathbb{Z}$ satisfies $\frac{(1+\delta)^{\ell-1}}{(1-\delta)^\ell} \le x \le \frac{(1+\delta)^\ell}{(1-\delta)^{\ell+1}}$, otherwise $\mathrm{lgq}_\delta(x) = 0$ if $x = 0$ or $\mathrm{lgq}_\delta(x) = -\mathrm{lgq}_\delta(-x)$ if $x < 0$.
Note that smaller values of the parameter δ correspond to more accurate logarithmic quantizers lgq_δ. For $\delta \in]0,1[$, the *logarithmic coder/decoder* is defined by the state space $\Xi = R$, initial state $\xi_0 = 0$, the alphabet $\mathcal{A} = S_\delta$, and by the maps

$$\begin{aligned} \xi(t+1) &= \xi(t) + \alpha(t), \\ \alpha(t) &= \mathrm{lgq}_\delta(x(t) - \xi(t)), \\ \hat{x}(t) &= \xi(t) + \alpha(t). \end{aligned} \tag{7.5}$$

The coder/decoder pair is analyzed as follows. One can observe that $\xi(t+1) = \hat{x}(t)$ for $t \in \mathbb{N}$, that is, the coder/decoder state contains the

estimate of the data x. The transmitted messages contain a quantized version of the estimate error $x - \xi$. The estimate $\hat{x} : N \to \mathbb{R}$ satisfies the recursive relation

$$\hat{x}(t+1) = \hat{x}(t) + \mathrm{lgq}_\delta\left(x(t+1) - \hat{x}(t)\right),$$

with initial condition $\hat{x}(0) = \mathrm{lgq}_\delta(x(0))$ determined by $\xi(0) = 0$. Finally, define the function $r : \mathbb{R} \to \mathbb{R}$ by $r(y) = \frac{\mathrm{lgq}_\delta(y) - y}{y}$ for $y \neq 0$ and $r(0) = 0$. Some elementary calculations show that $|r(y)| \leq \delta$ for all $y \in \mathbb{R}$. Accordingly, if we define the trajectory $\omega : \mathbb{N} \to [-\delta, +\delta]$ by $\omega(t) = r(x(t+1) - \hat{x}(t))$, then we obtain that

$$\hat{x}(t+1) = \hat{x}(t) + (1 + \omega(t))\big(x(t+1) - \hat{x}(t)\big). \tag{7.6}$$

This is called the *multiplicative noise* model for the logarithmic quantizer.

Remark 7.2 Note that, when communicating through digital channels, the use of the logarithmic quantizer described in the above Section, presents an evident drawback with respect to the zoom in- zoom out strategy, due to the fact that the logarithmic set of quantization levels S_δ is countable and not finite as the uniform set of quantization levels. This implementation issue could be overcome by truncating the map lgq_δ as follows. Let $a, b \in \mathbb{R}$ be such that $0 < a < b$; if $a \leq |x| \leq b$ then

$$\mathrm{lgq}_\delta(x) = \mathrm{sgn}(x)\left(\frac{1+\delta}{1-\delta}\right)^\ell,$$

where $\ell \in \mathbb{Z}$ is such that $\frac{(1+\delta)^{\ell-1}}{(1-\delta)^\ell} \leq |x| \leq \frac{(1+\delta)^\ell}{(1-\delta)^{\ell+1}}$, otherwise

$$\mathrm{lgq}_\delta(x) = \begin{cases} 0 & \text{if } |x| < a \\ \mathrm{sgn}(x)\ \mathrm{lgq}_\delta(b) & \text{if } |x| > M \end{cases}$$

Again, if m denotes the number of quantization levels, it is possible to see (see [55]) that, for the truncated logarithmic quantizer,

$$m = \frac{2\log C}{\log \frac{1+\delta}{1-\delta}}$$

We will come back on this remark later on.

7.4 Consensus algorithm with exchange of quantized information

We consider now the same algorithm previously illustrated with the assumption that the agents can communicate only through digital channels. Precisely in this Section, we adopt the logarithmic coder/decoder scheme (7.3)

described in Subsection 7.3.2; we analyze the zoom in - zoom out strategy via simulations in Section 7.6.

Here is an informal description of our proposed scheme. We envision that along each communication edge we implement a logarithmic coder/decoder; in other words, each agent transmits through a dynamic encoding scheme to all its neighbors the quantized information regarding its position. Once state estimates of all node's neighbors are available, each node will then implement the average consensus algorithm.

Next, we provide a formal description of the proposed algorithm. Let $P \in \mathbb{R}^{N\times N}$ be a stochastic symmetric matrix with positive diagonal elements and with connected induced graph $\mathcal{G}_{\mathcal{P}}$. Assume there are digital communication channels along all edges of $\mathcal{G}_{\mathcal{P}}$ capable of carrying a countable number of symbols. Pick an accuracy parameter $\delta \in]0,1[$. The *consensus algorithm with dynamic coder/decoder* is defined as follows:

Processor states: For each $i \in \{1,\dots,N\}$, node i has a state variable $x_i \in \mathbb{R}$ and state estimates $\hat{x}_j \in \mathbb{R}$ of the states of all neighbors j of i in $\mathcal{G}_{\mathcal{P}}$. Furthermore, node i maintains a copy of $\hat{x}_i$.

Initialization: The state $x(0) = (x_1(0),\dots,x_N(0))^* \in \mathbb{R}^N$ is given as part of the problem. All estimates $\hat{x}_j(0)$, for $j \in \{1,\dots,N\}$, are initialized to 0.

State iteration: At time $t \in \mathbb{N}$, for each i, node i performs three actions in the following order:
(1) Node i updates its own state by

$$x_i(t) = x_i(t-1) + \sum_{j=1}^{N} P_{ij}\left(\hat{x}_j(t-1) - \hat{x}_i(t-1)\right). \tag{7.7}$$

(2) Node i transmits to all its neighbors the symbol

$$\alpha_i(t) = \mathrm{lgq}_\delta(x_i(t) - \hat{x}_i(t-1)).$$

(3) Node i updates its estimates

$$\hat{x}_j(t) = \hat{x}_j(t-1) + \alpha_j(t), \tag{7.8}$$

for j being equal to all neighbors of i and to i itself.

Before the algorithm analysis, we clarify a few points.

Remark 7.3 Robot i and all its neighbors j maintain in memory an estimate $\hat{x}_i$ of the state x_i. We denote all these estimates by the same symbol because they are all identical: they are initialized in the same manner and they are updated through the same equation with the same information. On the other hand, it would be possible to adopt distinct quantizer accuracies δ_{ij} for each communication channel (i, j). In such a case then we would have to introduce variables $\hat{x}_{ij}$ that node i and j would maintain for the estimate of x_i.

Remark 7.4 We could define a different state update equation where each node i uses the exact knowledge of its own state x_i instead of the estimate $\hat{x}_i$, that is, we could adopt

$$\begin{aligned} x_i(t) &= x_i(t-1) + \sum_{j=1}^{N} P_{ij} \left(\hat{x}_j(t-1) - x_i(t-1)\right), \\ &= P_{ii} x_i(t-1) + \sum_{j \neq i} P_{ij} \hat{x}_j(t-1), \end{aligned} \tag{7.9}$$

instead of equation (7.7). We will discuss the drawback of this choice below.

We now analyze the algorithm. First, we write the closed-loop system in matrix form. Equation (7.7) is written as

$$x(t+1) = x(t) + (P - I)\hat{x}(t). \tag{7.10}$$

The N-dimensional vector of state estimates $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_N)^*$ is updated according to the multiplicative-noise model in equation (7.6). In other words, there exist $\omega_j \colon \mathbb{N} \to [-\delta, +\delta]$, for $j \in \{1, \ldots, N\}$, such that

$$\hat{x}_j(t+1) = \hat{x}_j(t) + (1 + \omega_j(t))\big(x_j(t+1) - \hat{x}_j(t)\big),$$

and, for $\Omega(t) := \operatorname{diag}\{\omega_1(t), \ldots, \omega_N(t)\}$,

$$\hat{x}(t+1) = \hat{x}(t) + (I + \Omega(t))\big(x(t+1) - \hat{x}(t)\big). \tag{7.11}$$

Equations (7.10) and (7.11) with multiplicative noise Ω determine the closed-loop system.

Next, we define the estimate error $e = \hat{x} - x \in \mathbb{R}^N$ and rewrite the close-loop system in terms of the quantities x and e. Straightforward calculations show that, for $t \in \mathbb{Z}_{\geq 0}$,

$$\begin{bmatrix} x(t+1) \\ e(t+1) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \Omega(t) \end{bmatrix} \begin{bmatrix} P & P - I \\ P - I & P - 2I \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}. \tag{7.12}$$

Initial conditions are $x(0)$ and $e(0) = -x(0)$.

Finally, we are ready to state the main properties of our quantized consensus algorithm.

Theorem 7.5 *Assume $P \in \mathbb{R}^{N\times N}$ satisfies Assumption 7.1 and define $\bar{\delta} \in \mathbb{R}$ by*

$$\bar{\delta} := \frac{1 + \lambda_{\min}(P)}{3 - \lambda_{\min}(P)}. \tag{7.13}$$

The solution $t \mapsto (x(t), e(t))$ of the consensus algorithm with dynamic coder-decoder has the following two properties:

(i) the state average is maintained constant by the algorithm, that is, defined $x_{ave}(t) = 1/N\mathbb{1}^* x(t)$,

$$x_{ave}(t) = x_{ave}(0)$$

for all $t \in \mathbb{N}$;

(ii) if $0 < \delta < \bar{\delta}$, then the state variables converge to their average value and the estimate error vanishes, that is,

$$\lim_{t\to\infty} x(t) = x_{ave}(0)\mathbb{1}$$

and

$$\lim_{t\to\infty} e(t) = 0.$$

Proof: Observe that

$$\begin{aligned}\mathbb{1}^* x(t+1) &= \mathbb{1}^* P x(t) + \mathbb{1}^*(P - I)e(t)\\ &= \mathbb{1}^* P x(t)\end{aligned}$$

where the second equality holds since $\mathbb{1}^*(P - I) = 0$. This proves the first part of the Theorem. The second part follows directly from Theorem 7.11 stated in Section 7.4.1, where we analyze the asymptotic properties of (7.5). ■

We here consider some remarks and examples.

Remark 7.6 Note that $\bar{\delta}$ is a increasing function on $\lambda_{\min}(P)$ and that $\bar{\delta} = 0$, if $\lambda_{\min}(P) = -1$, and $\bar{\delta} = 1$, if $\lambda_{\min}(P) = 1$ (see Figure 7.3).

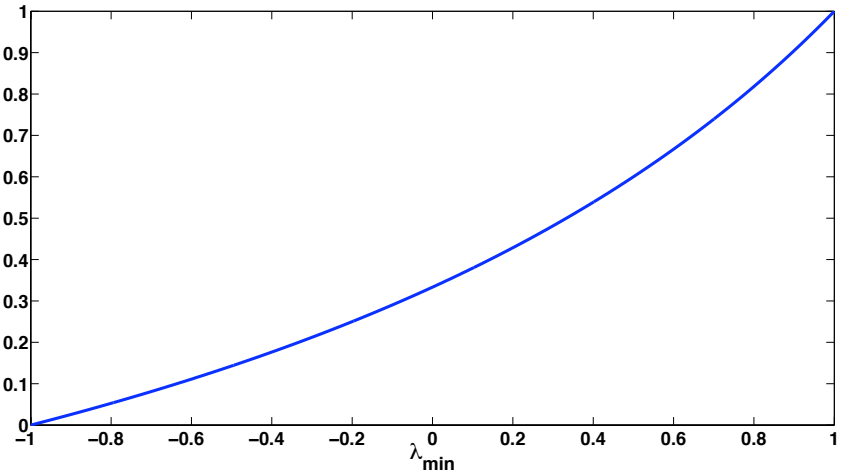

Figure 7.3: Behavior of $\bar{\delta}$.

Remark 7.7 The state update in equation (7.9) does not maintain the average. This fact motivates the choice of state update equation (7.7).

Example 7.8 Consider the sequence of circulant matrices $P_N \in \mathbb{R}^{N\times N}$ defined by

$$P_N = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \frac{1}{3} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}. \tag{7.14}$$

For this sequence of symmetric stochastic matrices we know, from Chapter 3, that $\lambda_{\min}(P_N) = \frac{1}{3} - \frac{2}{3}\cos\left(\frac{2\pi}{N}\left\lfloor\frac{N}{2}\right\rfloor\right)$. Hence $\lambda_{\min}(P_N) \geq -\frac{1}{3}$, implying therefore that $\bar{\delta} \geq \frac{1}{5}$ for all N. This shows that $\bar{\delta}$ is uniformly bounded away from 0. This is a remarkable property of scalability on the dimension of the network.

Remark 7.9 The fact that the critical accuracy sufficient to guarantee convergence is independent on the network dimension is more general than what seen in the previous example. Indeed, assume that $P_N \in \mathbb{R}^{N\times N}$ is a sequence of matrices of increasing size, where each P_N satisfies Assumption 7.1 and where each P_N has all the diagonal elements greater than a positive real number $\bar{p}$. Then, by Gershgorin's Theorem we have that $\lambda_{\min}(P_N) \geq -1 + 2\bar{p}$

and hence $\bar{\delta} \geq \frac{\bar{p}}{2-\bar{p}}$ for all N. It follows that the critical accuracy sufficient to guarantee convergence is bounded away from zero uniformly on the dimension of the network.

7.4.1 Convergence analysis

In this section we provide the analysis of the asymptotic properties of system (7.12), by treating it as a LPV system. For the sake of the notational convenience, let us define the matrix, belonging to $\mathbb{R}^{2N\times 2N}$,

$$\mathcal{A}(t) = \begin{bmatrix} I & 0 \\ 0 & \Omega(t) \end{bmatrix} \begin{bmatrix} P & P-I \\ P-I & P-2I \end{bmatrix}. \tag{7.15}$$

Consider now the system

$$z(t+1) = \mathcal{A}(t)z(t), \tag{7.16}$$

where $z(t) \in \mathbb{R}^{2N}$ for all $t \geq 0$ and where $z(0)$ is any vector in $\mathbb{R}^{2N}$. We start our analysis by rewriting (7.15) in a more suitable way. Let

$$\mathcal{E} = \left\{ \text{diag}\left\{e_1, \ldots, e_N\right\} \in \mathbb{R}^{N\times N} \,:\, e_i \in \{-1,+1\}, i \in \{1,\ldots,N\} \right\}.$$

Notice that $\mathcal{E}$ contains 2^N elements. Hence, we can write $\mathcal{E} = \{E_1, \ldots, E_{2^N}\}$, where we are assuming that some suitable way to enumerate the matrices inside $\mathcal{E}$ has been used. We assume that $E_1 = I$. We define now $\mathcal{E}_\delta = \{\delta E_1, \ldots, \delta E_{2^N}\}$. Observe that $\Omega(t) \in Co\{\mathcal{E}_\delta\}$ for all $t \geq 0$, where $Co\{\mathcal{E}_\delta\}$ denotes that convex hull of the set $\mathcal{E}_\delta$. By means of the above definitions we can introduce an another set of matrices

$$\mathcal{R} = \left\{ R_i = \begin{bmatrix} I & 0 \\ 0 & \delta E_i \end{bmatrix} \begin{bmatrix} P & P-I \\ P-I & P-2I \end{bmatrix} : E_i \in \mathcal{E} \right\}. \tag{7.17}$$

Accordingly to the definition of E_1 we have that

$$R_1 = \begin{bmatrix} I & 0 \\ 0 & \delta I \end{bmatrix} \begin{bmatrix} P & P-I \\ P-I & P-2I \end{bmatrix}. \tag{7.18}$$

The set $\mathcal{R}$ is useful because it is easy to see that the matrix $\mathcal{A}(t)$, belongs to $Co\{\mathcal{R}\}$ for all $t \geq 0$, where $Co\{\mathcal{R}\}$ denote the convex hull of the set $\mathcal{R}$. In other words, for all $t \geq 0$, there exist $\nu_1(t), \ldots, \nu_{2^N}(t)$ nonnegative real numbers such that $\sum_{i=1}^{2^N} \nu_i(t) = 1$ and

$$\mathcal{A}(t) = \sum_{i=1}^{2^N} \nu_i(t) R_i.$$

We state the following result that will permit us to analyze the system (7.16) by means of Theorem C.2 (see Appendix C).

Lemma 7.10 *For $v = [\mathbb{1}^* \;\; \mathbf{0}^*]^*$, we have*

$$R_i v = v, \quad \text{and} \quad v^* R_i = v^*, \quad \text{for all } i \in \{1, \dots, 2^N\}.$$

Moreover, for $\bar{\delta}$ as in equation (7.13), the following facts are equivalent:

(i) *1 is the only eigenvalue of unit magnitude of the matrix R_1, and all its other eigenvalues are strictly inside the unit disc;*

(ii) $0 \leq \delta < \bar{\delta}$.

Proof: The first part of the lemma is easily proved by observing that

$$\begin{bmatrix} I & 0 \\ 0 & \delta E_i \end{bmatrix} \begin{bmatrix} P & P-I \\ P-I & P-2I \end{bmatrix} \begin{bmatrix} \mathbb{1} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \delta E_i \end{bmatrix} \begin{bmatrix} \mathbb{1} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbb{1} \\ \mathbf{0} \end{bmatrix},$$

and

$$\begin{aligned} [\mathbb{1}^* \;\; \mathbf{0}^*] \begin{bmatrix} I & 0 \\ 0 & \delta E_i \end{bmatrix} \begin{bmatrix} P & P-I \\ P-I & P-2I \end{bmatrix} &= [\mathbb{1}^* \;\; \mathbf{0}^*] \begin{bmatrix} P & P-I \\ P-I & P-2I \end{bmatrix} \\ &= [\mathbb{1}^* \;\; \mathbf{0}^*]. \end{aligned}$$

Consider now R_1; to compute its eigenvalues we calculate

$$\det(sI - R_1) = \det \begin{bmatrix} sI - P & -(P-I) \\ -\delta(P-I) & sI - \delta(P-2I) \end{bmatrix}.$$

Since each block of the above matrix commute with each other block, we have from [124] that

$$\begin{aligned} \det(sI - R_1) &= \det\left[(sI-P)(sI-\delta(P-2I)) - \delta(P-I)^2\right] \\ &= \det\left[s^2 I - s\left(\delta(P-2I)+P\right) + \delta\left(P^2 - 2P - P^2 - I + 2P\right)\right] \\ &= \prod_{i=0}^{N-1} \left[s^2 - (\delta(\lambda_i - 2) + \lambda_i)\, s - \delta\right] \\ &= \left(s^2 - (1-\delta)s - \delta\right) \prod_{i=1}^{N-1} \left(s^2 - (\delta(\lambda_i - 2) + \lambda_i)s - \delta\right). \end{aligned}$$

Hence the eigenvalues of R_1 are given by the solution of the following N second order equations

$$s^2 - (1-\delta)s - \delta = 0, \tag{7.19}$$

and

$$s^2 - (\delta(\lambda_i - 2) + \lambda_i)s - \delta = 0, \qquad i \in \{1, \dots, N-1\}. \tag{7.20}$$

The solutions of (7.19) are 1 and $-\delta$. Consider now (7.20). Given i, let $s_1^{(i)}$ and $s_2^{(i)}$ denote the two solutions of (7.20). We have that

$$s_1^{(i)} = \frac{\delta(\lambda_i - 2) + \lambda_i - \sqrt{(\delta(\lambda_i - 2) + \lambda_i)^2 + 4\delta}}{2}$$

and

$$s_2^{(i)} = \frac{\delta(\lambda_i - 2) + \lambda_i + \sqrt{(\delta(\lambda_i - 2) + \lambda_i)^2 + 4\delta}}{2}.$$

Now we have to analyze the conditions $|s_1^{(i)}| < 1$ and $|s_2^{(i)}| < 1$, for all $i \in \{1, \dots, N-1\}$. To this purpose, we consider the bilinear transformation of the equation (7.20), i.e., we substitute to s the term $\frac{1+\tilde{s}}{1-\tilde{s}}$. We obtain the new equation

$$(1+\delta)(1-\lambda_i)\tilde{s}^2 + 2(1+\delta)\tilde{s} + 1 + \lambda_i + \delta(\lambda_i - 3) = 0. \tag{7.21}$$

Let $\tilde{s}_1^{(i)}$ and $\tilde{s}_2^{(i)}$ denote the two solutions of (7.21). From the property of the bilinear transformation, we have that $|s_1^{(i)}| < 1$ and $|s_2^{(i)}| < 1$ if and only if $\tilde{s}_1^{(i)} < 0$ and $\tilde{s}_2^{(i)} < 0$. Since $1 + \delta > 0$ and $(1+\delta)(1-\lambda_i) > 0$ for $i \in \{1, \dots, N-1\}$, we obtain, from the Cartesian rule, that $\tilde{s}_1^{(i)} < 0$ and $\tilde{s}_2^{(i)} < 0$ for all $i \in \{1, \dots, N-1\}$, if and only if $1 + \lambda_i + \delta(\lambda_i - 3) > 0$ for all $i \in \{1, \dots, N-1\}$. This last condition is verified if and only if $\delta < \bar{\delta}$. ■

We are able now to state the following theorem characterizing the asymptotic stability of the system (7.16).

Theorem 7.11 *Consider the system (7.16). The following facts are equivalent:*

(a) $\delta < \bar{\delta}$;

(b) for each initial condition $z(0) \in \mathbb{R}^{2N}$ *and for any sequence* $\{\Omega(t)\}_{t=0}^{+\infty}$ *with* $\Omega(t) \in Co\{\mathcal{E}_\delta\}$ *for all* $t \geq 0$, *we have*

$$\lim_{t \to +\infty} z(t) = \begin{bmatrix} \alpha \mathbb{1} \\ 0 \end{bmatrix}, \tag{7.22}$$

for $\alpha = \frac{1}{N}\left[\mathbb{1}^* \ \mathbf{0}^*\right] z(0)$.

Proof: We start by proving that (b) implies (a). To this aim, we consider the sequence $\mathcal{A}(0) = \mathcal{A}(1) = \mathcal{A}(2) = \ldots = R_1$. In this case $z(t)$ is the evolution of an autonomous linear time invariant discrete-time systems with updating matrix R_1. Therefore, by Lemma 7.10, (7.22) holds true if and only if and only if $\delta < \bar{\delta}$.

We prove now that (a) implies (b). We will show that, for $\delta < \bar{\delta}$, there exists a suitable symmetric matrix $L \in \mathbb{R}^{2N \times 2N}$ satisfying the following three properties

$$L \, [\mathbb{1}^* \;\; \mathbf{0}^*]^* = 0, \tag{7.23}$$

$$z^* L z > 0, \tag{7.24}$$

$$z^* \left(\frac{1}{2} \left(R_i^* L R_j + R_j^* L R_i \right) - L \right) z < 0, \quad \text{for all } R_i,\, R_j \in \mathcal{R}, \tag{7.25}$$

$\forall \; z \in < [\mathbb{1}^* \;\; \mathbf{0}^*]^* >^{\perp}$. This fact, together with Lemma 7.10 and Theorem C.2, ensures that fact (a) implies (b). As candidate matrix L we select

$$L = \begin{bmatrix} I - P & 0 \\ 0 & \gamma I \end{bmatrix}, \tag{7.26}$$

where γ is a suitable positive scalar to be determined. Observe that the eigenvalues of $I - P$ are 0 and $1 - \lambda_i$ for $i \in \{1, \ldots, N-1\}$, where it is immediate to see that $1 - \lambda_i > 0$ for $i \in \{1, \ldots, N-1\}$. Since $\sigma(L) = \sigma(I-P) \cup \sigma(\gamma I)$ it follows that also L has an eigenvalue equal to 0 and all other eigenvalues positive. Moreover, since $L \, [\mathbb{1}^* \;\; \mathbf{0}^*]^* = [((I-P)\mathbb{1})^* \;\; \mathbf{0}^*]^* = 0$, we have that the eigenspace associated to the eigenvalue 0 is generated by the vector $[\mathbb{1}^* \;\; 0^*]^*$. Hence L satisfies (7.23) and (7.24). Moreover, by the structure of L, it is easy to check that $R_i^* L R_j = R_j^* L R_i$ for all $R_i, R_j \in \mathcal{R}$. Thus, verifying (7.25) is equivalent to verify

$$z^* \left(R_i^* L R_j - L \right) z < 0, \quad \text{for all } R_i,\, R_j \in \mathcal{R}, \tag{7.27}$$

for any nonzero $z \in < [\mathbb{1}^* \;\; \mathbf{0}^*]^* >^{\perp}$. We have that

$$R_i^* L R_j - L = R_1 L R_1 - L + Q,$$

where

$$R_1^* L R_1 - L = \\ = \begin{bmatrix} (I-P)^2(\gamma\delta^2 I - I - P) & (I-P)(P(P-I) - \gamma\delta^2(P-2I)) \\ (I-P)(P(P-I) - \gamma\delta^2(P-2I)) & (I-P)^3 + \gamma\delta^2(P-2I)^2 - \gamma I \end{bmatrix}$$

and

$$Q = -\gamma\delta^2 \begin{bmatrix} (P-I)K^2(P-I) & (P-I)K^2(P-2I) \\ (P-2I)(P-I)+(P-2I)K^2(P-I) & (P-2I)K^2(P-2I) \end{bmatrix}$$
$$= -\gamma\delta^2 \begin{bmatrix} (P-I)K \\ (P-2I)K \end{bmatrix} [K(P-I) \;\; K(P-2I)],$$

with K such that $K^2 = I - E_i E_j{}^2$. Clearly, $Q = Q^* \leq 0$ and $Q\,[\mathbb{1}^* \;\; \mathbf{0}^*]^* = 0$. If (7.27) is satisfied for $i = j = 1$, then (7.27) holds also for any pair R_i, R_j belonging to $\mathcal{R}$.
Observe that, by Lemma C.4 (see Appendix C), we immediately have that

$$z^*(R_1^* L R_1 - L)z < 0, \qquad \forall \; z \in < [\mathbb{1}^* \;\; \mathbf{0}^*]^* >^{\perp} \tag{7.28}$$

if we choose

$$\gamma = \frac{1 + \lambda_{\min} + \delta^2 (\lambda_{\min} - 3)}{2\delta^2}.$$

■

7.5 Exponential convergence

The objective of this section is to understand how much the quantization affects the performance of the consensus algorithm. To this aim, by means of a Lyapunov analysis, we will provide a characterization of the asymptotic speed of the convergence toward the consensus of both the ideal algorithm (7.2) and the algorithm (7.12).

We start by introducing some definitions. A function $f : \mathbb{N} \to \mathbb{R}$ converges to 0 *exponentially fast* if there exist a constant $C > 0$ and another constant $\xi \in [0, 1[$ such that $|f(t)| \leq C\xi^t$, for all t; the infimum among all numbers $\xi \in [0, 1[$ satisfying the exponential convergence property is called the *exponential convergence factor* of f. In other words, the exponential convergence factor of f is given by (see Section 2.2.2)

$$\limsup_{t\to\infty} |f(t)|^{\frac{1}{t}}.$$

Consider first the system (7.2). To quantify the speed of convergence of (7.2) toward consensus, we introduce the following variable

$$\bar{x}(t) := x(t) - x_{ave}(0)\mathbb{1},$$

[2]Of note is that $I - E_i E_j$ is a positive semidefinite matrix and hence the matrix K is well-defined.

where $x_{ave}(0) = \frac{1}{N}\mathbb{1}^* x(0)$. Note that the i-th component of $\bar{x}(t)$ represents the distance of the state of the i-th system from the initial average. Clearly, $\lim_{t\to\infty} x(t) = x_{ave}(0)\mathbb{1}$ if and only if $\lim_{t\to\infty} \bar{x}(t) = 0$. It is easy to see that the variable $\bar{x}$ satisfies the same recursive equation of the variable x, that is,

$$\bar{x}(t+1) = P\bar{x}(t). \tag{7.29}$$

Moreover note that $\mathbb{1}^*\bar{x}(t) = 0$, for all $t \geq 0$. We define the exponential convergence factor of $\bar{x}(t)$, for a given initial condition $\bar{x}_0 \in < \mathbb{1} >^{\perp}$, to be

$$\rho(P, \bar{x}_0) := \limsup_{t\to\infty} ||\bar{x}(t)||^{\frac{1}{t}}$$

We can get rid of the initial condition and define the *exponential convergence factor* of the system (7.2) as follows

$$\rho(P) := \sup_{\bar{x}_0 \in <\mathbb{1}>^{\perp}} \rho(P, \bar{x}_0) \tag{7.30}$$

Consider now the positive semidefinite matrix $I - P$. Notice that

$$\rho(P, \bar{x}_0) = \limsup_{t\to\infty} (\bar{x}(t)^*(I-P)\bar{x}(t))^{\frac{1}{2t}}$$

and so we can characterize the speed of convergence to 0 of the variable $\bar{x}$ by studying the exponential convergence factor of the Lyapunov function $\bar{x}(t)^*(I-P)\bar{x}(t)$.

Theorem 7.12 *Consider* (7.29) *with* $P \in \mathbb{R}^{N\times N}$ *satisfing Assumption 7.1. Then the function* $t \mapsto (\bar{x}(t)^*(I-P)\bar{x}(t))^{1/2}$*, defined along any trajectory* $t \mapsto \bar{x}(t)$*, converges exponentially fast to* 0*. Moreover, the factor* $\rho(P)$*, defined in equation* (7.30)*, satisfies*

$$\rho(P) = \max\left\{\lambda_{\max}(P), -\lambda_{\min}(P)\right\}.$$

Proof: Let $\alpha := \max\left\{\lambda^2_{\max}(P), \lambda^2_{\min}(P)\right\}$ so that $z^*P^2z \leq \alpha z^*z$ for all $z \in < \mathbb{1} >^{\perp}$ and, in turn,

$$z^*(P(I-P)P)z \leq \alpha z^*(I-P)z, \tag{7.31}$$

for all $z \in < \mathbb{1} >^{\perp}$. This shows that the map $t \mapsto \bar{x}(t)^*(I-P)\bar{x}(t)$ converges exponentially fast to 0 along any trajectory $t \mapsto \bar{x}(t)$ and that $\rho(P) \leq \sqrt{\alpha}$. Moreover, observe that, if z is equal to the eigenvector corresponding to the eigenvalue defining β, then (7.31) holds true as equality. Then, if $\bar{x}_0$ is equal

to this eigenvector, we obtain a trajectory $t \mapsto \bar{x}(t)$ along which the function $t \mapsto \bar{x}(t)^*(I-P)\bar{x}(t)$ has exponential convergence factor equal to $\sqrt{\alpha}$. ■

This concludes the analysis of the algorithm (7.2). In the sequel of this section, we provide a similar analysis of the system (7.12). To this aim we consider again the system (7.16), that is

$$z(t+1) = \mathcal{A}(t)z(t), \tag{7.32}$$

where $z(0) = z_0$ is any vector in $\mathbb{R}^{2N}$. To perform a Lyapunov analysis of (7.32), it is convenient to introduce the variable

$$\bar{z}(t) = \begin{bmatrix} I - \frac{1}{N}\mathbb{1}\mathbb{1}^* & 0 \\ 0 & I \end{bmatrix} z(t).$$

Clearly, condition (b) of Theorem 7.11 holds true if and only if $\lim_{t\to\infty} \bar{z}(t) = 0$. It is straightforward to see that $\bar{z}$ satisfies the same recursive equation of $z(t)$, i.e.,

$$\bar{z}(t+1) = \mathcal{A}(t)\bar{z}(t) \tag{7.33}$$

and that $[\mathbb{1}^* \ \mathbf{0}^*]^* \bar{z}(t) = 0$ for all $t \geq 0$. Consider now the matrix $L \in \mathbb{R}^{2N\times 2N}$, introduced along the proof of Theorem 7.11 and defined as

$$L = \begin{bmatrix} I - P & 0 \\ 0 & \gamma I \end{bmatrix}.$$

For each $\gamma > 0$ define

$$\tilde{\rho}\left(P, \delta, \gamma; \bar{z}_0, \{\mathcal{A}(t)\}_{t=0}^{\infty}\right) := \limsup_{t\to\infty} (\bar{z}(t)^* L \bar{z}(t))^{\frac{1}{2t}} \tag{7.34}$$

We can get rid of the initial conditions $\bar{z}_0$ and the sequences $\{\mathcal{A}(t)\}_{t=0}^{\infty}$ by considering

$$\tilde{\rho}(P, \delta, \gamma) := \sup_{\bar{z}_0 \{\mathcal{A}(t)\}_{t=0}^{\infty}} \tilde{\rho}\left(P, \delta, \gamma; \bar{z}_0, \{\mathcal{A}(t)\}_{t=0}^{\infty}\right) \tag{7.35}$$

where the initial conditions $\bar{z}_0$ belong to the set of vectors orthogonal to $[\mathbb{1}^* \ \mathbf{0}^*]^*$ and the sequences $\{\mathcal{A}(t)\}_{t=0}^{\infty}$ are such that $\mathcal{A}(t) \in Co\{\mathcal{R}\}$ for all $t \geq 0$. It can be shown that $\tilde{\rho}(P, \delta, \gamma)$ is independent of γ and for this reason we denote it as $\tilde{\rho}(P, \delta)$.

We characterize now $\tilde{\rho}(P, \delta, \gamma)$. To this aim, consider the following semidefinite programming problem

$$\bar{\beta}(P, \delta, \gamma) := \begin{array}{c} \max \ \beta \\ \text{such that } R_1^* L R_1 - L \leq -\beta L \end{array} \tag{7.36}$$

We have the following result.

Theorem 7.13 *Consider* (7.33) *with the matrix* P *satisfing Assumption 7.1. Let* $\bar{\delta}$ *be defined as in* (7.13) *and let* $\delta \in \mathbb{R}$ *be such that* $0 \leq \delta < \bar{\delta}$. *Moreover let* $\gamma \in \mathbb{R}$ *be such that* $\gamma > 0$, *and let* $\bar{\beta}(P, \delta, \gamma)$ *be defined as in* (7.36). *Then, the function* $t \to (\bar{z}(t)^* L \bar{z}(t))^{1/2}$, *defined along any trajectory* $t \to \bar{z}(t)$ *converges exponentially fast to* 0 *and the factor* $\tilde{\rho}(P, \delta)$, *defined in equation* (7.35), *satisfies*

$$\tilde{\rho}(P, \delta) \leq \sqrt{1 - \bar{\beta}(P, \delta, \gamma)}.$$

Proof: We start by recalling, that since $\mathcal{A}(t)$ belongs to $Co(\mathcal{R})$, we can write that $\mathcal{A}(t) = \sum_{i=1}^{2^N} \nu_i(t) R_i$, where $\nu_1(t), \ldots, \nu_{2^N}(t)$ are nonnegative real numbers such that $\sum_{i=1}^{2^N} \nu_i = 1$. Along the proof of Theorem 7.11, we have seen that

$$z^*(R_i^* L R_j - L)z \leq z^*(R_1^* L R_1 - L)z < 0,$$

for all $z \in \mathbb{R}^{2N}$ such that $z \in < [\mathbb{1}^* \ \mathbf{0}^*]^* >^{\perp}$ and for any pair of matrices R_i, R_j belonging to $\mathcal{R}$. Hence we have that

$$\begin{aligned}
z^*(\mathcal{A}^*(t) L \mathcal{A}(t) - L)z &= z^* \left(\left(\sum_{i=1}^{2^N} \nu_i(t) R_i \right)^* L \left(\sum_{j=1}^{2^N} \nu_j(t) R_j \right) - L \right) z \\
&= z^* \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \Big(\nu_i(t) \nu_j(t) R_i^* L R_j - \nu_i(t) \nu_j(t) L \Big) \right) z \\
&\leq z^* \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \nu_i(t) \nu_j(t) (R_1^* L R_1 - L) \right) z \\
&= z^*(R_1^* L R_1 - L)z,
\end{aligned}$$

for all $z \in \mathbb{R}^{2N}$ such that $z \in < [\mathbb{1}^* \ \mathbf{0}^*]^* >^{\perp}$. Observe finally that $z^*(R_1^* L R_1 - L)z \leq \bar{\beta} z^* L z < 0$, from which we can argue that $\bar{z}(t+1)^* L \bar{z}(t+1) \leq (1 - \bar{\beta}) \bar{z}(t)^* L \bar{z}(t)$ and so the theses follow. ■

It is worth noting that the above Theorem relates $\tilde{\rho}(P, \delta)$ to the resolution of a LMI [21]. It is well known that the computational effort required by the resolution of a LMI strictly depends on its dimensionality. However, we can observe that Lemma C.3 (see Appendix C) provides an efficient way of solving (7.36), that drastically reduces its computational complexity. Indeed, we have that $\bar{\beta}(P, \delta, \gamma) = \min\{\beta_{min}^-(\delta, \gamma), \beta_{max}^-(\delta, \gamma)\}$, where $\beta_{min}^-(\delta, \gamma), \beta_{max}^-(\delta, \gamma)$ are defined in Lemma C.3. This means that one has to

calculate only the value of the two variables $\beta^-_{min}(\delta,\gamma), \beta^-_{max}(\delta,\gamma)$ and evaluate the minimum between them. Differently from the method based on the LMI, the complexity of this method is independent of N.

Example 7.14 In this example we consider a connected random geometric graph generated by choosing $N = 30$ points at random in the unit square, and then placing an edge between each pair of pints at distance less than 0.4. The matrix P is built using the Metropolis weights [149]. In this case we have that $\lambda_{min} = -0.013$ and $\bar{\delta} = 0.327$. In figure 7.4, we plot the behavior of β^-_{min} and β^-_{max} as functions of γ. The value of δ is assumed constant and precisely equal to 0.25.

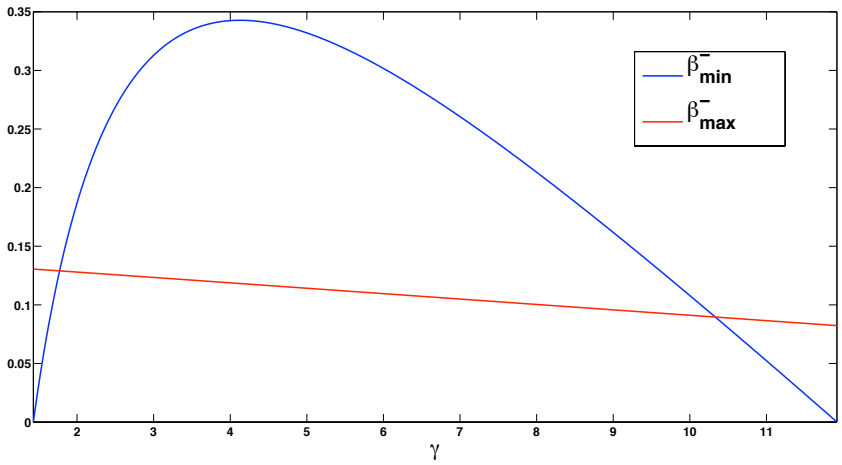

Figure 7.4: Behavior of $\tilde{\rho}$ as function of γ for P and δ fixed.

In general, assigned the matrix P and the value of the accuracy parameter δ, one could be interested in determining the maximum value of $\bar{\beta}$, as function of γ. Clearly, the best bound on $\tilde{\rho}(P,\delta)$ corresponds to to the maximum value of $\bar{\beta}$, namely

$$\tilde{\rho}(P,\delta) \leq \sqrt{1 - \bar{\beta}_{opt}(P,\delta)}$$

where

$$\bar{\beta}_{opt}(P,\delta) := \max_{\gamma>0} \bar{\beta}(P,\delta,\gamma).$$

We illustrate this discussion in the following example.

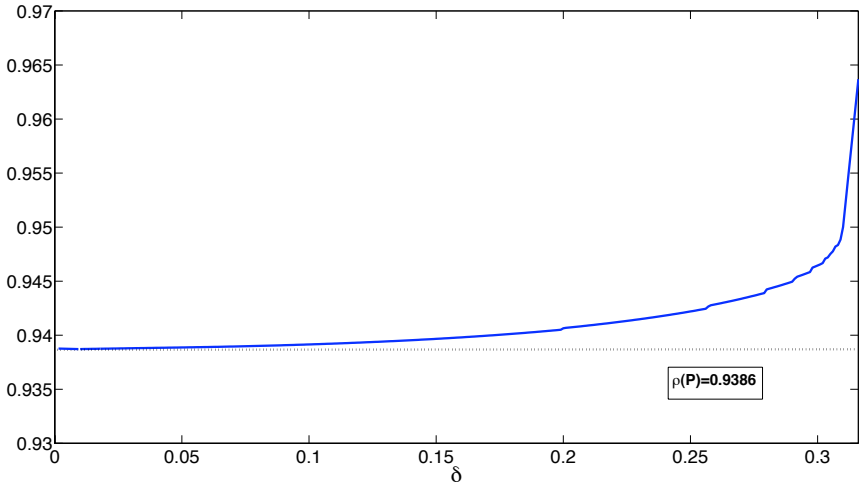

Figure 7.5: Behavior of $\sqrt{1-\bar{\beta}_{opt}(P,\delta)}$.

Example 7.15 We consider the same matrix P generated in the previous example. In Figure 7.5, we depict the behavior of $\sqrt{1-\bar{\beta}_{opt}(P,\delta)}$ as a function of δ. The dotted line represents the value of $\rho(P)$, that is, the convergence factor of the ideal algorithm (7.29). Notice that the convergence factor $\sqrt{1-\bar{\beta}_{opt}(P,\delta)}$ depends smoothly on the accuracy parameter δ and that

$$\lim_{\delta\to 0}\sqrt{1-\bar{\beta}_{opt}(P,\delta)}=\rho(P).$$

An interesting characterization of $\tilde{\rho}$ can be provided when considering a family of matrices $\{P_N\}$ of increasing size whose maximum eigenvalue converges to 1. It is worth noting that this situation is encountered in many practical situations [86, 22, 35]. We formalize this situation as follows.

Assumption 7.16 (Vanishing spectral gap) *Assume we have a sequence of symmetric stochastic matrices $\{P_N\}\subset\mathbb{R}^{N\times N}$ satisfying Assumption 7.1 and the following conditions*

(i) $\lambda_{\min}(P_N)>c$ *for some* $c\in\,]-1,1[$ *and for all* $N\in\mathbb{N}$*;*

(ii) $\lambda_{\max}(P_N)=1-\epsilon(N)+o(\epsilon(N))$ *as* $N\to\infty$*, where* $\epsilon:\mathbb{N}\to\mathbb{R}$ *is a positive function such that* $\lim_{N\to\infty}\epsilon(N)=0$*.*

According to Theorem 7.12, as $N\to\infty$, we have that $\rho\left(P_N\right)=1-\epsilon(N)+o\left(\epsilon(N)\right)$. In considering the quantized version of the consensus algorithm,

together with the sequence $\{P_N\}$, we have also to fix the sequence $\{\delta_N\}$. For simplicity, in the following we will assume that, $\{\delta_N\}$ is a constant sequence, i.e., $\delta_N = \delta$ with suitable δ such that $\delta < \frac{1+c}{3-c}$ which ensures the stability for all N.

Theorem 7.17 *Let $\{P_N\} \subset \mathbb{R}^{N\times N}$ be a family of matrices of increasing size satisfying Assumptions 7.1 and 7.16. Let $\delta \in \mathbb{R}$ be such that $\delta < \frac{1+c}{3-c}$. Then, as $N \to \infty$, we have that*

$$\tilde{\rho}(P_N, \delta) \leq 1 - \left(1 - \frac{1 + c + \delta^2(c-3)}{4(1-\delta^2)}\right) \epsilon(N) + o\left(\epsilon(N)\right).$$

Proof: We choose

$$\gamma = \frac{1 + c + \delta^2\,(c-3)}{2\delta^2}.$$

Consider the polynomial f defined in (C.8). Let $\beta^-_{min}(\delta, \gamma, N)$ and $\beta^-_{max}(\delta, \gamma, N)$ be as defined in Lemma C.3 relatively to the matrix P_N.
Notice that $f(1, \delta, \gamma, \beta) = \gamma\beta^2 + (\gamma\delta^2 - \gamma)\,\beta$. Then the equation $f(1, \delta, \gamma, \beta) = 0$ has solutions $\beta = 0$ and $\beta = 1 - \delta^2$. This implies that, since $\lambda_{max}(P_N) \to 1$, then $\beta^-_{max}(\delta, \gamma, N) \to 0$ as $N \to \infty$. This implies that for N big enough we have that

$$\min\{\beta^-_{min}(\delta, \gamma, N), \beta^-_{max}(\delta, \gamma, N)\} = \beta^-_{max}(\delta, \gamma, N)$$

and hence, from Theorem 7.13 and Lemma C.3 (see Appendix C), it follows that for N big enough we have that

$$\tilde{\rho}(P_N, \delta) \leq \sqrt{1 - \beta^-_{max}(\delta, \gamma, N)}$$

Let $\lambda_N := \lambda_{max}(P_N)$ and $\beta_N := \beta^-_{max}(\delta, \gamma, N)$ so the we have that $\lambda_N \to 1$ and $\beta_N \to 0$. We know that $f(\lambda_N, \delta, \gamma, \beta_N) = 0$. As $N \to \infty$, from the implicit function theorem, we have that

$$\beta_N = \left[\frac{\frac{\partial}{\partial\lambda} f}{\frac{\partial}{\partial\beta} f}\right]_{|\lambda=1,\beta=0} \epsilon(N) + o(\epsilon(N)).$$

Now notice that

$$\begin{aligned}\frac{\partial f}{\partial\lambda} = \left(-3(1-\lambda)^2 + 2\gamma\delta^2(\lambda-2)^2 - \gamma^2\delta^2 + 2\gamma\delta)\right)\beta + \gamma^2\delta^2 + (1-\lambda)^2 + \\ - \gamma(1 + \lambda + \delta^2(\lambda - 3)) + (1-\lambda)(\gamma - 2\gamma\delta^2 + 2(1-\lambda))\end{aligned}$$

and that

$$\frac{\partial f}{\partial \beta} = 2\gamma\beta + (1-\lambda)^3 + \gamma\delta^2(\lambda-2)^2 - \gamma + \gamma(1-\lambda)(\gamma\delta^2 - 1 - \lambda).$$

which lead to

$$\frac{\partial f}{\partial \lambda}_{|\lambda=1,\beta=0} = -(2\gamma - 2\gamma^2\delta^2 - \gamma\delta^2)$$

and

$$\frac{\partial f}{\partial \beta}_{|\lambda=1,\beta=0} = \gamma\delta^2 - 1.$$

Then

$$\beta_N = \left(2 - \frac{\gamma\delta^2}{1-\delta^2}\right)\epsilon(N) + o(\epsilon(N)).$$

The thesis follows by expanding in Taylor's series the function $\sqrt{1-\beta_N}$. ■

Notice that the coefficient in front of $\epsilon(N)$ is negative. Indeed, it can be seen that that coefficient is negative if and only if

$$\delta^2 < \frac{3-c}{1+c}$$

and this is true since we have chosen $\delta < \frac{1+c}{3-c}$ and since $\delta < 1$.

7.6 Numerical simulations

In this section we consider two examples providing some numerical results illustrating the performance respectively of the Zoom in -Zoom out strategy and of the truncated version of the logarithmic quantizer discussed in Remark 7.2.

Example 7.18 In this example we consider a connected random geometric graph generated by choosing N points at random in the unit square, and then placing an edge between each pair of points at distance less than 0.25. We assume that $N = 30$ and that the initial conditions has been generated randomly inside the interval $[-100, 100]$. Again, the matrix P is built using the Metropolis weights (see Section 2.2.1). For all the experiments, we set the parameters k_{in} and k_{out} to the values 1/2 and 2 respectively, and initialized the scaling factor f of each agent to the value 50. Moreover we run simulations for two different values of m, $m = 5$ and $m = 10$. The results

obtained are reported in Figure 7.6. The variable plotted is the normalized Euclidean norm of the vector $\bar{x}(t) := x(t) - x_{ave}(0)\mathbb{1}$, that is,

$$s(t) = \sqrt{\frac{1}{N}\sum_{i=1}^{N} \bar{x}_i^2(t)}.$$

Note that, as depicted in Figure 7.6, also the zoom in- zoom out uniform coder- decoder strategy seems to be very efficient in achieving the consensus. In particular it is remarkable that this strategy works well even if the uniform quantizer has a low number of quantization levels ($m = 5$). Finally it is worth observing, that as theoretically proved in the logarithmic coder-decoder strategy, also in this case the performance degrades smoothly as the quantization becomes coarser.

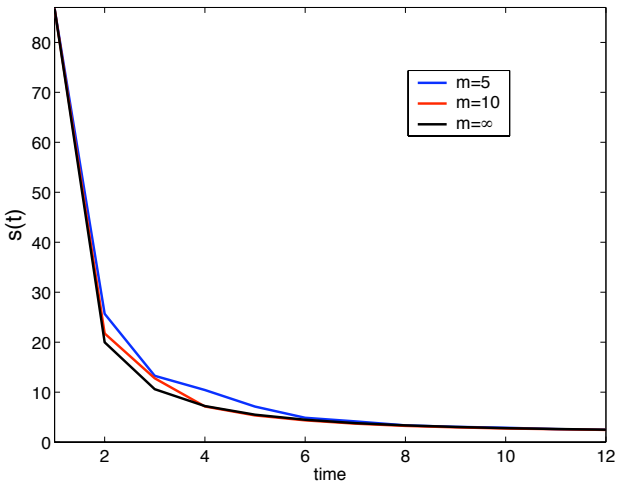

Figure 7.6: Zoom in- zoom out strategy

It is worth noting that some preliminary results regarding the convergence of the zoom in- zoom out strategy are present in [11].

Example 7.19 In this example we consider the same matrix P of the previous example. Moreover we assume again that the initial conditions have been generated randomly inside the interval $[-100, 100]$. The information exchanged between the systems is quantized by the *truncated* logarithmic quantizer discussed in Remark 7.2. More precisely, we assume that the real numbers a, b introduced in Remark 7.2 are equal respectively to 0.5 and 100. The result obtained is reported in Figure 7.7.

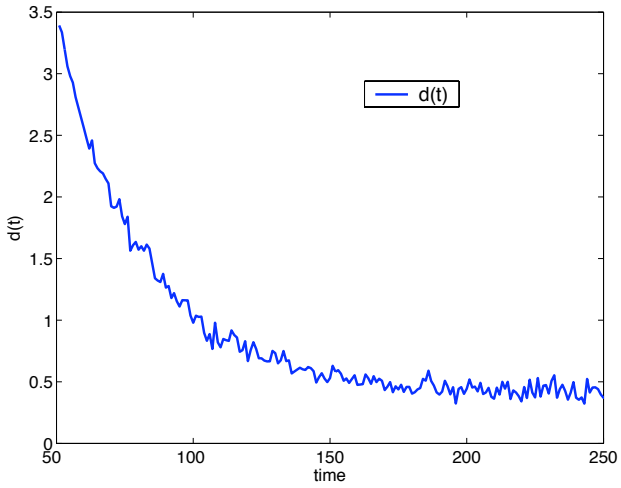

Figure 7.7: Truncated logarithmic quantizer

The variable plotted is

$$d(t) := \|\bar{x}(t)\|_\infty.$$

One can see that $d(t)$ does not converge asymptotically to 0. However, at the steady state, $d(t)$ oscillates inside an interval whose amplitude is comparable to 0.5, that is, the lower value at which we have truncated the logarithmic quantizer.

This numerical observations leads to the following consideration. Assume that our goal is to have convergence of the initial states $x_i(0) \in [-M, M]$ to a target configuration $x_i(\infty) \in [\alpha-\epsilon, \alpha+\epsilon]$, where α is a constant depending only on the initial condition $x(0)$ and ϵ describes the desired agreement precision. This is a "practical stability" requirement. In this case the contraction rate is $C := M/\epsilon$. Assume that, as in [75], the exact data transmission are substituted by transmissions of precision ϵ uniformly quantized data. In this framework it is well known [56] that each uniform quantizer needs C different levels and so the transmission of its data needs an alphabet of C different symbols. Assume now that the information is encoded by truncated logarithmic quantizers where $a = \epsilon$ and $b = M$. We have seen in Remark 7.2 that in such case each logarithmic quantizer needs

$$\frac{2 \log C}{\log \frac{1+\delta}{1-\delta}}$$

different symbols. Note that for C sufficiently large, with the logarithmic communications we obtain a significantly improvement in terms of the communication effort required. It will be the subject of future research to analyze

the tradeoff between the steady state of $d(t)$ and the values of the parameters a, b at which we truncate the logarithmic quantizers.

7.7 Conclusions

In this chapter we presented a new approach solving the average consensus problem in presence of only quantized exchanges of information. In particular we considered two strategies, one based on logarithmic quantizers, and the other one based on a zooming in-zooming out strategy. We studied them with theoretical and experimental results proving that using these schemes the average consensus problem can be efficiently solved even if the agents can share only quantized information. Additionally, we show that the convergence factors depend smoothly on the accuracy parameter of the quantized and that, remarkably, that the critical quantizer accuracy sufficient to guarantee convergence is independent from the network dimension. A field of future research will be to look for encoding and decoding methods which are able to solve the average problem also with noisy digital channels.

Chapter 8

Distributed Kalman filtering

8.1 Introduction

The main objective of this chapter deals with a possible application of the consensus ideas to the wide field of the distributed estimation. It is worth mentioning that, since eighties, the problems related to the distributed estimation have attracted intensively the attention of the scientific community producing, along the years, a very rich literature. Nevertheless in the last period, the interest toward this kind of problems is renewing. This is mainly due to the recent technological advances in wireless communication that, combined to the decreasing in cost and size of electronic devices, are promoting the appearance of large inexpensive interconnected systems, each with computational and sensing capabilities. These complex systems of agents, for instance, can be used for monitoring very large scale areas with fine resolution. However, in this application, collecting measurements from distributed wireless sensors nodes at a single location for on-line data processing may not be feasible due to several reasons among which long packet delay (e.g. due to multi-hop transmission) and/or limited bandwidth of the wireless network, due e.g. to energy consumption requirements.

This problem is particularly relevant in wireless ad-hoc sensor networks where information needs to be multi-hopped from one node to another using closer neighbors. Therefore there is a growing need for in-network data processing tools and algorithms that provide high performance in terms of on-line estimation while

(i) reducing the communication load among all sensor nodes,

(ii) being very robust to sensor node failures or replacements and packet losses, and

(iii) being suitable for distributed control applications.

As previously said, the literature is very rich of contributions addressing several aspects of distributed estimation. We can classify these works into few large classes based on the modeling adopted: *static vs dynamic* estimation, *distributed vs hierarchical* estimation, and *all-to-all vs multi-hop* communication networks. The distinction between static and dynamic estimation depends on whether the quantities to be estimated are constant or time-varying. The second distinction between hierarchical and distributed depends on whether the global estimate is required to be computed at a specific location on the network or at many sensing locations. The last distinction between all-to-all vs multi-hop depends on whether one node can directly send a message to any other node, or it has to route it through intermediate nodes. The work of this chapter belongs to the class of dynamic distributed estimation under multi-hop communication, while most of the works in the literature focus on different combinations of the three classes mentioned above.

For example, [142] derives conditions under which one can reconstruct the global sufficient statistics from local sufficient statistics; [85] investigates how much information two sensors (say S_1 and S_2) have to transmit regarding their measurements (say y_1 and y_2) in order for a fusion center to be able to evaluate certain functions of the measured data y_1 and y_2; this latter paper and [64] introduce the concept of *communication complexity* since computation efficiency depends also on the underlying communication graph, thus shedding some new light on well-known data fusion formulas. These works however, are limited to static hierarchical estimation in rooted-tree communication networks which is a scenario where communication delay is irrelevant. There is also a vast literature dealing with dynamic estimation with all-to-all communication where the main goal is to find the minimal representation of sufficient statistics to reduce either computation load in the central node [78] [144] or to reduce bandwidth requirements [132, 116]. However delay due to multi-hop topologies is not considered in these works.

An interesting approach dealing with distributed estimation and control in a multi-hop setting can be found in [82] where the authors propose to design linear distributed estimators and controllers which use only local information. However, this approach leads to a nonlinear optimization problem affected by well known problems of multiple local minima. Nonetheless, this is

very similar, in spirit, to the approach followed in this paper. In fact, we will focus on distributed estimation of dynamical systems for which sensor nodes are not physically co-located and exchange information only with their neighbors. For example, suppose that we want to track a dynamic quantity that changes according to a random walk, i.e $x(t+1) = x(t)+w(t)$, where $w(t)$ is a zero mean white noise process with covariance q, and we have N sensors that can measure this quantity corrupted by some noise, i.e. $y_i(t) = x(t) + n_i(t)$, where $n_i(t)$ are uncorrelated zero mean white noise processes with same covariance r. If all measurements were instantaneously available to a single location, it is well known from the centralized Kalman filter that the sufficient statistics necessary to reconstruct the optimal estimate would be given by the mean of all measurements, i.e. $\text{mean}(y(t)) := \frac{1}{N}\sum_{i=1}^{N} y_i(t)$. In a multi-hop setting, it is not possible to assume that all measurements are instantaneously available at a specific location, since data from distant nodes, at best, arrives with some delay which depends on the specific network topology. Moreover, in a distributed setting where each sensor node is required to compute a global state estimate, the number of messages from every node to every other node can congest the network.

However, if it was possible to provide an algorithm that computes the mean of a set of numbers only through local communication, then the optimal estimate could be computed at each sensor node as follows:

$$\begin{aligned}\hat{x}_i(t+1) &= (1-\ell_0)\text{mean}(\hat{x}(t)) + \ell_0\text{mean}(y(t)) \\ &= \text{mean}\big((1-\ell_0)\hat{x}(t) + \ell_0 y(t)\big)\end{aligned}$$

where $\hat{x}_i$ is the local estimate of $i-$th sensor. Algorithms able to compute the average of a set of numbers in a distributed way are known as *average consensus algorithms*, whose one popular class, based on linear iterations $z^+ = Pz$, where z is the vector whose entries are the quantities to be averaged and P is a *quasi-doubly stochastic matrix*, has been illustrated in Chapter 2. It is worth recalling that, under some weak connectivity properties, see Section 2.2.1, these algorithms guarantee that $\lim_{m\to\infty}[P^m z]_i = \text{mean}(z)$, i.e. all elements of vector $P^m z$ converge to their initial mean $\text{mean}(z)$. Therefore, provided that it is possible to communicate sufficiently fast within two subsequent sensor measurements, i.e. $m \gg 1$, then intuitively we can assume that the following distributed estimation strategy yields the optimal global state estimate:

$$\begin{aligned} z &= (1-\ell_0)\hat{x}_i(t) + \ell_0 y_i(t) && \textbf{measur. \& predict.} \\ \hat{x}_i(t+1) &= [P^m z]_i && \textbf{consensus}\end{aligned}$$

To our's knowledge Olfati-Saber [102] and Spanos et al. [129] were the first ones to propose this two-stage strategy based on computing first the mean of

the sensor measurements via consensus algorithms, and then to update and predict the local estimates using the centralized Kalman optimal gains. This approach has been extended to static multivariable systems in [149] and to dynamic multivariable systems in [130][96]. In this context of fast communication, i.e. $m \gg 1$, it is natural to optimize P for fastest convergence rate of P^m, which corresponds to the second largest singular value of P, for which there are already very efficient optimization tools available [147] [148]. The assumption $m \gg 1$ is reasonable in applications for which communication is inexpensive as compared to sensing. This is the case, for example, in rendezvous control or coordination of mobile sensors where moving and sensing is energetically more expensive than communicating. However, there are many other important applications in which the number m of messages exchanged per sampling time per node needs to be small, as required in battery-powered wireless sensor networks. Therefore the assumption that $[P^m z]_i \approx \text{mean}(z)$ is not valid. In this context, for example, it is not clear whether maximizing the rate of convergence of P is the best strategy. Moreover, also the optimal gain ℓ becomes a function of the matrix P and the number of exchanged messages m; this will unlikely coincide with the optimal centralized Kalman gain, which is the strategy proposed in all the aforementioned papers [102][129][130][96][149].

Recently, Alriksson at al. [5] and Speranzon et al. [131], considered the case $m = 1$, i.e. sensors are allowed to communicate only once between sampling instants. In particular, in [5], the authors consider a general MIMO scenario where the matrix $P = P(t)$ and the gain $\ell = \ell_i(t)$ (W and K, respectively, in their terminology) are selected at each time step in order to minimize the estimation error covariance of each sensor for the next time step, with the only constraint to maintain the estimate unbiased. In [131] the authors consider a single update equation $\hat{x}(t+1) = K\hat{x}(t) + Hy(t)$, and similarly to [5], they minimize the sum of all covariance errors at each time step, but differently they simultaneous optimize the gains K and H by enforcing stability of the matrix K. Both these iterative estimation algorithms seem to converge in numerical simulations and provide good performance, but no proof of stability and global optimality nor insights about the effect of connectivity of the underlying graph on the overall performance were given.

It is worth noting that the algorithms based on stochastic matrices represent just one class among all the average consensus algorithms [52] [134]. For example, there has been recent work (see e.g. [40, 143]) on *finite-time* consensus algorithms. These approaches guarantee that consensus can be reached in finite time; however these schemes are nonlinear and hence more difficult

to analyze. Moreover, they are limited to the continuous time framework, making them less attractive in the presence of communication constraints. Another class of average consensus algorithms includes iterative distributed gradient descent algorithms based on Lagrange multipliers [15], which have been recently proposed by Skizas et al. [121] in the context of static distributed estimation. The authors proved that these algorithms converge to the centralized optimal estimator in the ideal scenario of fast communication and that they maintain good performance even under quantization, non-gaussian noise and small number of consensus iterations.

In this chapter, we want to study the interaction between the consensus matrix Q, the number of messages per sampling time m, and the gain ℓ. With respect with the aforementioned works, we consider a simpler scenario with a scalar state which can be measured by N identical and independent sensors, a setup which still captures some of the most important features of the problem. In fact, also in this simple setup the joint optimization of Q and ℓ is not convex, as discussed in Section 8.5. Our goal is to provide better insights about the problem of distributed estimation using consensus matrices, rather then posing it as a black-box optimization algorithm. Therefore, we explore some important regimes, namely fast communication $m \to \infty$, "small" measurement noise ($r/q \to 0$) and "small" process noise ($q/r \to 0$).

This analysis provides useful guidelines for choosing the local filter gain ℓ and the consensus matrix Q also for more general scenarios. As a side result of our analysis, we also see that the standard recipe of choosing Q optimizing the second largest eigenvalue is not necessarily the best thing to do; similarly choosing the centralized optimal gain ℓ_c is not necessarily the optimal strategy.

Finally we provide some numerical examples to clarify the proposed analytical results.

8.2 Problem formulation

Consider a set V of N sensor nodes which are labeled $i = 1, 2, \ldots, N$. These sensors can communicate over a network modeled as a direct graph $\mathcal{G} = (V, E)$, where the edge (i, j) is in E if and only if the node i can transmit its information to the node j. We assume that the graph $\mathcal{G}$ is time-invariant. A physical process with state $x \in \mathbb{R}$ evolves according to the continuous-time system

$$\dot{x}(t) = v(t) \tag{8.1}$$

where $v(t)$ is a continuous-time white noise[1] of zero mean and intensity $q \geq 0$, that is $\mathbb{E}[q(t)q(s)] = q\delta_{ts}$, where δ_{hk} is the Kronecker delta. The initial condition is also assumed to be a random variable with mean x_0 and variance σ. Each sensor take measurements of the physical process according to the equation

$$y_i(kT) = x(kT) + n_i(kT), \qquad k \in \mathbb{Z}, \tag{8.2}$$

where T is the sampling time. Note that $y_i \in \mathbb{R}$, $\forall\, i$. We denote $y(kT) = [y_1(kT), \ldots, y_N(kT)]^*$ and $n(kT) = [n_1(kT), \ldots, n_N(kT)]^*$. Moreover the noise processes $n_i(kT) \in \mathbb{R}$ are such that $\mathbb{E}[n(kT)] = 0$, $E[n(kT)n(hT)] = rI\delta_{hk}$. Note also that (8.2) can be rewritten in the following vector form

$$y(kT) = x(kT)\mathbb{1} + n(kT). \tag{8.3}$$

From now on, without loss of generality, we shall assume that $T = 1$. Suppose also that, between each pair of subsequent measurement update indexes k and $k+1$, each node exchanges m messages; we assume that these transmissions take place at the following times $k+\delta, k+2\delta, \ldots, k+(m-1)\delta, k+m\delta$, where $\delta = \frac{1}{m}$. Note that $k + m\delta = k + 1$.

We shall denote with $\hat{x}_i(k + h\delta|k)$ the estimate, at node i, of the state x at time $k + h\delta$ given measurements up to time k. In compact form we shall also denote with $\hat{x} := [\hat{x}_1, \ldots, \hat{x}_N]^*$ the vector of estimates throughout the network; more precisely, making the dependence upon time explicit:

$$\hat{x}\,(k + h\delta|k) := \begin{bmatrix} \hat{x}_1\,(k + h\delta|k) \\ \vdots \\ \hat{x}_N\,(k + h\delta|k) \end{bmatrix}.$$

We assume that these estimates are updated, for $k \geq 0$, according to the following rule

$$\begin{cases} \hat{x}\,(k|k) = (1 - \ell(k))\,\hat{x}(k|k-1) + \ell(k)y(k) \\ \hat{x}\,(k + h\delta|k) = P(k,h)\hat{x}\,(k + (h-1)\delta|k)\,, \end{cases} \tag{8.4}$$

where $h = 1, \ldots, m$ and where $P(k,h)$ is a suitable matrix compatible with the communication graph and where $0 < \ell(k) < 1$, $\forall\, k \geq 0$. From now on we shall use constant "gains" $\ell(k)$ and $P(k,h)$, i.e. $\ell(k) = \ell$ and $P(k,h) = P$; furthermore, we shall also assume the the recursions (8.4) are initialized by

$$\hat{x}(0|-1) := x_0. \tag{8.5}$$

[1]We recall that what is commonly referred to as "continuous time white noise" can be thought of as the "derivative" of a Wiener process which, unfortunately, is nowhere differentiable. More rigorously $x(t)$ is a Wiener process.

A natural request is also that $\hat{x}_i$ be an unbiased estimator for all i's and for all times, i.e. $\mathbb{E}\left[\hat{x}_i(k+h\delta|k)\right] = x_0$, $\forall\, i$, $\forall\, k \geq 0$, $\forall\, h \in [1, m]$. This leads to the condition

$$P\mathbb{1} = \mathbb{1}. \tag{8.6}$$

In fact, from the update rule (8.4) it follows that

$$x_0\mathbb{1} = \mathbb{E}\left[\hat{x}\left(k+(h+1)\delta|k\right)\right] = \mathbb{E}\left[P\hat{x}\left(k+h\delta|k\right)\right] = x_0 P\mathbb{1}.$$

For $x_0 \neq 0$, the equality $x_0 P\mathbb{1} = x_0\mathbb{1}$ implies (8.6). Hence it turns out that P has to be a quasi-stochastic matrix, which we shall assume from now on. Now we define the new variable $\tilde{x}\left(k+h\delta|k\right) = x\left(k+h\delta\right)\mathbb{1} - \hat{x}\left(k+h\delta|k\right)$ which represents the estimation error. In order to analyze the structure of the recursive equations that $\tilde{x}\left(k+h\delta|k\right)$ satisfies, it is convenient to discretize (8.1) in the following way

$$x\left(k+(h+1)\delta\right) = x\left(k+h\delta\right) + w\left(k+h\delta\right)$$

where

$$w\left(k+h\delta\right) = \int_{k+h\delta}^{k+(h+1)\delta} v(\tau)d\tau.$$

Note that w is a discrete time white noise with mean zero and variance q/m, i.e. $\mathbb{E}\left[w\left(k+h\delta\right)\right] = 0$ and $\mathbb{E}\left[w^2\left(k+h\delta\right)\right] = \frac{q}{m}$. By straightforward calculations, for $h = 0$, we get

$$\tilde{x}(k|k) = (1-\ell)\,\tilde{x}(k|k-1) - \ell\, n(k)$$

and, for $1 \leq h \leq m$,

$$\tilde{x}\left(k+h\delta|k\right) = P^h\tilde{x}\left(k|k\right) + \left(\sum_{i=0}^{h-1} w\left(k+i\delta\right)\right)\mathbb{1}$$

We shall be concerned with the *second order* properties of the error, represented by the covariance matrices

$$\Sigma\left(k+h\delta|k\right) = \mathbb{E}\left[\tilde{x}\left(k+h\delta|k\right)\tilde{x}\left(k+h\delta|k\right)^*\right],$$

defined for $0 \leq h \leq m$. One can show that $\Sigma\left(k+h\delta|k\right)$ satisfies, for $h = 0$,

$$\Sigma(k|k) = (1-\ell)^2\,\Sigma(k|k-1) + \ell^2 rI \tag{8.7}$$

and, for $h = m$,

$$\Sigma(k+1|k) = P^m\Sigma(k|k)(P^m)^* + q\mathbb{1}\mathbb{1}^*. \tag{8.8}$$

Plugging (8.7) into (8.8) we obtain the recursive equation

$$\Sigma(k+1|k) = (1-\ell)^2 P^m \Sigma(k|k-1)\,(P^*)^m + \ell^2 r P^m\,(P^*)^m + q\mathbb{1}\mathbb{1}^* \qquad (8.9)$$

while, inserting (8.8) into (8.7) evaluated at the index $k+1$ we get:

$$\Sigma(k+1|k+1) = (1-\ell)^2\, P^m \Sigma(k|k)\,(P^*)^m + (1-\ell)^2\, q\mathbb{1}\mathbb{1}^* + \ell^2 r I. \quad (8.10)$$

From (8.5) it follows that the error covariance $\Sigma(0|-1)$ satisfies $\Sigma(0|-1) = \sigma\mathbb{1}\mathbb{1}^*$; similarly $\Sigma(0|0)$ is given by $\Sigma(0|0) = (1-\ell)^2\sigma\mathbb{1}\mathbb{1}^* + \ell^2 r I$. Iterating the update rule (8.9), starting from the initial condition $\Sigma(0|-1)$, we obtain

$$\Sigma(k|k-1) = (1-\ell)^{2k} P^{km} \Sigma(0|-1)\,(P^*)^{km} + \\ + \ell^2 r \sum_{i=0}^{k-1} (1-\ell)^{2i} P^{(i+1)m}\,(P^*)^{(i+1)m} + q\sum_{i=0}^{k-1}(1-\ell)^{2i}\mathbb{1}\mathbb{1}^*\,;$$

similarly, iterating (8.10) with initial condition $\Sigma(0|0)$, we get

$$\Sigma(k|k) = (1-\ell)^{2k} P^{km} \Sigma(0|0)\,(P^*)^{km} + q\sum_{i=0}^{k-1}(1-\ell)^{2i+2}\mathbb{1}\mathbb{1}^* + \\ + \ell^2 r \sum_{i=0}^{k-1}(1-\ell)^{2i} P^{im}\,(P^*)^{im}$$

In this chapter we shall be concerned with the asymptotic ($k \to \infty$) behavior of the error covariance. Hence we consider the limits

$$\lim_{k\to\infty} \Sigma(k|k-1) = r\ell^2 \sum_{i=0}^{\infty}(1-\ell)^{2i} P^{(i+1)m}\,(P^*)^{(i+1)m} + q\frac{1}{1-(1-\ell)^2}\mathbb{1}\mathbb{1}^*$$

and

$$\lim_{k\to\infty} \Sigma(k|k) = q\frac{(1-\ell)^2}{1-(1-\ell)^2}\mathbb{1}\mathbb{1}^* + r\ell^2 \sum_{i=0}^{\infty}(1-\ell)^{2i} P^{im}\,(P^*)^{im}$$

and define the cost functions[2]

$$J_1(P,\ell;m,r,q) = \mathrm{tr}\left\{\lim_{k\to\infty}\Sigma(k+1|k)\right\}$$

[2]In the remainder of the paper, when there is no risk of confusion, we might drop some arguments of the cost (e.g. denote $J_1(P,\ell)$ rather than $J_1(P,\ell;m,r,q)$).

and

$$J_2(P,\ell;m,r,q) = \text{tr}\left\{\lim_{k\to\infty}\Sigma(k|k)\right\}$$

Note that some restrictions on P need to be imposed to ensure that the limits converge. From now given a quasi-stochastic matrix, we will denote its essential spectral by $\rho(P)$[3] If $\rho(P) \leq 1$ and $0 < \ell < 1$, then convergence is guaranteed. In the following of the chapter we will restrict to parameters which satisfy these constraints.

The costs defined above lead to the formulation of the following minimization problem:

Problem 8.1 *Given a graph $\mathcal{G}$ and a nonnegative integer m, find a real $\ell \in (0,1)$ and a matrix $P \in \mathcal{P}$, where $\mathcal{P}$ is the set of the quasi-stochastic matrices compatible with the graph $\mathcal{G}$, minimizing J_1 or J_2.*

In the sequel the set $\mathcal{P}$ may be further restricted while always being compatible with the topology of the communication network.

Remark 8.2 In the sequel we will consider only J_1. The reason will be clear in the next sections where the minimization on J_1 will permit us to retrieve, for some particular cases, the results already known in the literature regarding Kalman filtering. For the sake of simplicity, we will denote this cost function simply by J in place of J_1. Hence

$$J = r\ell^2\text{tr}\left\{\sum_{i=0}^{\infty}(1-\ell)^{2i}P^{(i+1)m}\left(P^*\right)^{(i+1)m}\right\} + q\frac{1}{1-(1-\ell)^2}N \tag{8.11}$$

Remark 8.3 Let us denote with λ_i, $i = 0,..,N-1$ the eigenvalues[4] of P; since P is quasi-stochastic, we can set, without loss of generality, $\lambda_0 = 1$, i.e. $\sigma(P) = \{1, \lambda_1, \lambda_2, \ldots, \lambda_{N-1}\}$. Note that, if P is a normal matrix, i.e. $PP^* = P^*P$, then $\text{tr}\{P^{im}(P^*)^{im}\} = \sum_{h=0}^{n-1}|\lambda_h|^{2im}$; it is hence easy to see that formula (8.11) can be rewritten as follows:

$$J = \frac{r\ell^2 + qN}{1-(1-\ell)^2} + r\ell^2\sum_{i=1}^{N-1}\frac{|\lambda_i|^{2m}}{1-(1-\ell)^2|\lambda_i|^{2m}} \tag{8.12}$$

[3]In Chapter 3 we denote the essential spectral by ρ_{ess}. Since there is no risk of confusion we use here the simpler notation ρ.

[4]Multiple eigenvalues are counted as many times as their algebraic multiplicity.

Also note that, if P is normal and quasi-stochastic, then it is also quasi-doubly stochastic. Relevant subclasses of normal matrices are, for instance, Abelian Cayley matrices [10], circulant matrices and symmetric matrices.

The previous remark suggests that the following assumption is both useful and reasonable.

Assumption 8.4 *The set $\mathcal{P}$ in problem 8.1 is the subset of normal quasi-stochastic matrices with $\rho(P) \leq 1$ which are compatible with the graph $\mathcal{G}$.*

Example 8.5 Assume that $\mathcal{P}$ is the set of the circulant stochastic matrices of the form

$$P_k = \begin{bmatrix} 1-k & k & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1-k & k & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ k & 0 & 0 & 0 & \cdots & 0 & 1-k \end{bmatrix}$$

where $k \in [0,1]$ and assume that $m = 1$. Let

$$P^{opt},\ \ell^{opt} \in \underset{\ell \in (0,1);\, P \in \mathcal{P}}{\arg\min}\ J(P, \ell; m, r, q) \tag{8.13}$$

It is well known in the literature [46] that the eigenvalues of P can be expressed in the following form $\lambda_h = 1 - k + ke^{j\frac{2\pi}{N}h}$, $0 \leq h \leq N-1$. Notice that $\left|1 - k + ke^{j\frac{2\pi}{N}h}\right| \geq \left|1/2 + 1/2e^{j\frac{2\pi}{N}h}\right|$, $\forall\, h\ : 0 \leq h \leq N-1$. Hence it follows that

$$P_{\frac{1}{2}} = \underset{P \in \mathcal{P}}{\arg\min}\ J(P, \ell; 1, r, q)$$

for all $\ell \in (0,1)$, i.e. $P^{opt} = P_{\frac{1}{2}}$. In order to calculate the optimal gain we have to solve

$$\underset{\ell \in (0,1)}{\arg\min}\ \frac{qN}{1-(1-\ell)^2} + r\ell^2 \sum_{h=0}^{N-1} \frac{|\lambda_h|^2}{1-(1-\ell)^2|\lambda_h|^2} \tag{8.14}$$

where $\lambda_h = \frac{1}{2} + \frac{1}{2}e^{j\frac{2\pi}{N}h}$. Unfortunately it is not possible to give, in general, a closed form expression for the optimizing gain ℓ^{opt}, which has to be found using numerical search techniques.

However a simple expression for (8.14) can be obtained when the number of sensors N goes to infinity, which allows to study the behavior of the optimal

ℓ for large networks. To this purpose it is convenient to define the *normalized* cost

$$\begin{aligned}\bar{J}_N := \frac{J}{N} &= \frac{q}{1-(1-\ell)^2} + \frac{r\ell^2}{N}\sum_{h=0}^{N-1}\frac{|\lambda_h|^2}{1-(1-\ell)^2|\lambda_h|^2} \\ &= \frac{q}{1-(1-\ell)^2} + \frac{r\ell^2}{N}\tilde{J}_N\end{aligned}$$

where the last equation defines $\tilde{J}_N$. Define also the function $f : \mathbb{C} \to \mathbb{C}$

$$f(z) = \frac{1}{2} + \frac{1}{2}z.$$

and note that $\lambda_h = f\left(e^{j\frac{2\pi}{N}h}\right)$. As $N \to \infty$, $\tilde{J}_N$ converges to

$$\begin{aligned}\lim_{N\to\infty}\tilde{J}_N = \tilde{J}_\infty &= \frac{1}{2\pi}\int_0^{2\pi}\frac{|f(e^{j\phi})|}{1-(1-\ell)^2|f(e^{j\phi})|^2}d\phi \\ &= \frac{1}{2\pi j}\oint_\gamma \frac{z^{-1}(2+z+z^{-1})}{4-(1-\ell)^2(2+z+z^{-1})}dz\end{aligned}$$

where γ is the unit circle. It is straightforward to see that the poles of $\frac{z^{-1}(2+z+z^{-1})}{4-(1-\ell)^2(2+z+z^{-1})}$ inside γ are $z_1 = \frac{2-(1-\ell)^2}{(1-\ell)^2} - \sqrt{\left(\frac{2-(1-\ell)^2}{(1-\ell)^2}\right)^2 - 1}$, and $z_2 = 0$; the integral can be computed explicitly using the residue theorem yielding:

$$\tilde{J}_\infty = \frac{1}{(1-\ell)^2}\left(\frac{1}{\sqrt{1-(1-\ell)^2}} - 1\right)$$

Hence

$$\lim_{N\to\infty}\bar{J}_N\left(P_{\frac{1}{2}}, \ell; 1, r, q\right) = \frac{q}{1-(1-\ell)^2} + \frac{r\ell^2}{(1-\ell)^2}\left(\frac{1}{\sqrt{1-(1-\ell)^2}} - 1\right).$$

value of ℓ^{opt} can be found by minimizing the above limiting expression for $\bar{J}_N$.

8.3 Optimal consensus matrix P for fixed gain ℓ

In this section we assume that the estimation gain ℓ is fixed, and thus consider the optimization problem:

$$P^{opt}(\ell; m) = \arg\min_{P\in\mathcal{P}} J(P, \ell; m). \tag{8.15}$$

Although the analysis of this problem is quite hard in general, a detailed study can be carried out in some interesting situations. In particular in the following we will restrict to the three special cases:

- the communication graph $\mathcal{G}$ is undirected
- the sensors can communicate arbitrarily fast within two subsequent measurements, i.e., $m \to \infty$
- the estimation gain ℓ is sufficiently large, i.e. $\ell \to 1$; this intuitively corresponds to the situation in which the variance of the measurement noise is negligible with respect to the variance of the process, i.e $\frac{r}{q} \approx 0$.

Before proceeding, we observe that

$$\min_{P \in \mathcal{P}} J = \frac{r\ell^2 + qN}{1 - (1-\ell)^2} + \min_{P \in \mathcal{P}} r\ell^2 \sum_{j=1}^{N-1} \frac{|\lambda_j|^{2m}}{1 - (1-\ell)^2 |\lambda_j|^{2m}}$$

and hence only the second term on the right hand side play a role in the optimization. We can therefore restrict to consider only this latter quantity which, for convenience of notation, we denote as

$$S(P, \ell; m) = \sum_{j=1}^{N-1} \frac{|\lambda_j|^{2m}}{1 - (1-\ell)^2 |\lambda_j|^{2m}} \tag{8.16}$$

8.3.1 Undirected communication graph $\mathcal{G}$

We start by observing that, imposing that a matrix P is compatible with a graph is a convex constraint. Observe moreover that, since we assume that $\mathcal{P}$ is a subset of the normal matrices, for all $P \in \mathcal{P}$ we have that $\|P\|_2 = \max_i |\lambda_i|$ and so also the condition $\rho(Q) \leq 1$ is a convex constraint as well.
Unfortunately, matrix normality is not a convex constraint in general. This difficult can be overcome when the graph is undirected as the lemma below shows. Indeed note that, if the communication graph $\mathcal{G}$ is undirected, $P \in \mathcal{P}$ implies that also $P^* \in \mathcal{P}$. For any $P \in \mathcal{P}$, consider its symmetric part $P_{sym} := (P + P^*)/2$. Clearly, P_{sym} is normal and it is compatible with $\mathcal{G}$, therefore $P_{sym} \in \mathcal{P}$. The following lemma provides an interesting comparison between $J(P, \ell; m)$ and $J(P_{sym}, \ell; m)$ showing that the former is always greater or at most equal to the latter.

Lemma 8.6 *Let P be any matrix in $\mathcal{P}$ and let P_{sym} be defined as above. Then*

$$J(P_{sym}, \ell; m) \leq J(P, \ell; m).$$

Proof: Let λ_i be any eigenvalue of P. Then $Re\{\lambda_i\}$ is an eigenvalue of P_{sym}, where $Re\{\lambda_i\}$ denotes the real part of λ_i. Clearly $|Re\{\lambda_i\}| \leq |\lambda_i|$ which implies that

$$\frac{|Re\{\lambda_i\}|^{2m}}{1-(1-\ell)^2|Re\{\lambda_i\}|^{2m}} \leq \frac{|\lambda_i|^{2m}}{1-(1-\ell)^2|\lambda_i|^{2m}}.$$

Therefore, from (8.16), it follows that $S(P_{sym}, \ell; m) \leq S(P, \ell; m)$; hence also $J(P_{sym}, \ell; m) \leq J(P, \ell; m)$ holds true. ■

Remark 8.7 It is important to note that normality plays a fundamental role in the previous lemma which cannot be generalized to quasi-stochastic matrices P. In fact, one can find a non-normal P for which the symmetric part P_{sym} gives a larger cost index.

An immediate consequence of Lemma 8.6 is that, when the communication graph is undirected, the minimum of the cost function J is reached by symmetric matrices. Thus, if $\mathcal{P}_{sym}$ is the subset of $\mathcal{P}$ containing the symmetric matrices, i.e. $\mathcal{P}_{sym} = \{P \in \mathcal{P} : P = P^*\}$, solving (8.15) is equivalent to solve

$$\arg\min_{P \in \mathcal{P}_{sym}} J(P, \ell; m). \tag{8.17}$$

The following result provides a powerful characterization of (8.17) which has important implications when it comes to performing optimization.

Theorem 8.8 *Let $\mathcal{P}_{sym}$ be as above. Then the cost function $J(P, \ell; m)$ defined on $\mathcal{P}_{sym}$ is a convex function.*

Proof: Consider the function $f : \mathcal{B} \subseteq \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x) = \sum_{i=1}^{n} \frac{x_i^{2m}}{1-\alpha x_i^{2m}},$$

where $m \in \mathbb{N}$, $0 < \alpha < 1$, $x = [x_1, x_2, \ldots, x_n]^T$ and where $\mathcal{B} = \{x \in \mathbb{R}^n : |x_i| \leq 1\}$. It is easy to verify that the function f is convex and symmetric, i.e. it is invariant to any permutation of the vector entries x_i. Hence, it follows from

the theory of convex spectral functions that also J is a convex function [18]. ■

Theorem 8.8 states that (8.17) is a convex problem thus implying that the solution of (8.17) can be performed efficiently by suitable numerical algorithms. In fact, Xiao et al. [148] adopted this strategy to optimize similar performance costs over symmetric stochastic matrices.

8.3.2 Fast communication ($m \to \infty$)

We have seen in Chapter 2 that the speed of consensus algorithms is governed by the essential spectral radius of the consensus matrix.

Indeed, the essential spectral radius plays also an important role in distributed estimation provided communication is "sufficiently fast". In fact the following theorem shows that optimizing the essential spectral radius improves the performance provided that the number of message exchanges is larger than a specified bound. However, when comparing the performance of two consensus matrices P_1 and P_2, this bound is a function of $\rho(P_1)$ and $\rho(P_2)$ as explained in the following theorem.

Theorem 8.9 *Let P_1 and P_2 be two matrices such that $\rho(P_1) > \rho(P_2)$. Then there exists $\bar{m}$ (depending only on $\rho(P_1) - \rho(P_2)$) such that*

$$J(P_1, \ell; m) > J(P_2, \ell; m), \qquad \forall\, m > \bar{m}.$$

Proof: Let $\rho_1 = \rho(P_1)$, $\rho_2 = \rho(P_2)$ and $\epsilon = \rho(P_1) - \rho(P_2) = \rho_1 - \rho_2$. Observe that

$$\begin{aligned} S(P_1, \ell; m) &\geq \frac{\rho_1^{2m}}{1 - (1-\ell)^2 \rho_1^{2m}} \\ &> \frac{(\rho_2 + \epsilon)^{2m}}{1 - (1-\ell)^2 \rho_2^{2m}} \end{aligned}$$

and that

$$S(P_2, \ell; m) \leq N \frac{\rho_2^{2m}}{1 - (1-\ell)^2 \rho_2^{2m}}.$$

Hence, if $(\rho_2 + \epsilon)^{2m} > N \rho_2^{2m}$, we have that $S(P_1, \ell; m) > S(P_2, \ell; m)$. Straightforward calculations show that this last condition is satisfied if and only if $m > \frac{\log N}{2 \log\left(1 + \frac{\epsilon}{\rho_2}\right)}$. Note that $\frac{\log N}{2 \log(1+\epsilon)} \geq \frac{\log N}{2 \log\left(1 + \frac{\epsilon}{\rho_2}\right)}$. Therefore by letting

$\bar{m} = \left\lceil \frac{\log N}{2\log(1+\epsilon)} \right\rceil$ the statement of the theorem follows. Note that $\bar{m}$ depends only on ϵ. ■

8.3.3 Large gain ($\ell \to 1$)

In this section we shall instead consider the number of message exchanges m as fixed and assume the gain ℓ is "large"; this, as we shall also see in Section 8.5.2, is what happens when the measurement noise is small as compared to process noise, a situation frequently encountered in practice. It is remarkable that in this case the Frobenius norm is instead the "right" way to compare consensus matrices. This is made precise by the following theorem.

Theorem 8.10 *Let P_1, P_2 be two matrices such that $\|P_1^m\|_F > \|P_2^m\|_F$. Then there exists $\bar{\ell}$ (depending only on $\|P_1^m\|_F - \|P_2^m\|_F$) such that*

$$J(P_1, \ell; m) > J(P_2, \ell; m), \qquad \forall\, \ell > \bar{\ell}. \tag{8.18}$$

Proof: Let $f(\ell, m) = S(P_1, \ell; m) - S(P_2, \ell; m)$ and $\epsilon = \|P_1^m\|_F - \|P_2^m\|_F$. We have that $f(\ell, m)$ can be written in the following way

$$f(\ell, m) = \sum_{i=0}^{\infty} \left(\|P_1^{(i+1)m}\|_F^2 - \|P_2^{(i+1)m}\|_F^2 \right) (1-\ell)^{2i} = \sum_{i=0}^{\infty} \alpha_i (1-\ell)^{2i}$$

where $\alpha_i = \|P_1^{(i+1)m}\|_F^2 - \|P_2^{(i+1)m}\|_F^2$. It is straightforward to see that $\alpha_0 > \epsilon^2$ and that, for $i \geq 1$, $\alpha_i \geq -(N-1)\rho^{2m(i+1)}$, where $\rho = \rho(P_2)$. Hence, it follows that

$$\begin{aligned} f(\ell, m) &> \epsilon^2 - (N-1) \sum_{i=1}^{\infty} \rho^{2m(i+1)} (1-\ell)^{2i} \\ &> \epsilon^2 - \frac{(1-\ell)^2 (N-1)}{1-(1-\ell)^2} \end{aligned}$$

Clearly, if the last term is positive, then also $f(\ell, m) > 0, \forall\, m$. This condition is satisfied if and only if

$$\ell \geq 1 - \sqrt{\frac{\epsilon^2}{N - 1 + \epsilon^2}} = \bar{\ell}.$$

Therefore, $S(P_1, \ell; m) - S(P_2, \ell; m) > 0$, $\forall\, \ell > \bar{\ell}$ and consequently $J(P_1, \ell; m) > J(P_2, \ell; m)$, thus proving the claim of the theorem. ■

Remark 8.11 At first sight, Theorem 8.9 and Theorem 8.10 seem in contradiction. However, this can be explained by observing that $\|P^m\|_F^2 = 1 + \alpha\rho^{2m}(P) + o(\rho^{2m}(P))$, where α is the algebraic multiplicity of the essential spectral radius of P. Therefore, for large m, minimizing the Frobenius norm of P^m or the essential spectral radius of P is almost equivalent.

8.4 Optimal gain ℓ for fixed consensus matrix P

In this section we assume that the consensus matrix P is fixed. Hence the problem we want to solve is the following

$$\ell^{opt} \arg\min_{\ell\in(0,1)} \; J(P,\ell;m) \tag{8.19}$$

The previous optimization problem is convex in ℓ. This fact can be easily checked by observing that the cost functional J can be written as sum of functions of the form:

$$g(\ell) = \frac{x\ell^2}{1-x(1-\ell)^2}, \quad h(\ell) = \frac{x}{1-x(1-\ell)^2}, \quad x \in [0,1]$$

which are convex in $\ell \in (0,1)$.

Given any $P \in \mathcal{P}$ we define

$$J^{opt}(m,r,q|P) \stackrel{\triangle}{=} \min_{\ell} J(P,\ell;m,r,q) \tag{8.20}$$

$$\ell^{opt}(m,r,q|P) \in \arg\min{}_{\ell} J(P,\ell;m,r,q) \tag{8.21}$$

Note that $J^{opt}(m,r,q|P) = J(P,\ell^{opt}(m,r,q|P);m,r,q)$ and that $P^{opt}(m,r,q) \in \arg\min{}_P J^{opt}(m,r,q|P)$. In the sequel, without risk of confusion, we shall omit arguments which are kept fixed.

Convexity of J allows easy computations of $\ell^{opt}(m|P)$. In the remaining of this section we shall see that the sequence $\{\ell^{opt}(m|P)\}_{m=0}^{\infty}$ is monotonically non-decreasing in m. Moreover, it is bounded below and above by ℓ_d^{opt} and ℓ_c^{opt}, which are the optimal gains minimizing J respectively when $P = I$ and when $P = \frac{1}{N}\mathbb{1}\mathbb{1}^*$, namely

$$\ell_d^{opt} = \arg\min_{\ell\in(0,1)} \; J(I,\ell;m,r,q), \; \ell_c^{opt} = \arg\min_{\ell\in(0,1)} \quad J\left(\frac{1}{N}\mathbb{1}\mathbb{1}^*,\ell;m,r,q\right)$$

Note that $P = I$ and $P = \frac{1}{N}\mathbb{1}\mathbb{1}^*$ represent the two extreme cases in modeling the flow of information between the sensors. Indeed, $P = I$ corresponds to the situation in which the sensors do not communicate; in such a case there are N Kalman filters running separately (the subscript "d" in ℓ_d^{opt} means *decentralized*, i.e. no communication). In the other case, instead, we have that the underlying communication graph is complete and this means that each sensor has full knowledge about the estimates of all the other sensors (the subscript "c" in ℓ_c^{opt} means *centralized*, i.e. full communication). Since $I^m = I$ and $(1/N\mathbb{1}\mathbb{1}^*)^m = 1/N\mathbb{1}\mathbb{1}^*$ for all $m \geq 1$, we have that ℓ_c^{opt} and ℓ_d^{opt} are independent on m. The following proposition characterizes precisely ℓ_d^{opt} and ℓ_c^{opt}.

Proposition 8.12 *Let ℓ_d^{opt} and ℓ_c^{opt} be as above. Then*

$$\ell_d^{opt} = \frac{-q + \sqrt{q^2 + 4qr}}{2r}, \ \ell_c^{opt} = \frac{-q + \sqrt{q^2 + 4q\bar{r}}}{2\bar{r}}$$

where $\bar{r} = \frac{r}{N}$.

Proof: The proof follows from standard results known in Kalman filtering. ■

The following theorem shows also that ℓ_d^{opt} and ℓ_c^{opt} play the role of, respectively, lower and upper bounds for $\ell^{opt}(m|P)$. Indeed, a stronger result can be obtained, which characterizes the sequence $\{\ell^{opt}(m|P)\}_{m=0}^{\infty}$.

Theorem 8.13 *Let $P \in \mathcal{P}$. Let $\ell^{opt}(m|P)$ be defined as above. Then the following chain of inequalities holds true*

$$\begin{aligned}\ell_d^{opt} = \ell^{opt}(0|P) \leq \ell^{opt}(1|P) \leq \ldots \leq \ell^{opt}(m|P) \leq \\ \leq \ell^{opt}(m+1|P) \leq \ldots \leq \ell^{opt}(\infty|P) \leq \ell_c^{opt}.\end{aligned}$$

The achieved (optimal) cost satisfies the reversed chain of inequalities:

$$\begin{aligned}J\left(P, \ell_d^{opt}; 0\right) \geq J\left(P, \ell^{opt}(1|P); 1\right) \geq \ldots \geq J\left(P, \ell^{opt}(m|P); m\right) \geq \\ > J\left(P, \ell^{opt}(m+1|P); m+1\right) \ldots \geq J\left(P, \ell_c^{opt}; \infty\right)\end{aligned}$$

Moreover $\ell^{opt}(\infty|P) = \ell_c^{opt}$ if and only if $\rho(P) < 1$.

Proof: Note that the equality $\ell^{opt}(0|P) = \ell_d^{opt}$ follows directly from the fact that $P^0 = I$. We prove now that $\ell^{opt}(m|P) \leq \ell^{opt}(m+1|P)$. We start

by calculating the partial derivative

$$\frac{\partial J(P,\ell;m)}{\partial \ell} = 2\sum_{i=1}^{N-1} \frac{r\ell|\lambda_i|^{2m} - r\ell(1-\ell)|\lambda_i|^{4m}}{\left[1-(1-\ell)^2|\lambda_i|^{2m}\right]^2} - 2\frac{Nq(1-\ell) - r\ell^2}{\left[1-(1-\ell)^2\right]^2}$$

Consider now the functions $p : [0,1] \times [0,\infty) \to \mathbb{R}$ defined as

$$p(x,m) = \frac{x^{2m} - (1-\ell)x^{4m}}{\left(1-(1-\ell)^2 x^{2m}\right)^2}.$$

We have that

$$\frac{\partial p}{\partial m} = 2x^{2m}\frac{1-(1-\ell^2)x^{2m}}{\left[1-(1-\ell)^2x^{2m}\right]^3}\log x.$$

It is easy to check that $\frac{\partial p}{\partial m} < 0\ \forall \ell \in (0,1)$ and $\forall x \in [0,1]$. This implies that

$$\begin{aligned}
&\frac{\partial J(P,\ell;m)}{\partial \ell} - \frac{\partial J(P,\ell;m+1)}{\partial \ell} = \\
&= 2r\ell\sum_{i=0}^{N-1} \frac{|\lambda_i|^{2m} - (1-\ell)|\lambda_i|^{4m}}{\left[1-(1-\ell)^2|\lambda_i|^{2m}\right]^2} - 2r\ell\sum_{i=0}^{N-1} \frac{|\lambda_i|^{2(m+1)} - (1-\ell)|\lambda_i|^{4(m+1)}}{\left[1-(1-\ell)^2|\lambda_i|^{2(m+1)}\right]^2} \\
&\geq 0.
\end{aligned}$$

Since we already know that for all m the function $J(P,\ell;m)$ is convex in $\ell \in (0,1)$, it follows that $\ell^{opt}(m+1|P) > \ell^{opt}(m|P)$. In order to show that $\ell^{opt}(m|P) \leq \ell_c^{opt}$, we remark that the matrix $\frac{1}{N}\mathbb{1}\mathbb{1}^*$ has an eigenvalue equal to 1 and $N-1$ eigenvalues equal to 0. This implies that, for all m,

$$\frac{\partial J\left(\frac{1}{N}\mathbb{1}\mathbb{1}^*,\ell;m\right)}{\partial \ell} = -2\frac{Nq(1-\ell) - r\ell^2}{\left[1-(1-\ell)^2\right]^2}.$$

Thus we have that

$$\frac{\partial J(P,\ell;m)}{\partial \ell} - \frac{\partial J\left(\frac{1}{N}\mathbb{1}\mathbb{1}^*,\ell;m+1\right)}{\partial \ell} = 2r\ell\sum_{i=1}^{N-1} \frac{|\lambda_i|^{2m} - (1-\ell)|\lambda_i|^{4m}}{\left[1-(1-\ell)^2|\lambda_i|^{2m}\right]^2} > 0$$

where the last inequality follows from the fact that $x - \alpha x^2 > 0$, $\forall x \in (0,1)$ and $\forall \alpha \in (0,1)$. From this last inequality $\ell^{opt}(m|P) < \ell_c^{opt}$ follows.

We prove now $\ell^{opt}(m|P) = \ell_c^{opt}$ if and only if $\rho(P) < 1$. If $\rho(P) < 1$, then $\lim_{k\to\infty} P^k = \frac{1}{N}\mathbb{1}\mathbb{1}^*$. Conversely, if $\rho(P) = 1$, then there exists λ_i, $1 \leq i \leq N-1$ such that $|\lambda_i| = 1$ implying that

$$\frac{\partial J(P,\ell;m)}{\partial \ell} - \frac{\partial J\left(\frac{1}{N}\mathbb{1}\mathbb{1}^*,\ell;m\right)}{\partial \ell} \geq \frac{2r\ell^2}{\left[1-(1-\ell)^2\right]^2} > 0.$$

Thus $\ell^{opt}(\infty|P) < \ell_c^{opt}$.

The chain of inequalities on the cost functionals follows from the fact that $J(P, \ell; m)$ is a decreasing function on m. ■

The following two lemmas provide some interesting properties of $J^{opt}(m, r, q|P)$ as a function of m and as a function of r, q. They will be useful in the next Section.

Lemma 8.14 *Let $P, \bar{P} \in \mathcal{P}$ be such that $\rho(\bar{P}) + \epsilon \leq \rho(P)$, where $\epsilon > 0$. Then there exists $\bar{m}_\epsilon$, depending only on ϵ, such that the cost function (8.20) satisfies*

$$J^{opt}(m|P) > J^{opt}(m|\bar{P}), \qquad \forall\, m > \bar{m}_\epsilon. \tag{8.22}$$

Proof: Let $\rho = \rho(P)$, $\bar{\rho} = \rho(\bar{P})$ and λ_i and $\bar{\lambda}_i$ be the eigenvalues of P and $\bar{P}$, respectively. Observe that

$$\begin{aligned}\sum_{j=1}^{N-1} \frac{|\lambda_j|^{2m}}{1-(1-\ell)^2|\lambda_j|^{2m}} &\geq \frac{\rho^{2m}}{1-(1-\ell)^2\rho^{2m}} \\ &\geq \frac{(\bar{\rho}+\epsilon)^{2m}}{1-(1-\ell)^2\bar{\rho}^{2m}}\end{aligned}$$

and that

$$\sum_{j=1}^{N-1} \frac{|\bar{\lambda}_j|^{2m}}{1-(1-\ell)^2|\bar{\lambda}_j|^{2m}} \leq N \frac{\bar{\rho}^{2m}}{1-(1-\ell)^2\bar{\rho}^{2m}}.$$

Hence, if $(\bar{\rho}+\epsilon)^{2m} > N\bar{\rho}^{2m}$, we have that

$$\sum_{j=1}^{N-1} \frac{|\lambda_j|^{2m}}{1-(1-\ell)^2|\lambda_j|^{2m}} > \sum_{j=1}^{N-1} \frac{|\bar{\lambda}_j|^{2m}}{1-(1-\ell)^2|\bar{\lambda}_j|^{2m}} \tag{8.23}$$

Straightforward calculations show there exists $\bar{m}_\epsilon$ depending only on ϵ such that $(\bar{\rho}+\epsilon)^{2m} > N\bar{\rho}^{2m}$ for all $m \geq \bar{m}_\epsilon$ so ensuring that Eqn. (8.23) holds. This implies that $J(P, \ell; m) > J(\bar{P}, \ell; m)$ for all $m > \bar{m}_\epsilon$. The lemma follows from the fact that

$$\begin{aligned}J^{opt}(m|\bar{P}) = J(\bar{P}, \ell^{opt}(m|\bar{P}); m) \leq J(\bar{P}, \ell^{opt}(m|P); m) < \\ < J(P, \ell^{opt}(m|P); m) = J^{opt}(m|P)\end{aligned}$$

■

Lemma 8.15 *Let $P, \bar{P} \in \mathcal{P}$ and assume that $\|\bar{P}^m\|_F + \epsilon \leq \|P^m\|_F$, where $\epsilon > 0$. Then there exists δ_ϵ depending only on ϵ such that the cost function (8.20) satisfies*

$$J^{opt}(r, q|P) > J^{opt}\left(r, q|\bar{P}\right), \qquad \forall\, r/q < \delta_\epsilon.$$

Proof: Let

$$f(\ell, m) := \sum_{j=1}^{N-1} \frac{|\lambda_j|^{2m}}{1-(1-\ell)^2|\lambda_j|^{2m}} - \sum_{j=1}^{N-1} \frac{|\bar{\lambda}_j|^{2m}}{1-(1-\ell)^2|\bar{\lambda}_j|^{2m}}$$

where λ_i and $\bar{\lambda}_i$ are the eigenvalues of P and $\bar{P}$, respectively. We have that $f(\ell, m)$ can be written in the following way

$$\begin{aligned} f(\ell, m) &= \sum_{i=0}^{\infty} \left(\|P^{(i+1)m}\|_F^2 - \|\bar{P}^{(i+1)m}\|_F^2\right)(1-\ell)^{2i} \\ &= \sum_{i=0}^{\infty} \alpha_i (1-\ell)^{2i} \end{aligned}$$

where $\alpha_i = \|P^{(i+1)m}\|_F^2 - \|\bar{P}^{(i+1)m}\|_F^2$. It is straightforward to see that $\alpha_0 > \epsilon^2$. Also since all eigenvalues of P^k and $\bar{P}^k$ are inside the unit circle for all k then $||P^{(i+1)m}||_F \geq 1$ and $||\bar{P}^{(i+1)m}||_F \leq N$ for $i \geq 1$, therefore $\alpha_i \geq -(N-1)$. Hence, it follows that

$$f(\ell, m) > \epsilon^2 - (N-1)\sum_{i=1}^{\infty}(1-\ell)^{2i} = \epsilon^2 - \frac{(1-\ell)^2(N-1)}{1-(1-\ell)^2}$$

Clearly, if the last term is non-negative, then also $f(\ell, m) > 0, \forall\, m$. This condition is satisfied if and only if

$$\ell \geq 1 - \sqrt{\frac{\epsilon^2}{N-1+\epsilon^2}} =: \ell_\epsilon.$$

Therefore, we have that $J(P, \ell; r, q) > J(\bar{P}, \ell; r, q)$ for all $\ell \geq \ell_\epsilon$. Now observe that if $\frac{r}{q} \to 0$ then $\ell_d^{opt} \to 1$. Since $\ell_\epsilon < 1$, then there exists $\delta_\epsilon > 0$ such that if $r/q < \delta_\epsilon$ then $\ell_d^{opt} > \ell_\epsilon$ and so, by Theorem 8.13, $\ell^{opt}(r, q|P) > \ell_\epsilon$. By the previous arguments this implies that

$$\begin{aligned} J^{opt}(r, q|P) &= J(P, \ell^{opt}(r, q|P); r, q) \\ &> J\left(\bar{P}, \ell^{opt}(r, q|P); r, q\right) \\ &\geq J\left(\bar{P}, \ell^{opt}(r, q|\bar{P}); r, q\right) \\ &= J^{opt}(r, q|\bar{P}) \end{aligned}$$

which completes the proof. ∎

8.5 Joint optimization of P and ℓ: special cases

We have shown in the previous two sections that the cost functional J is a convex function, both in $P \in \mathcal{P}_{sym}$ for ℓ fixed and in $\ell \in (0,1)$ for P fixed. Unfortunately, as simple examples demonstrate, J is not a convex function jointly in ℓ and $P \in \mathcal{P}_{sym}$. Therefore, the joint minimization of J

$$(P^{opt}(m,r,q), \ell^{opt}(m,r,q)) \in \underset{\ell \in (0,1);\, P \in \mathcal{P}_{sym}}{\arg\min} J(P,\ell;m,r,q) \tag{8.24}$$

results to be quite hard in general. Nevertheless, an analytical characterization is possible when restricting to some "asymptotic cases" on the values of m, r and q. In particular we will consider the following situations:

- the sensors can communicate arbitrarily fast within two subsequent measurements, i.e., $m \to \infty$
- $\frac{r}{q} \approx 0$, i.e. the variance of the measurement noise is negligible with respect to the variance of the process
- $\frac{q}{r} \approx 0$, i.e. the variance of the process is negligible with respect to the variance of the measurement noise

First note that $P^{opt}(m,r,q)$ and $\ell^{opt}(m,r,q)$ are indeed only functions of m and r/q. In the sequel, without risk of confusion, we shall omit arguments which are kept fixed.

8.5.1 Fast communication ($m \to \infty$)

The results of this section parallel those of section 8.3.2. Indeed it will be shown that when $m \to \infty$, optimizing P for "fast convergence", i.e. minimizing the essential spectral radius is the "right" thing to do. Moreover, as expected, the optimal gain converges to the centralized gain ℓ_c^{opt}.

Theorem 8.16 *Let $P^{opt}(m)$, $\ell^{opt}(m)$ be a solution of (8.24). Then*

$$\lim_{m\to\infty} \rho(P^{opt}(m)) = \min_{P\in\mathcal{P}} \rho(P).$$

and

$$\lim_{m\to\infty} \ell^{opt}(m) = \ell_c^{opt}.$$

Moreover, if $\arg\min_{P\in\mathcal{P}} \rho(P)$ *is a singleton, then also*

$$\lim_{m\to\infty} P^{opt}(m) = \arg\min_{P\in\mathcal{P}} \rho(P).$$

Proof: Consider a sequence $\{\rho(P^{opt}(m))\}_{m=0}^{\infty}$. Let $\bar{P}$ be any matrix minimizing the essential spectral radius inside $\mathcal{P}$. We shall now prove that

$$\lim_{m\to\infty} \rho\left(P^{opt}(m)\right) = \rho(\bar{P}) \tag{8.25}$$

For notational convenience, we denote along the proof $\rho\left(P^{opt}(m)\right)$ by ρ_m and $\rho(\bar{P})$ by $\bar{\rho}$. Assume by contradiction that $\lim_{m\to\infty}\rho_m \neq \bar{\rho}$. This means that there exists $\epsilon > 0$ and a sequence of integers $m_1 < m_2 < m_3 < \ldots$, such that $\rho_{m_i} \geq \bar{\rho} + \epsilon$, $\forall\, i \in \mathbb{N}$ and consequently that $\inf\{\rho_{m_i}\} \geq \bar{\rho} + \epsilon$. This implies, by Lemma 8.14, that there exist $\bar{m}_\epsilon$, depending only on ϵ, such that $J^{opt}(m|P^{opt}(m_i)) > J^{opt}(m|\bar{P})$, $\forall\, i \in \mathbb{N}$ and $\forall m > \bar{m}_\epsilon$. Therefore, if $\bar{i}$ is such that $m_{\bar{i}} > \bar{m}_\epsilon$, we get

$$J^{opt}(m_i|P^{opt}(m_i)) > J^{opt}(m_i|\bar{P}),\ \ \forall\, i \geq \bar{i},$$

contradicting the fact that $P^{opt}(m_i) \in \arg\min_P J^{opt}(m_i|P)$ and thus proving Equation (8.25).

Consider now any sequence of integers $\tilde{m}_1 < \tilde{m}_2 < \tilde{m}_3 < \tilde{m}_4 \ldots$ such that $\lim_{i\to\infty} P^{opt}(\tilde{m}_i) = \tilde{P}$ where $\tilde{P}$ is a suitable matrix inside $\mathcal{P}$. It follows, by the continuity of ρ that $\lim_{i\to\infty}\rho(P^{opt}(\tilde{m}_i)) = \rho(\tilde{P})$. Clearly $\rho(\tilde{P}) = \bar{\rho}$. Suppose now that $\arg\min_{P\in\mathcal{P}}\rho(P)$ is a singleton and call $\bar{P}$ the unique element in this set. It follows that $\tilde{P} = \bar{P}$, thus implying, from the compactness of $\mathcal{P}$, that

$$\lim_{m\to\infty} P^{opt}(m) = \arg\min_{P\in\mathcal{P}} \rho(P).$$

Consider now a sequence $\{\ell^{opt}(m)\}_{m=0}^{\infty}$ and let $\tilde{\rho}$ be such that $\bar{\rho} < \tilde{\rho} < 1$. Let us introduce a matrix $\tilde{P}$ such that $\sigma(\tilde{P}) = \{1, \tilde{\rho}, \ldots, \tilde{\rho}\}$, that is $\tilde{P}$ has $N-1$ eigenvalues equal to $\tilde{\rho}$. It follows that

$$J\left(\tilde{P}, \ell; m\right) = \frac{r\ell^2 + qN}{1-(1-\ell)^2} + (N-1)\frac{r\ell^2\tilde{\rho}^{2m}}{1-(1-\ell)^2\tilde{\rho}^{2m}},$$

and, by recalling the expression of $\frac{\partial J(P,\ell;m)}{\partial \ell}$, that

$$\frac{\partial J\left(\tilde{P}, \ell; m\right)}{\partial \ell} = 2r\ell(N-1)\frac{\tilde{\rho}_m^{2m} - (1-\ell)\tilde{\rho}_m^{4m}}{[1-(1-\ell)^2\tilde{\rho}_m^{2m}]} - 2\frac{Nq(1-\ell) - r\ell^2}{[1-(1-\ell)^2]^2}$$

Let now ϵ be any real such that $0 < \epsilon < \tilde{\rho} - \bar{\rho}$. We have just proved that there exists a positive integer $\bar{m}_\epsilon$ such that $\forall\, m > \bar{m}_\epsilon$ we have that $\rho_m < \tilde{\rho} - \epsilon$. Consider now the function $f : [0,1] \times (0,1) \to \mathbb{R}$ defined as

$$f(x,\alpha) = \frac{x^{2m} - \alpha x^{4m}}{(1 - \alpha^2 x^{2m})^2},$$

where $m \in \mathbb{N}$. By straightforward calculations we have that

$$\frac{\partial f(x,\alpha)}{\partial x} = \frac{2mx^{2m}\left(1 + \alpha^2 x^{2m} - 2\alpha x^{2m}\right)}{x\left(1 - \alpha^2 x^{2m}\right)^3}.$$

It is possible to show that $\frac{\partial f(x,\alpha)}{\partial x} > 0,\ \forall\, x \in (0,1], \forall \alpha \in (0,1)$. Since, by the definition of essential spectral radius, we have that all the eigenvalues of $P^{opt}(m)$ different from 1, are, in absolute value, smaller than ρ_m and therefore of $\tilde{\rho}$, it follows that

$$\frac{\partial J\left(P(m), \ell; m\right)}{\partial \ell} \leq \frac{\partial J\left(\tilde{P}, \ell; m\right)}{\partial \ell}, \qquad \forall\, \ell \in (0,1). \tag{8.26}$$

Let now $\tilde{\ell}^{opt}(m) = \arg\min_{\ell \in (0,1)} J\left(\tilde{P}, \ell; m\right)$. From Theorem 8.13 we have that $\lim_{m\to\infty} \tilde{\ell}^{opt}(m) = \ell_c^{opt}$. On the other hand, by the convexity of J on ℓ and by (8.26) it follows that $\tilde{\ell}^{opt}(m) \leq \ell^{opt}(m)$, $\forall m > \bar{m}_\epsilon$. Therefore $\lim_{m\to\infty} \ell^{opt}(m) = \ell_c^{opt}$. ■

8.5.2 Small measurement noise ($r/q \to 0$)

In this subsection we treat the case in which the variance of the measurement noise is negligible with respect of the variance of the process, that is $r/q \to 0$. This parallels the case analyzed in Section 8.3.3. Also here, as in Section 8.3.3, it is the Frobenius norm of P which plays a crucial role.
Indeed, while in Section 8.3.3 the gain was *assumed* to be large, here ℓ is shown to converge to 1 as r/q goes to zero.

Theorem 8.17 *Let $P^{opt}(r,q)$, $\ell^{opt}(r,q)$ be a solution of Equation (8.24) and let*

$$\bar{P} \in \arg\min_{P \in \mathcal{P}} \|P^m\|_F.$$

Then

$$\lim_{r/q \to 0} \|P^{opt}(r,q)^m\|_F = \|\bar{P}^m\|_F.$$

Moreover

$$\ell^{opt}(r,q) = 1 - \frac{\|\bar{P}\|_F^2}{N}\frac{r}{q} + o\left(r/q\right).$$

In addition, if $\arg\min_{P\in\mathcal{P}} \|P^m\|_F$ *is a singleton also*

$$\lim_{r/q\to 0} P^{opt}(r,q) = \arg\min_{P\in\mathcal{P}} \|P^m\|_F$$

holds.

Proof: Let $J^{opt}(r,q|P)$ and $\ell^{opt}(r,q|P)$ be quantities defined in (8.20) and (8.21). We shall first prove that

$$\lim_{r/q\to 0} \|(P^{opt}(r,q))^m\|_F = \|\bar{P}^m\|_F. \tag{8.27}$$

Assume by contradiction that $\lim_{r/q\to 0} \| \left(P^{opt}(r,q)\right)^m \|_F \neq \|\bar{P}^m\|_F$. This means that there exists $\epsilon > 0$ and a sequence $\frac{r_1}{q_1} > \frac{r_2}{q_2} > \frac{r_3}{q_3} > \ldots$, such that $\| \left(P^{opt}(r_i,q_i)\right)^m \|_F \geq \|\bar{P}^m\|_F + \epsilon$, $\forall i \in \mathbb{N}$. By Lemma 8.15 this implies that there exists δ_ϵ, depending only on ϵ, such that

$$J^{opt}(r,q|P^{opt}(r_i/q_i)) > J^{opt}(r,q|\bar{P}),$$

for each pair r,q such that $\frac{r}{q} < \delta_\epsilon$ and $\forall\, i \in \mathbb{N}$. Therefore, if $\bar{i}$ is such that $\frac{r_{\bar{i}}}{q_{\bar{i}}} < \delta_\epsilon$, we get

$$J^{opt}(r_i,q_i|P^{opt}(r_i,q_i)) > J^{opt}(r_i,q_i|\bar{P}), \quad \forall\, i \geq \bar{i},$$

which yields a contradiction and hence proves (8.27).

Consider now any sequence $\frac{r_{i_1}}{q_{i_1}} > \frac{r_{i_2}}{q_{i_2}} > \frac{r_{i_3}}{q_{i_3}} > \ldots$, such that

$$\lim_{h\to\infty} P^{opt}(r_{i_h},q_{i_h}) = \tilde{P},$$

where $\tilde{P}$ is a suitable matrix inside $\mathcal{P}$. It follows that

$$\lim_{h\to\infty} \| \left(P^{opt}(r_{i_h},q_{i_h})\right)^m \|_F = \|\tilde{P}^m\|_F.$$

Clearly $\|\tilde{P}^m\|_F = \|\bar{P}^m\|_F$. Suppose now that $\arg\min_{P\in\mathcal{P}} \|P^m\|_F$ is a singleton and call $\bar{P}$ the unique element in this set. It follows that $\tilde{P} = \bar{P}$, thus implying, for the compactness of $\mathcal{P}$, that

$$\lim_{r/q\to 0} P^{opt}(r,q) = \bar{P} = \arg\min_{P\in\mathcal{P}} \|P^m\|_F$$

Consider now $\ell^{opt}(r,q)$. Preliminarily we observe that $\lim_{r/q\to 0}\ell_d^{opt}=1$. Since by Theorem 8.13 we have that $\ell^{opt}(r,q|P)\geq \ell_d^{opt}$, it follows that also $\lim_{r/q\to 0}\ell^{opt}(r,q|P)=1$ and $\lim_{r/q\to 0}\ell^{opt}(r,q)=1$. Using the implicit function theorem it follows that $\ell^{opt}(r,q|P)$ is differentiable around $r/q=0$, i.e.

$$\ell^{opt}(r,q|P)=1+\alpha(P)\frac{r}{q}+o\left(\frac{r}{q}\right)$$

where the coefficient $\alpha(P)$ is given by

$$\alpha(P)=-\left[\left(\frac{\partial^2 J}{\partial \ell^2}\right)^{-1}\frac{\partial^2 J}{\partial \ell \partial r/q}\right]_{\ell=1,P,r/q=0}=-\frac{\|P^m\|_F^2}{N}$$

Since $\ell^{opt}(r,q)=\ell^{opt}(r,q|P^{opt}(r,q))$, then $\ell^{opt}(r,q)=1+\alpha(P^{opt}(r,q))r/q+o(r/q)$. Using the fact that $\lim_{r/q\to 0}\|P^{opt}(r,q)\|_F=\|\bar{P}\|_F$, then

$$\ell^{opt}(r,q)=1+\alpha(\bar{P})r/q+o(r/q).$$

■

Remark 8.18 It is interesting to observe that the communication graph $\mathcal{G}$ determines the coefficient of the first order expansion of ℓ in r/q. Note that, in the extreme cases $P^{opt}=I$ (no communication, decentralized estimation) and $P^{opt}=\frac{\mathbb{1}\mathbb{1}^*}{N}$ (communication graph fully connected, centralized estimation) one recovers, respectively, $\ell_d^{opt}=1-\frac{r}{q}+o(\frac{r}{q})$ and $\ell_c^{opt}=1-\frac{1}{N}\frac{r}{q}+o(\frac{r}{q})$, which can also be easily obtained from the standard expressions of ℓ_d^{opt} and ℓ_c^{opt}.

Remark 8.19 Observe that $\|P^m\|_F^2=1+\mu\rho^{2m}(P)+o(\rho^{2m}(P))$, where μ is the algebraic multiplicity[5] of the second largest eigenvalue. Therefore, for large m, minimizing the Frobenius norm of P^m is equivalent to minimizing the essential spectral radius of P (up to $o(\rho^{2m}(P))$ terms).

Remark 8.20 It is interesting to observe that the communication graph $\mathcal{G}$ determines the coefficient of the first order expansion of ℓ in r/q. Note that, in the extreme cases $P^{opt}=I$ (no communication, decentralized estimation) and $P^{opt}=\frac{\mathbb{1}\mathbb{1}^*}{N}$ (communication graph fully connected, centralized estimation) one recovers, respectively, $\ell_d^{opt}=1-\frac{r}{q}+o(\frac{r}{q})$ and $\ell_c^{opt}=1-\frac{1}{N}\frac{r}{q}+o(\frac{r}{q})$, which can also be easily obtained from the standard expressions of ℓ_d^{opt} and ℓ_c^{opt} .

[5]We assume here $\rho(P)<1$.

8.5.3 High measurement noise ($q/r \to 0$)

Similarly to the previous section, we now consider the other limiting case for $q/r \approx 0$. The following result holds.

Theorem 8.21 *Let $P^{opt}(r,q)$, $\ell^{opt}(r,q)$ be defined as above and denote with $p(P)$ the number of eigenvalues of P on the unit circle. Then*

$$\lim_{q/r\to 0} p(P^{opt}(r,q)) = \min_{P\in\mathcal{P}} p(P) =: p^{opt}.$$

Moreover

$$\ell^{opt}(r,q) = \sqrt{\frac{N}{p^{opt}}}\sqrt{\frac{q}{r}} + o\left(\sqrt{q/r}\right).$$

Proof: We start by observing that $\lim_{q/r\to 0} \ell_d^{opt} = 0$ and $\lim_{q/r\to 0} \ell_c^{opt} = 0$. More precisely it is possible to see $\ell_d^{opt} = \sqrt{\frac{q}{r}} + o\left(\frac{q}{r}\right)$ and $\ell_c^{opt} = \sqrt{N}\sqrt{\frac{q}{r}} + o\left(\frac{q}{r}\right)$. From the previous observation and from Theorem 8.13 it follows that

$$\lim_{q/r\to 0} \ell^{opt}(r,q|P) = 0. \tag{8.28}$$

Assume now by contradiction that $\lim_{q/r\to 0} p\left(P^{opt}(r,q)\right) \neq p^{opt}$. This means that there exists a sequence $\frac{q_1}{r_1} > \frac{q_2}{r_2} > \frac{q_3}{r_3} > \dots$, such that $p\left(P^{opt}(r_i,q_i)\right) \geq p^{opt} + 1,\ \forall\, i \in \mathbb{N}$. Let us rewrite J in the following way

$$\begin{aligned} J(P,\ell;r,q) &= \frac{qN + r\ell^2 p(P)}{1-(1-\ell)^2} + r\ell^2 \sum_{i:|\lambda_i|\neq 1} \frac{|\lambda_i|^{2m}}{1-(1-\ell)^2|\lambda_i|^{2m}} \\ &= \frac{qN + r\ell^2 p(P)}{1-(1-\ell)^2} + \frac{r\ell^2}{1-(1-\ell)^2} f(\ell) \end{aligned}$$

where

$$f(\ell) = \left(1-(1-\ell)^2\right) \sum_{i:|\lambda_i|\neq 1} \frac{|\lambda_i|^{2m}}{1-(1-\ell)^2|\lambda_i|^{2m}}.$$

Note that $\lim_{\ell\to 0} f(\ell) = 0$. Hence

$$J(P,\ell;r,q) = \frac{qN + r\ell^2 p(P)}{1-(1-\ell)^2} + o\left(\frac{r\ell^2}{1-(1-\ell)^2}\right) \tag{8.29}$$

Let now $\bar{P}$ be such that $p(\bar{P}) = p^{opt}$. From (8.29) it follows that

$$J(P,\ell;r,q) - J(\bar{P},\ell;r,q) = \ (p(P) - p^{opt})\frac{r\ell^2}{1-(1-\ell)^2} + o\left(\frac{r\ell^2}{1-(1-\ell)^2}\right).$$

Equation (8.28) implies that also $\lim_{i\to\infty}\ell^{opt}(r_i,q_i|P)=0$ so that for i large enough

$$J(P^{opt}(q_i,r_i),\ell^{opt}(q_i,r_i);r_i,q_i)>J(\bar{P},\ell^{opt}(q_i/r_i);r_i,q_i)$$

which yields a contradiction.
Consider now $\ell^{opt}(r,q)$. Let us first compute the partial derivative

$$\frac{\partial J(P^{opt}(r,q),\ell;r,q)}{\partial\ell}=\frac{r\left(2p(P^{opt}(r,q))\ell^2-2\gamma^2N+2\gamma^2N\ell\right)}{(1-(1-\ell)^2)}+$$
$$+\frac{r\left(+2\ell^2f(\ell)+2\ell^3\dot{f}(\ell)-\ell^4\dot{f}(\ell)\right)}{(1-(1-\ell)^2)}$$

where $\gamma=\sqrt{\frac{q}{r}}$. Let now $F(\gamma,\ell)=2p(P^{opt}(r,q))\ell^2-2\gamma^2N+2\gamma^2Nl+2\ell^2f(\ell)+2\ell^3\dot{f}(\ell)-\ell^4\dot{f}(\ell)$ and consider the equation $F(\gamma,\ell)=0$. We adopt an argument similar to the proof of the implicit function theorem. By applying the Taylor's expansion around $(0,0)$ we get

$$F(\gamma,\ell)=-4N\gamma^2+4p(P^{opt}(r,q))\ell^2+o\left(\|\gamma^2\|+\|\ell^2\|\right).$$

and equating this expression to 0 we obtain that $\ell^{opt}(r,q)=\ell^{opt}(r,q|P^{opt}(r,q))$ satisfies

$$\ell^{opt}(r,q)=\gamma\sqrt{\frac{N}{p(P^{opt}(r,q))}}+o(\gamma)$$
$$=\sqrt{\frac{N}{p(P^{opt}(r,q))}}\sqrt{\frac{q}{r}}+o\left(\sqrt{\frac{q}{r}}\right)$$

Therefore, using an argument similar to the one adopted in the proof of Theorem 8.17, it follows that

$$\ell^{opt}(r,q)=\sqrt{\frac{N}{p^{opt}}}\sqrt{\frac{q}{r}}+o\left(\sqrt{\frac{q}{r}}\right),$$

which concludes the proof. ■

Remark 8.22 Also in this case the constant in the first order expansion of the optimal gain as a function of q/r depends on the communication graph; the extreme cases are, respectively, $\ell_c^{opt}(r,q)=\sqrt{N}\sqrt{q/r}+o(\sqrt{q/r})$ and $\ell_d^{opt}(r,q)=\sqrt{q/r}+o(\sqrt{q/r})$. In fact, if sensors cannot communicate, then necessarily $P^{opt}=I$, therefore $\ell^{opt}(r,q)=\ell_d^{opt}(r,q)$, while if the communication graph is fully connected, then $P^{opt}=\frac{1}{N}\mathbb{1}\mathbb{1}^*$, therefore $\ell^{opt}(r,q)=\ell_c^{opt}(q/r)$.

The previous theorem shows also that for $q << r$ the optimizing P, while being consistent with the communication graph, has to minimize the number of unitary eigenvalues.

8.6 Illustrative examples

In this section we provide some examples illustrating the approach proposed in this paper. In particular in Example 8.23 we solve analytically the minimization problem formulated in Section 8.3. In Example 8.24 and in Example 8.25 we provide a numerical comparison between the approach presented here and the method proposed in [5]. In Example 8.27, inspired by [90], we propose a new scheme for updating the estimates which seems to be very promising in order to improve the performance. In particular we allow the sensors to keep in memory also the previously received estimates, and not only the current one, and to use them to build an updated estimate. More precisely the estimate $\hat{x}(t+(h+1)\delta)$ will be a suitable weighted combination of the current estimate $\hat{x}(t+h\delta)$ and of the previous estimate $\hat{x}(t+(h-1)\delta)$. It has been shown by Muthukrihnan et al. [90] in the context of load balancing, and by Cao et al.[28] in the context of gossip algorithms, that the use of memory permits to speed up convergence and to improve performance. We show here, by means of simulative results, that for the set of matrices considered in Example 8.23 and 8.24 the presence of one level of memory permits to reach better performance in terms of the cost function J.

Example 8.23 Assume that $\mathcal{Q}$ is the set of the circulant stochastic matrices of the form

$$P_k = \begin{bmatrix} 1-2k & k & 0 & 0 \cdots 0 & k \\ k & 1-2k & k & 0 \cdots 0 & 0 \\ 0 & k & 1-2k & k \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \cdots \vdots & \vdots \\ k & 0 & 0 & 0 \cdots k & 1-2k \end{bmatrix} \tag{8.30}$$

where k is a real in $\left[0, \frac{1}{2}\right]$ and assume that $m = 1$. We want to solve the minimization problem considered in Section 8.3, i.e.

$$P^{opt} = \arg\min_{P \in \mathcal{P}} J = \arg\min_{P \in \mathcal{P}} \sum_{i=1}^{N-1} \frac{|\lambda_i|^{2m}}{1-(1-\ell)^2|\lambda_i|^{2m}}. \tag{8.31}$$

Unfortunately, even if $\mathcal{P}$ exhibits a particular structure, it is not possible to give, in general, a closed form expression for the optimizing matrix P^{opt},

which has to be found using numerical search techniques.
However, similarly to Example 8.5, a more detailed study of (8.31) can be provided when the number of sensors N goes to infinity. To this purpose it is convenient to consider the normalized cost:

$$\bar{J}_N := \frac{1}{N}\sum_{i=0}^{N-1}\frac{|\lambda_i|^2}{1-(1-\ell)^2|\lambda_i|^2}$$

and the function $f : \mathbb{C} \to \mathbb{C}$

$$f(z) = 1 - 2k + kz + kz^{-1}.$$

The eigenvalues of $\mathcal{P}_k$ can be expressed as $\lambda_h = 1 - 2k + ke^{j\frac{2\pi}{N}h} + ke^{-j\frac{2\pi}{N}h}$ (see e.g. [46]). Note that $\lambda_h = f\Big(e^{j\frac{2\pi}{N}h}\Big)$. As the number of agents grows, the normalized cost J_N converges to:

$$\lim_{N\to\infty} J_N = \bar{J}_\infty = \frac{1}{2\pi}\int_0^{2\pi}\frac{|f(e^{j\phi})|^2}{1-(1-\ell)^2|f(e^{j\phi})|^2}d\phi = \frac{1}{2\pi j}\oint_\gamma \frac{z^{-1}f^2(z)}{1-(1-\ell)^2 f^2(z)}dz$$

where γ is the unit circle. It is straightforward to see that the poles of $\frac{z^{-1}f^2(z)}{1-(1-\ell)^2f^2(z)}$ inside γ are

$$z_1 = \frac{\ell + 2(1-\ell)k - \sqrt{\ell^2 + 4\ell(1-\ell)k}}{2(1-\ell)k},$$

$$z_2 = \frac{\ell - 2 - 2(1-\ell)k + \sqrt{(2-\ell)^2 + 4(1-\ell)(2-\ell)k}}{2(1-\ell)k}$$

and

$$z_3 = 0.$$

The integral can be computed explicitly using the residue theorem yielding:

$$\bar{J}_\infty = -\frac{1}{(1-\ell)^2} + \frac{1}{2(1-\ell)^2}\left(\frac{1}{\sqrt{\ell^2 + 4\ell(1-\ell)k}} - \frac{1}{\sqrt{(2-\ell)^2 - 4(1-\ell)(2-\ell)k}}\right)$$

By equating the first derivative of $\bar{J}_\infty$ to zero, we get that P^{opt} tends to the matrix $P_{k^{opt}}$ for $N \to \infty$, where

$$k^{opt} = \frac{2 - 2\ell + \sqrt[3]{(2-\ell)\ell^2} - \sqrt[3]{\ell(2-\ell)^2}}{4(1-\ell)}.$$

Example 8.24 In [5] the authors analyze a general MIMO scenario where the gain $\ell = \ell(t)$ (K in their terminology) and the consensus matrix $P = P(t)$ are time varying matrices which are chosen recursively at each time step. In order to compare the results in [5] with our approach we assume that the averaging matrix W in [5] corresponds to performing m consensus iterations using the matrix P, i.e. $W = P^m$. In [5] the gain ℓ is chosen to minimize the estimation error covariance of the local estimators (i.e. in a decentralized fashion) and it is different for each sensor. Moreover the consensus matrix P is chosen so that the estimation error covariance of the local estimators is minimized after consensus (weighted averaging in [5])[6].

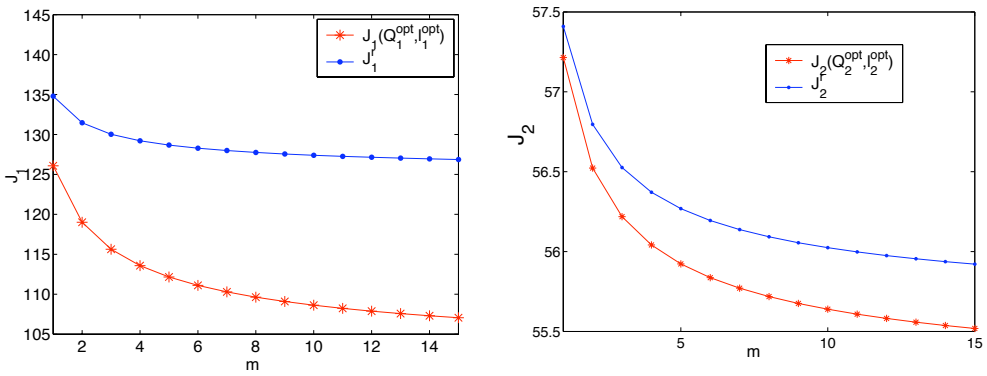

Figure 8.1: Comparison between $J_1(P_1^{opt}(m), \ell_1^{opt}(m), m)$,and $J_1^r(m)$ ***(left)***. Comparison between $J_2(P_2^{opt}(m), \ell_2^{opt}(m), m)$, and $J_2^r(m)$ ***(right)***.

In the simulation reported in this example, we assume that $N = 100$, $q = 1$ and $r = 1$. Moreover we assume that $\mathcal{P}$ is the same set of circulant matrices defined in the previous example. We consider the minimization of both J_1 and J_2, with J_1, J_2 defined as in the Section II. We use the following notational conventions: $P_1^{opt}(m)$, $\ell_1^{opt}(m)$ and $P_2^{opt}(m)$, $\ell_2^{opt}(m)$ are the optimal consensus matrices and the optimal gains respectively for J_1 and J_2 obtained by solving numerically the problem formulated at the end of Section II, given by:

$$\begin{aligned}\left(P_1^{opt}(m),\, \ell_1^{opt}(m)\right) &\in \arg\min_{\ell\in(0,1),\, P\in\mathcal{P}} J_1(P, \ell; m, r, q)\\ \left(P_2^{opt}(m),\, \ell_2^{opt}(m)\right) &\in \arg\min_{\ell\in(0,1),\, P\in\mathcal{P}} J_2(P, \ell; m, r, q)\end{aligned}$$

As mentioned earlier, in [5] the optimal gain and the optimal consensus matrix are found *recursively* at each time step t. The fact that the consensus matrix is of the form (8.30) implies that the gain is the same for all the

[6]Technically, the approach proposed in [5] was applied only for the case $m = 1$.

sensors: we denote it by $\ell^r(t,m)$. Moreover $P^r(t,m)$ represents the optimal consensus matrix (the superscript "r" means *recursively*). The asymptotic values (in t) of $\ell^r(t,m)$ and $P^r(t,m)$ are denoted respectively by $\ell^r(m)$ and $P^r(m)$, i.e. $\lim_{t\to\infty} \ell^r(t,m) = \ell^r(m)$ and $\lim_{t\to\infty} P^r(t,m) = P^r(m)$. Similarly, we indicate by $J_1^r(m)$ and $J_2^r(m)$ the asymptotic cost values[7] to which J_1 and J_2 converge using the method proposed in [5]. Finally, since the set $\mathcal{P}$ is parameterized by k, we shall identify a matrix P_k with the corresponding value of the parameter k. Hence, we use $k_1^{opt}(m)$, $k_2^{opt}(m)$ and $k^r(m)$ in lieu of $P_1^{opt}(m)$, $P_2^{opt}(m)$ and $P^r(m)$. Simulations for m ranging in the interval $[1,15]$ are shown in Figures 8.1 and 8.2.

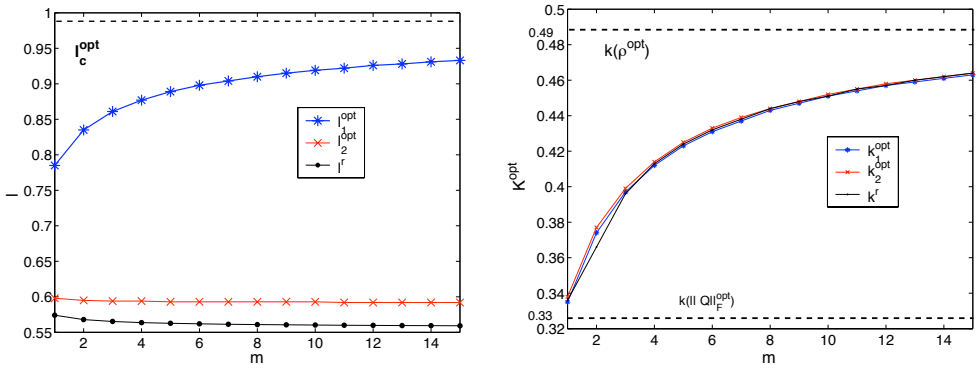

Figure 8.2: Optimal gains ***(left)*** and consensus matrices ***(right)*** as a function of number of exchanged messages m. Optimal consensus matrices P are parameterized by k. In the right panel we also report the values $k(\rho^{opt})$, which minimizes the spectral radius $\rho(P)$ and $k(||P||_F^{opt}$ which minimizes the Frobenius norm $||P||_F$.

From left panel of Figure 8.2 we can see that the iterative (local) optimization proposed in [5] converges to values of the parameters $\ell^r(m)$ which are different from the optimal values obtained minimizing the asymptotic cost proposed in this paper. Indeed, the recursive approach seems to give a worse performance than the approach proposed in this paper both in terms of the performance costs J_1 and J_2, as shown in Figure8.1. Moreover, for large m, the gain $\ell_1^{opt}(m)$ converges to the optimal centralized gain ℓ_c^{opt}, as shown in Section 8.4.

Note also from right panel of Figure 8.2 that the optimal consensus parameter k of the matrix P_k for small m is close to the value that minimizes the Frobenius norm $||P||_F$, while for large m it converges to the value that minimizes the spectral radius $\rho(P)$, as shown in Section 8.3.

[7]There is no proof of convergence in [5]; however this is observed experimentally.

Example 8.25 In this simulation we consider a strongly connected random geometric graph generated by choosing N points at random in the unit square, and then placing an edge between each pair of points at distance less than 0.3. We assume that $N = 30$, $q = 1$ and $r = 1$. It is worth noticing that in this case differently from the previous one, the gains, calculated by the method proposed in [5] recursively at each instant time t , are in general different for each sensor. Hence we report here only the comparison between the cost functions. We run simulations for m ranging in the interval $[1, 10]$. The results obtained are depicted in Figure 8.3. The notation is the same used previously. Note that $\ell_1^{opt}(m), P_1^{opt}(m)$ and $\ell_2^{opt}(m), P_2^{opt}(m)$ yield again

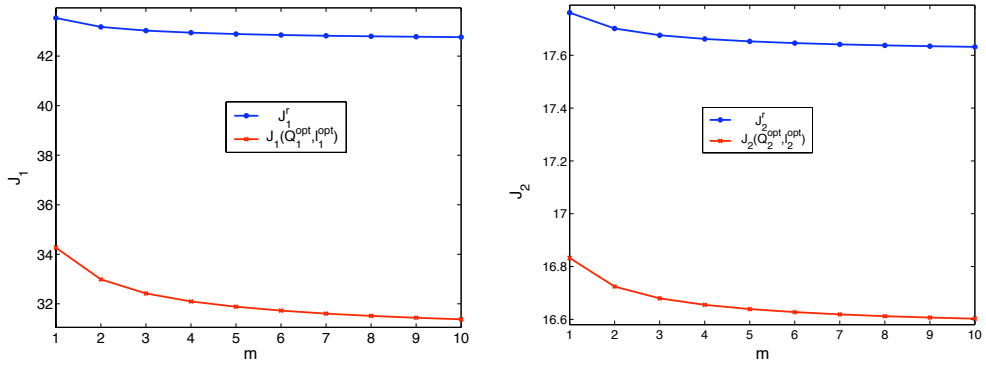

Figure 8.3: Comparison between $J_1(P_1^{opt}(m), \ell_1^{opt}(m), m)$, $J_1(P_2^{opt}(m), \ell_2^{opt}(m), m)$ and $J_1^r(m)$ ***(left)***. Comparison between $J_2(P_2^{opt}(m), \ell_2^{opt}(m), m)$, $J_2(P_1^{opt}(m), \ell_1^{opt}(m), m)$ and $J_2^r(m)$ ***(right)***.

values of J_1 and J_2 which are better respectively than J_1^r and J_2^r.

Remark 8.26 It is worth remarking that the optimization strategy proposed in [5] gives worse performance in terms of the asymptotic values for both costs J_1 and J_2 in both the previous examples. This is somewhat to be expected since the former approach recursively optimize the cost at the next time step, not the steady state cost. This recursive strategy gives the optimal steady state performance only for the centralized scenario, i.e. when the communication graph is fully connected, as well known from any standard textbook on optimal estimation and Kalman filtering [36]. When the graph is not fully connected, this strategy is not guaranteed to converge to the optimal value, as indeed shown in the previous numerical examples.

Example 8.27 Inspired by the work of Muthukrihnan et al. [90], in this example we propose the following scheme of updating the estimate $\hat{x}(k+h\delta|k)$

which adds a memory buffer,

$$\hat{x}(k+(h+1)\delta|k) = P\hat{x}(k+h\delta|k) \qquad \text{if } h=0,$$

and

$$\hat{x}(k+(h+1)\delta|k) = \nu P\hat{x}(k+h\delta|k)+(1-\nu)\hat{x}(k+(h-1)\delta|k) \qquad \text{if } h=1,\dots,m-1 \tag{8.32}$$

where ν is a weighting parameter. It has been proved in [90], that, when we are dealing with a consensus algorithm, if $\nu \notin [0,2]$ then (8.32) does not yield the convergence. Let $\tilde{x}\,(k+h\delta|k) = x(k+h\delta)\mathbb{1} - \hat{x}(k+h\delta|k)$ denote again the estimation error and let us introduce, for $1 \leq h \leq m$, the following quantities

$$z(k+h\delta|k) = \begin{bmatrix} \tilde{x}(k+h\delta|k) \\ \tilde{x}(k+(h-1)\delta|k) \end{bmatrix}$$

and

$$\Sigma(k+h\delta|k) = \mathbb{E}\left[z(k+h\delta|k)z^*(k+h\delta|k)\right] = \begin{bmatrix} \Sigma_{11}(k+h\delta|k) & \Sigma_{12}(k+h\delta|k) \\ \Sigma_{21}(k+h\delta|k) & \Sigma_{22}(k+h\delta|k) \end{bmatrix}.$$

where

$$\begin{aligned}
\Sigma_{11}(k+h\delta|k) &= \mathbb{E}[\tilde{x}\,(k+h\delta|k)\,\tilde{x}^*\,(k+h\delta|k)] \\
\Sigma_{12}(k+h\delta|k) &= \mathbb{E}[\tilde{x}\,(k+h\delta|k)\,\tilde{x}^*\,(k+(h-1)\delta|k)] \\
\Sigma_{21}(k+h\delta|k) &= \mathbb{E}[\tilde{x}\,(k+(h-1)\delta|k)\,\tilde{x}^*\,(k+h\delta|k)] \\
\Sigma_{22}(k+h\delta|k) &= \mathbb{E}[\tilde{x}\,(k+(h-1)\delta|k)\,\tilde{x}^*\,(k+(h-1)\delta|k)]
\end{aligned}$$

Note that $\Sigma_{12}(k+h\delta|k) = \Sigma_{21}^*(k+h\delta|k)$. It is possible to see that the above quantities satisfy the following recursive equations

$$\begin{aligned}
\Sigma_{11}\,(k+(h+1)\delta) &= \nu^2 P\Sigma_{11}\,(k+h\delta)\,P^* + 2\nu(1-\nu)\frac{q}{m}\mathbb{1}\mathbb{1}^* + \frac{q}{m}\mathbb{1}\mathbb{1}^* + \\
&\quad + \nu(1-\nu)\left[P\Sigma_{12}\,(k+h\delta) + \Sigma_{21}\,(k+h\delta)\,P^*\right] + \\
&\quad + (1-\nu)^2\Sigma_{22}\,(k+h\delta) + (1-\nu)^2\frac{q}{m}\mathbb{1}\mathbb{1}^* \\
\Sigma_{12}\,(k+(h+1)\delta) &= \nu P\Sigma_{11}\,(k+h\delta) + (1-\nu)\Sigma_{21}\,(k+h\delta) + (1-\nu)\frac{q}{m}\mathbb{1}\mathbb{1}^* \\
\Sigma_{22}(k+(h+1)\delta|k) &= \Sigma_{11}\,(k+h\delta)
\end{aligned}$$

In our simulation we assume that $N = 50$, $m = 5$, $q = 1$, $r = 1$ and that $P = P_{\frac{1}{3}}$ where $P_{\frac{1}{3}}$ is as the matrix introduced in Example 8.23. Moreover $\ell = \ell^{opt}$ where

$$\ell^{opt} = \underset{\ell\,\in\,(0,1)}{\arg\min}\; J\left(P_{\frac{1}{3}}, \ell; 5, 1, 1\right) \approx 0.879.$$

Section 8.3	undirected graph	$m \to \infty$	$\ell \to 1$
Fixed ℓ	$P^{opt} \in$ symmetric Section 8.3.1	$P^{opt} \in \arg\min_{P \in \mathcal{P}} \rho(P)$ Section 8.3.2	$P^{opt} \in \arg\min_{P \in \mathcal{P}} \|P^m\|_F$ Section 8.3.3
Section 8.4	$m = 0$	$0 < m < \infty$	$m \to +\infty$ P primitive
Fixed P	$\ell^{opt}(P,0) = \ell_d^{opt}$ Theorem 8.13	$\ell_d^{opt} < \ell^{opt}(P,m) \leq \ell^{opt}(P,m+1) < \ell_c^{opt}$ Theorem 8.13	$\ell^{opt}(P,\infty) = \ell_c^{opt}$ Theorem 8.13
Section 8.5	$m \to \infty$	$r/q \to 0$	$r/q \to \infty$
Optimal ℓ and P	$P^{opt} \in \arg\min_{P \in \mathcal{P}} \rho(P)$ $\ell^{opt} \to \ell_c^{opt}$ Section 8.5.1	$P^{opt} \in \arg\min_{P \in \mathcal{P}} \|P^m\|_F$, $\ell^{opt} = 1 - \frac{\|\bar{P}^m\|_F^2}{N} \frac{r}{q} + o\left(\frac{r}{q}\right)$ Section 8.5.2	$P^{opt} \in \arg\min_{P \in \mathcal{P}} p(P)$, $\ell^{opt} = \sqrt{\frac{N}{p^{opt}}} \sqrt{\frac{q}{r}} + o\left(\sqrt{\frac{q}{r}}\right)$ Section 8.5.3

Table 8.1: Summary of the results

Clearly, if $\nu = 1$, then $\lim_{k \to +\infty} \operatorname{tr}\left(\Sigma_{11}(k+1|k)\right) = J\left(P_{\frac{1}{3}}, \ell^{opt}; 5, 1, 1\right)$, which is the cost function introduced in Equation (8.12) for which no memory is added. In Figure 8.4 we depict the behavior of $\lim_{k \to +\infty} \operatorname{tr}\left(\Sigma_{11}(k+1|k)\right)$ for ν ranging in $[0, 2]$. It is remarkable to note that the minimum is reached when $\nu \approx 1.44$, meaning that the presence of memory can improve the performance.

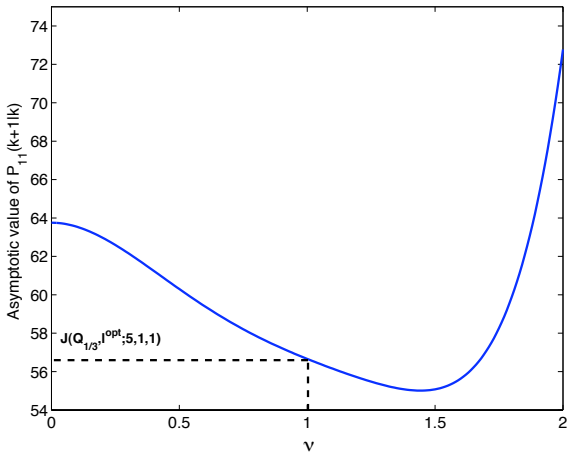

Figure 8.4: Steady state error $\operatorname{tr}\left(\Sigma_{11}(k+1|k)\right)$ of distributed optimal estimation with memory as a function of weighting parameter ν. Figure also indicates optimal cost in the absence of memory which corresponds to $\nu = 1$.

8.7 Conclusions

In this chapter we have studied a prototypical problem of distributed estimation for sensor networks; the state of a scalar linear system is estimated via a two stage procedure which consists in (i) a standard (and decentralized) Kalman-like update and (ii) information propagation using consensus strategies. To this purpose two design parameters, i.e. the Kalman gain ℓ and the consensus matrix P have to be designed. This choice is made by optimizing the steady state prediction (or estimation) error. We have discussed, under specific circumstances, the behavior of the "optimal" parameters. This is summarized in table of Figure 8.6.

Although these results have been obtained for simple scenarios where the state is scalar and all sensors are equal, they provide useful guidelines for choosing the local filter gain ℓ and the consensus matrix P also for more general scenarios. In particular, we showed that the common practise of finding algorithms that minimize the spectral radius for the consensus matrix P is not necessarily the optimal strategy in the context of optimal estimation of time-varying signals. In fact, we showed that depending on some specific regimes, it is more convenient to optimize the Frobenius norm $||P^m||_F$ or the sum of the unitary eigenvalues $p(P)$. Moreover, as discussed in Section V, we showed that the joint optimization of P and ℓ is not convex even in our simple setup. We also compared our approach with the recursive optimization proposed by Alriksson et al. [5], showing also that their strategy based on minimization of the estimation error at the next time step, fails to minimize the steady state cost (see Figures 8.1 and 8.3).

Many research avenues still deserve to be explored. The most promising one is the use of memory to improve performance. As shown in Example 8.27 in the previous section, memory can considerably improve estimation performance. However, the impact of memory length on performance and optimization algorithms for the weighting parameters are still open problems. Another important aspect is the extension of the results in this paper to the multivariable case where the gains for the sensors can be different. Finally, it is fundamental to find provably optimal strategies to simultaneously optimize the consensus matrix P and the update gains ℓ, since the steady state cost function is non-convex in these parameters.

Chapter 9

Conclusions

The *consensus problem* and the more specific *average consensus problem* represent the first step toward a comprehensive understanding of how, in complex networked systems, the dynamics of the individual systems can give rise to a group behavior. This is mainly motivated by two reasons. The first is that these problems are the simplest problems we can formulate in the new emerging area of the cooperative control, the second is related to the wide range of applications of the consensus algorithms in information processing in sensor networks, in load balancing and in Multi-Vehicle Cooperative Control.

In this dissertation we focused on the average consensus problem by providing some theoretical developments that, now, we briefly summarize.

In Chapter 3, we provided a mathematical characterization of the intuitive idea that the amount of information exchanged by the systems in the consensus algorithm has to influence the rate of convergence towards the asymptotic agreement. We did this for a class of graphs exhibiting particular symmetries: the Cayley graphs defined on Abelian groups. We showed that, if we impose symmetries in the communication network, and thus also in the consensus matrix, and we keep the number of incoming arcs in each node bounded, the convergence rate degrades as the number of systems increases.

In Chapter 4 we focused on *randomized consensus algorithms*. We reviewed the concepts of *probabilistic consensus* and *average probabilistic consensus*. In the first part of the chapter we introduced two random strategies, that illustrate a remarkable property of the randomized consensus algorithms, i.e., they allow to achieve better performance than deterministic ones with comparable complexity. In the second part of the chapter we considered the symmetric gossip algorithm, which is particularly suitable to model the

communication network in applications to sensors, peer-to-peer and *ad hoc* networks. We provided an interesting characterization of this algorithm when considering the Cayley graphs.

In Chapters 5, 6, 7, we analyzed the more realistic and practical situation in which the communication network between the systems is constituted of only rate-constrained digital links. We assumed that the nodes quantize their information before transmitting it. In particular, we considered two types of quantizers, well-known in the literature: the *deterministic* uniform quantizer and the *probabilistic* uniform quantizer.
In Chapter 5 we focused on the time-invariant situation by providing a simple and effective adaptation of the standard average consensus algorithm which does not converge to an asymptotic agreement but is able to preserve the average of states and to drive the systems reasonably near to the consensus. We studied this scheme by means of a worst-case model and a probabilistic model showing favorable convergence properties and providing performance bounds for the limit points of the iterates generated.
In Chapter 6 we analyzed the effects of the quantization on the symmetric gossip algorithm. In order to overcome these effects, we introduced two updating rule, the *globally quantized* strategy and the *partially quantized* strategy. In the former the nodes use only quantized information to update their state, instead in the latter they have access also to exact information regarding their own state. We saw that the partially quantized strategy, with both the quantizers, deterministic and probabilistic, do not reach the consensus in general, but maintain the average of the state at each iteration and drive all the states very close to the average of the initial condition. On the other hand, we showed that the globally strategy leads almost surely to a consensus which, however, do not coincide with the average of the initial condition. We provided some simulations characterizing the distance between the consensus point and the initial average. While using the deterministic quantizer this distance turns out to be not negligible, with the probabilistic quantizer the consensus is surprisingly reached very close to the average of the initial condition.
In Chapter 7 we introduced coding/decoding schemes that leads to a quantized strategy that permits both to maintain the initial average and to reach it asymptotically. More precisely we adapted coding/decoding stchemes, that have been proposed for centralized quantized control problems, to the distributed consensus problem. In particular, we presented two coding/decoding strategies, one based on the exchange of logarithmically quantized information, the other on a zoom in- zoom out strategy which involves the use of uniform quantizers. We provided analytical and numerical results illustrat-

ing the convergence properties of these strategies. In particular we showed that the convergence factors depend smoothly on the accuracy parameter of the quantizers used and that, remarkably, that the critical quantizer accuracy sufficient to guarantee convergence is independent from the network dimension.

Finally in Chapter 8 we considered a possible application of the consensus ideas to a prototypical problem of distributed estimation for sensor networks. More precisely we analyzed a two-stage procedure, estimating the state of a scalar linear system, which consists in (i) a standard (and decentralized) Kalman-like update and (ii) information propagation using consensus strategies. In this procedure two design parameters, i.e. the Kalman gain ℓ and the consensus matrix P have to be designed. This choice is made by optimizing the steady state prediction (or estimation) error. We discussed, under specific circumstances, the behavior of the "optimal" parameters.

Many research avenues still deserve to be explored. In particular, regarding the randomized consensus algorithms,

- to provide an analysis of randomized consensus algorithms for wide classes of graphs;
- to provide a detailed studied of the rate of convergence for the symmetric gossip algorithms for the Cayley graphs and for the random geometric graphs;

regarding the quantized consensus,

- to explain why and in which sense the probabilistic analysis seems to capture the main features of the quantized strategies also when the information is quantized by means of deterministic quantizers;
- to evaluate the speed of convergence of the *partially quantized* strategy and of the *globally quantized* strategy;
- to carry out a deeper analysis on the *globally* quantized strategy using probabilistic quantizers; in particular, to estimate, in some probabilistic way, the distance of the consensus point to which this strategy leads the systems from the average of the initial condition;
- to extend the analysis, carried out in Chapter 7, to the case of a general quasi-doubly stochastic matrix P;

- to analyze, from a theoretical point of view, the zoom-in/ zoom-out strategy;
- to look for encoding and decoding methods which are able to solve the average consensus problem also with noisy digital channel;

finally, regarding the distributed Kalman filtering,

- to explore the use of memory to improve the performance of the two-step procedure introduced in Chapter 8;
- to find provably optimal strategies to simultaneously optimize the consensus matrix P and the update gains ℓ, since the state cost function is non-convex in these parameter.

Appendix A

Harmonic analysis on finite group

A.1 Introduction

The main objective of this appendix is that of reviewing some concepts on harmonic analysis on finite groups. The presentation is mainly based on [14].

A.2 Finite Abelian groups

We start by recalling the definition of a finite Abelian group.

Definition A.1 *A finite Abelian group G of order N, is a set G of cardinality $|G| = N$, closed under a binary operation $+$, such that the following axioms are satisfied*[1]

(i) (Associativity) For all $g, h, \ell \in G$, we have that

$$(g + h) + \ell = g + (h + \ell).$$

(ii) (Commutativity) The binary operation $+$ is commutative, i.e. for any $g, h \in G$ we have that

$$g + h = h + g.$$

[1] In the following we will use $|.|$ to denote both the cardinality of a set and the absolute value operator.

(iii) (Neutral element) There exists a neutral element $e \in G$ *such that for all* $g \in G$

$$g + e = g.$$

In the following, by convention, we will indicate with 0 *such element.*

(iv) (Inverse element) Corresponding to each $g \in G$ *there is an element* $g^{'} \in G$ *such that*

$$g + g^{'} = 0.$$

In the following we will indicate such element with $-g$.

From now on when we refer to a group we will implicitly consider finite Abelian groups. We will assume the operation to be addition if not otherwise stated.

Example A.2 *Let us consider* $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$. *If we consider as binary operation the addition modulo* N, *all the previous axioms are satisfied, and thus it represents a finite Abelian group.*

Let G and H be two Abelian groups. The two groups, in general, need not to be finite. An *h*omomorphism is a map $\phi : G \to H$ satisfying

$$\phi(g + h) = \phi(g) + \phi(h)$$

with $g \in G$ and $h \in H$. An homomorphism that is bijective is called *i*somorphism. Two Abelian groups are isomorphic if there is an homomorphism between them. Isomorphic groups are regarded as "equal" from a structural or group-theoretic point of view, even though their elements might be quite different kinds of objects. In the following we will write $G \cong H$ to denote that G is isomorphic to H.

Example A.3 *Let us consider the finite additive group* $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ *and the binary group* $B = \{00, 01, 10, 11\}$ *with operation the* 2*-bits binary sum. If we consider the map*

$$\phi : G \to H : g \mapsto \phi(g) = binary_2(g)$$

where $binary_2(g)$ *is the binary representation in two bits of the integer* g, *then we see immediately that it is an isomorphism. Thus* $G \cong H$.

Given a finite group G, a subgroup H of G is a subset of G that also forms a group under the same binary operation $+$.

Example A.4 *Let us consider $\mathbb{Z}_8$, and let $H = \{0, 4\} \subset \mathbb{Z}_8$. It is easy to see that H is also a group with respect to the addition modulo eight.*

Given a finite group G let us consider a subset S of a group. Then $< S >$, the subgroup generated by S, is the smallest subgroup of G containing every element of S. Equivalently, $< S >$ is the subgroup of all elements of G that can be expressed as the finite product of elements in S and their inverses.

Example A.5 *Let us consider the group*

$$\mathbb{Z}_3 \times \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}.$$

Let us consider $S = \{(0,0)\}$. It is clear that $< S >= S$ is a subgroup of G. Let us consider $S = \{(0,0), (0,1)\}$ then we see that $S \subset< S >= \{(0,0), (0,1), (0,2)\}$ is the smallest subgroup of G containing S.

If $G =< S >$, then we say S generates G; and the elements in S are called generators or group generators.

Example A.6 *Let us consider again group $\mathbb{Z}_3 \times \mathbb{Z}_3$. Then we see that if we choose $S = \{(0,0), (0,1), (1,0)\}$ then $< S >= G$.*

Definition A.7 *A finite Abelian group G is called a cyclic group if its elements are all of the form kg for $k \in \mathbb{Z}$ for some fixed $g \in G$. The element g is called the generator of the group G, and we will write that $< g >= G$.*

It turns out that a finite cyclic group is a group generated by a single element. In a finite cyclic group of order N, the generator satisfies $Ng = 0$, and N is the smallest positive integer with this property, and N is called the order of the generator. Thus the order of the generator is equal to the order of the group (even if the sense of the word "order" is different). It is then easy to see that any two finite cyclic groups of the same order are isomorphic.

Example A.8 *Let us consider the group $\mathbb{Z}_4$. Then we have that*

$$< 1 >=< 3 >= \mathbb{Z}_4.$$

Notice that $< 2 >= \{0, 2\}$ which, clearly, is not $\mathbb{Z}_4$. The group $\mathbb{Z}_4$ is called the cyclic group of the integers modulo four.

The direct sum $G \oplus H$ of two Abelian groups G and H is the set of all ordered pairs (g, h), with $g \in G$ and $h \in H$. If we define the following addition operation

$$(g_1, h_1) + (g_2, h_2) = (g_1 + g_2, h_1 + h_2)$$

then it is easy to see that $G \oplus H$ is an Abelian group whose neutral element is $(0, 0)$ and the inverse of (g, h) is $(-g, -h)$. The definition of direct sum is easily extended to more than two Abelian groups.
We have the following fundamental result.

Theorem A.9 *[Fundamental theorem of finite Abelian groups] Let G be a finite Abelian group. Then there exists cyclic groups $\mathbb{Z}_{q_1}, \mathbb{Z}_{q_2}, \ldots, \mathbb{Z}_{q_r}$ of orders $q_1, q_2, \ldots, q_r > 1$, respectively, where the q_i are prime powers, for $1 \leq i \leq r$, such that*

$$G \cong \mathbb{Z}_{q_1} \oplus \mathbb{Z}_{q_2} \oplus \cdots \oplus \mathbb{Z}_{q_r}.$$

Thus G is isomorphic to the direct sum of cyclic groups.

Example A.10 *Let us consider the group $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. Using the previous theorem we have that*

$$\mathbb{Z}_6 = \mathbb{Z}_2 \oplus \mathbb{Z}_3.$$

Indeed, let us consider the following map

$$\phi : \mathbb{Z}_2 \oplus \mathbb{Z}_3 \rightarrow : (g, h) \mapsto (3g + 2h)\, mod 6,$$

with $g \in \mathbb{Z}_2$ and $h \in \mathbb{Z}_3$. It is easy to see that ϕ is an isomorphism, and thus $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

A.3 Group characters

Let G a finite Abelian group of order N. We can define the character of the group G as follows.

Definition A.11 *A character of G is a homomorphism $\chi : G \rightarrow \mathbb{C}^* = \mathbb{C} - \{0\}$, which maps G to the non-zero multiplicative group of complex numbers.*

Since χ is a homomorphism then we have that

$$\chi(g + h) = \chi(g)\chi(h), \qquad g, h \in G.$$

In particular, we have that

$$\chi(g)^N = \chi(Ng) = \chi(0) = 1, \quad g \in G,$$

and so the values of χ are the N^{th} roots of the unity. Notice moreover that

$$\chi(-g) = \chi(g)^{-1} = \bar{\chi(g)}$$

where the bar indicates the complex conjugate. The character defined by

$$\chi(g) = 1, \quad \forall\, g \in G,$$

is called the *t*rivial (or principal) character of the group G. All the others are called non-trivial characters.

Example A.12 *Consider the group $\mathbb{Z}_N$. We have that the characters of $\mathbb{Z}_N$ are*

$$\chi_\ell : g \mapsto \exp\left(i2\pi\frac{\ell}{N}g\right),$$

with $\ell = 0, 1, \ldots, N-1$ and $g \in \mathbb{Z}_N$. It is easy to see that the trivial character of group $\mathbb{Z}_N$ is $\chi_0(g)$.

The characters of a finite Abelian group have many properties. We summarize here some important facts.

Proposition A.13 *For any nontrivial character χ of G,*

$$\sum_{g \in G} \chi(g) = 0.$$

Proof: Let χ be a non-trivial character of G and let $h \in G$ be such that $\chi(h) \neq 1$. Let $L = \sum_{g \in G} \chi(g)$, then we have

$$\chi(h)L = \sum_{g \in G} \chi(h)\chi(g) = \sum_{g \in G} \chi(h+g) = L.$$

We then have

$$L(\chi(h) - 1) = 0$$

which implies $L = 0$ since $\chi(h) \neq 0$. This concludes the proof. ■

Let us now denote $\hat{G}$ the set of all characters. Let us define the following operation between two characters, χ and ϕ, of a group G

$$(\chi\phi)(g) = \chi(g)\phi(g). \tag{A.1}$$

It is easy to see that this set forms an Abelian group under the operation defined by $(A.1)$. The conjugation represent the inversion in the group $\hat{G}$. The group of the characters $\hat{G}$ is called the $d$ual group of G.

Proposition A.14 *Let χ and ϕ be two characters of the group G. Then we have*

$$\sum_{g \in G} \chi(g)\bar{\phi}(g) = \begin{cases} N & if \quad \chi = \phi \\ 0 & otherwise \end{cases}$$

Proof: If $\chi = \phi$ then this follows from the fact that $\bar{\chi}(g) = \chi(g)^{-1}$. If $\chi \neq \phi$, then $\chi\bar{\phi}$ is a nontrivial character of G and using Proposition A.13 the result follows. ■

Example A.15 *Consider the group $\mathbb{Z}_N$. The characters of the group are*

$$\chi_\ell = \exp\left(i2\pi\frac{\ell}{N}g\right)$$

$\ell = 0, \ldots, N-1$ and $g \in \mathbb{Z}_N$. Then the dual group of $\mathbb{Z}_N$ is

$$\hat{\mathbb{Z}}_N = \{\chi_0, \ldots, \chi_{N-1}\}.$$

Lemma A.16 $\hat{\mathbb{Z}}_N \cong \mathbb{Z}_N$.

Proof: Since $\mathbb{Z}_N$ is a cyclic group then $< 1 >= \mathbb{Z}_N$. For the characters of $\mathbb{Z}_N$ we have the form

$$\chi_\ell(k) = \exp\left(i2\pi\frac{\ell}{N}k\right)$$

where $k \in \mathbb{Z}, \ell = 0, \ldots, N-1$. But this shows that χ_1 is a generator of $\hat{Z}_N$. Since two cyclic groups of the same order are isomorphic then we can conclude. ■

Lemma A.17 *If the group G is expressed as direct sum, namely $G = G_1 \oplus G_2$ and $\phi_i : G_i \to \mathbb{C}^x$ is a character of G_i, $i = 1, 2$, then $\chi = \phi_1 \oplus \chi_2$, defined as*

$$\chi(g_1, g_2) = \phi_1(g_1)\phi_2(g_2), \tag{A.2}$$

is a character of G. Moreover, all characters of G are of this form. Thus we have that

$$\hat{G} \cong \hat{G}_1 \oplus \hat{G}_2.$$

Proof: It is easy to show that χ is a character of G since it defines an homomorphism between G and $\mathbb{C}^x \times \mathbb{C}^x$ and

$$\begin{aligned}
\chi(g + h) &= \chi((g_1, g_2) + (h_1, h_2)) \\
&= \chi(g_1 + h_1, g_2 + h_2) \\
&= \phi(g_1 + h_1)\phi(g_2 + h_2) \\
&= \phi(g_1)\phi(h_1)\phi(g_2)\phi(h_2) \\
&= \phi(g_1 + g_2)\phi(h_1 + h_2) \\
&= \phi(g)\phi(h)
\end{aligned}$$

where we used the fact that ϕ_i is a character of G_i. It is clear that $\hat{G}_1 \oplus \hat{G}_2 \to \hat{G}$ defined by (A.2) is injective. We need to show then that if we consider $\chi(\tilde{g}_1, g_2) = \chi(g_1, g_2)$ then $\phi_1(\tilde{g}_1) = \phi_1(g_1)$. It follows that

$$0 = \chi(\tilde{g}_1, g_2) - \chi(g_1, g_2) = \phi_1(\tilde{g}_1)\phi_2(g_2) - \phi_1(g_1)\phi_2(g_2) = \phi_2(g_2)\Big(\phi_1(\tilde{g}_1) - \phi_1(g_1)\Big).$$

Since $\phi_2(g_2) = 0$ then it follows that $\phi_1(\tilde{g}_1) = \phi_1(g_1)$ as we wanted. Let us now consider $\chi \in \hat{G}$. Then the restriction $\phi_i = \chi|_{G_i}$ is a character of G_i, and it is easy to verify that $\chi = \phi_1 \oplus \phi_2$. ■

Theorem A.18 *For arbitrary finite Abelian groups $\hat{G} \cong G$.*

Proof: From Theorem A.9 we have that $G \cong \mathbb{Z}_{q_1} \oplus \cdots \oplus \mathbb{Z}_{q_r}$. We know that $\hat{\mathbb{Z}}_{q_i} \cong \mathbb{Z}_{q_i}$ by Lemma A.16. From Lemma A.17 follows that the direct sum $\hat{G} \cong \hat{\mathbb{Z}}_{N_1} \oplus \cdots \oplus \hat{\mathbb{Z}}_{N_k} \cong G$. ■

Let $\mathbb{C}^G$ denote the space of functions $f : G \to \mathbb{C}$. This represents an N-dimensional linear space over $\mathbb{C}$. We introduce an inner product over this space

$$< f_1, f_2 >= \frac{1}{N} \sum_{g \in G} f_1(g) f_2\bar{(g)}, \qquad f_1, f_2 \in \mathbb{C}^G.$$

We then have the following result.

Theorem A.19 *The elements of the set $\hat{G}$ forms an orthonormal basis in $\mathbb{C}^G$.*

Proof: Orthogonality follows directly from Lemma A.17. Completeness follows from the fact the $G \cong \hat{G}$, which implies $|\hat{G}| = N = \dim \mathbb{C}^G$. ■

Let $\chi_0, \ldots, \chi_{N-1}$ be the characters of $G = \{g_0, \ldots, g_{N-1}\}$. We can then consider the following matrix

$$[C]_{ij} = \chi_i(g_j)$$

which is called the character table of G.

A.4 Fourier transform on groups

We now introduce the Fourier transform of functions defined on Abelian group G.

Definition A.20 *Let $f : G \to \mathbb{C}$ be any function. We define the Fourier transform $\hat{f} : \hat{G} \to \mathbb{C}$ of f by*

$$\hat{f}(\chi) = \sum_{g \in G} f(g)\bar{\chi}g, \qquad \chi \in \hat{G}. \tag{A.3}$$

Example A.21 *Let us consider again the group $\mathbb{Z}_N$. The characters, as we have seen before, are given by*

$$\chi_\ell(g) = e^{i\frac{2\pi}{N}\ell g}, \quad g \in \mathbb{Z}_N, \quad \ell = 0, \ldots, N-1.$$

The correspondence $\ell \to \chi_\ell$ yields an explicit isomorphism between $\mathbb{Z}_N$ and $\hat{\mathbb{Z}}_N$. Given any function $f : \mathbb{Z}_N \to \mathbb{C}$, its Fourier transform is given by

$$\hat{f}(\chi_\ell) = \sum_{g=0}^{N-1} f(g)e^{-i\frac{2\pi}{N}\ell g}.$$

This transformation is easily inverted, and we define the inverse Fourier transform as follows

$$f = \frac{1}{N}\sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi(g), \qquad g \in G. \tag{A.4}$$

An important fact about the Fourier transform is that it is an isometry with respect to a (suitably normalized) L_2-norm. Indeed we have the following result.

Theorem A.22 (Plancherel's formula) *For any* $f_1, f_2 \in \mathbb{C}^G$,

$$< \hat{f}_1, \hat{f}_2 >= N < f_1, f_2 > .$$

Proof: Let us define the following vectors

$$f_1 = (f_1(g_0), \ldots, f_1(g_{N-1}))$$
$$f_2 = (f_2(g_0), \ldots, f_2(g_{N-1}))$$

and

$$\hat{f}_1 = (\hat{f}_1(\chi_0), \ldots, \hat{f}_1(\chi_{N-1}))$$
$$\hat{f}_2 = (\hat{f}_2(\chi_0), \ldots, \hat{f}_2(\chi_{N-1})).$$

Let C be the character table of G. Then we have that $\hat{f}_1 = f_1 C$ and $\hat{f}_2 = f_2 C$, and thus

$$< \hat{f}_1, \hat{f}_2 >= \frac{1}{N} f_1 C C^* f_2^* = f_1 f_2^* = N < f_1, f_2 >,$$

where we used the fact that $CC^* = NI$, and where the star indicates the transpose and complex conjugation operator. ■

Corollary A.23 (Parceval's formula) *For any* $f \in \mathbb{C}^G$,

$$< \hat{f}, \hat{f} >= N < f, f > .$$

Appendix B

Graph Theory

B.1 Graph Theory

In this appendix we recall some notation and concepts on directed and undirected graphs (the reader can further refer to textbooks on graph theory such as [65] or [49]).

A *directed graph* is a pair $(V, \mathcal{E})$ where $V = \{1, \ldots, N\}$ is a finite nonempty node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is an edge set of ordered pairs of nodes, called edges. The notation (i, j) means an edge getting out from the node i and getting in the node j. For the edge (i, j) i is the *parent node* and j is the *child node*. The self-loops are denoted by (i, i).

The adjacency matrix A is a $\{0, 1\}$-valued square matrix indexed by the elements in V defined by letting $A_{ij} = 1$ if and only $(i, j) \in \mathcal{E}$. It is worth noting that, in this thesis we admit the presence of self-loops in the graph $\mathcal{G}$ and hence if $(i, i) \in \mathcal{E}$ then $A_{ii} = 1$; often in the literature the self-loops are not encountered in the adjacency matrix.

In contrast to a direct graph, the pairs of nodes in an *undirected graph* are unordered; an undirected graph can be viewed as a special case of a directed graph, where an edge (i, j) in the undirected graph corresponds to edges (i, j) and (j, i). Clearly in the case of an undirected graph the adjacency matrix is symmetric.

Define the in-degree of a vertex j as $\text{indeg}(j) := \sum_i A_{ij}$ and the out-degree of a vertex i as $\text{outdeg}(i) := \sum_j A_{ij}$. A graph is called in-regular (out-regular) of degree k if each vertex has in-degree (out-degree) equal to k.

Given a direct graph $\mathcal{G}$, a *directed graph* consists of a sequence of vertices

$i_1 i_2 \ldots\ldots i_r$ such that $(i_\ell, i_{\ell+1}) \in \mathcal{E}$ for every $\ell = 1, \ldots, r-1$; i_1 (resp. i_r) is said to be the initial (resp. terminal) vertex of the path. An *undirected graph* in an undirected graph is defined analogously.

In a directed graph, a *cycle* is a directed path in which the initial and the terminal vertices coincide. A vertex i is said to be connected to a vertex j if there exists a path with initial vertex i and terminal vertex j.

A directed graph is said to be *connected* if, given any pair of vertices i and j, either i is connected to j or j is connected to i. A directed graph is said to be *strongly connected* if, given any pair of vertices i and j, i is connected to j. An undirected graph is *connected* if there is an undirected path between every pair of distinct nodes. Note that, for an undirected graph, the notions of to be connected or strongly connected coincide.

A *direct tree* is a directed graph in which every node has exactly one parent except for one node, called the *root*, which has no parent and which has a direct path to every other node. Note that a directed tree has no cycle because every edge is oriented away from the root. In undirected graphs, a *tree* is a graph in which every pair of nodes is connected by exactly one undirected path.

A *subgraph* $(V_s, \mathcal{E}_s)$ of $(V, \mathcal{E})$ is a graph such that $V_s \subseteq V$ and $\mathcal{E}_s \subseteq \cap(V_s \times V_s)$. A *directed spanning tree* $(V_s, \mathcal{E}_s)$ of the directed graph $(V, \mathcal{E})$ is a subgraph of $(V, \mathcal{E})$ such that $(V_s, \mathcal{E}_s)$ is a directed tree and $V_s = V$. An *undirected spanning tree* of an undirected graph is defined analogously. Note that in undirected graphs, the existence of an undirected spanning tree is equivalent to being connected, while in directed graphs is a weaker condition than being strongly connected.

Given a direct graph $\mathcal{G} = (V, E)$, a node $k \in V$ is said to be connected to a node $\ell \in V \setminus \{k\}$ if there is a path from k to ℓ in the graph with respects the orientation of the arcs. Note that the directed graph $\mathcal{G}(V, \mathcal{E})$ has a direct spanning tree if and only if $(V, \mathcal{E})$ has at least one node connected to all the other nodes. Finally, given a sequence of direct graphs $\mathcal{G}(t)\,(V, E(t))$, $t \in \mathbb{N}$, a node $k \in V$ is said to be connected to a node $\ell \in V \setminus \{k\}$ across an interval $I \subseteq \mathbb{N}$ if k is connected to ℓ for the directed graph $(V, \cup_{t \in I} E(t))$.

Finally, given the graphs $\mathcal{G}_1 = (V, \mathcal{E}_1)$ and $\mathcal{G}_2 = (V, \mathcal{E}_2)$, the union of $\mathcal{G}_1$ and $\mathcal{G}_2$ is defined by

$$\mathcal{G}_1 \cup \mathcal{G}_2 = (V, \mathcal{E}_1 \cup \mathcal{E}_2)\,.$$

Appendix C

Some useful algebraic results

C.1 Stability of discrete time linear parameter varying (LPV) systems

Given $A_1, \ldots, A_k \in \mathbb{R}^{n\times n}$, we let $\{A(t)\}_{t=0}^{+\infty} \subset \mathrm{Co}\{A_1, \ldots, A_k\}$ denote a sequence of matrices taking values in the convex hull of $\{A_1, \ldots, A_k\}$. We consider the dynamical system

$$x(t+1) = A(t)x(t). \tag{C.1}$$

The following result is an extension to the discrete-time system (C.1) of the classical result stated in [21], establishing a sufficient condition for the stability of continuous-time LPV systems.

Theorem C.1 (Common Lyapunov function) *For $A_1, \ldots, A_k \in \mathbb{R}^{n\times n}$, if there exists a symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$\frac{A_i^* P A_j + A_j^* P A_i}{2} - P < 0, \quad \textit{for all } i, j \in \{1, \ldots, k\},$$

then, for all initial conditions $x(0) \in \mathbb{R}^n$ and sequences

$$\{A(t)\}_{t=0}^{+\infty} \subset \mathrm{Co}\{A_1, \ldots, A_k\},$$

the solution to (C.1) *satisfies*

$$\lim_{t\to+\infty} x(t) = 0.$$

The objective of this Subsection is to generalize this classic result as follows.

Theorem C.2 (Common Lyapunov function for convergence to eigenspace) *Assume that* 1 *is a simple eigenvalue with left and right eigenvector* $v \in \mathbb{R}^n$ *for each matrix* $A_1, \ldots, A_k \in \mathbb{R}^{n\times n}$. *If there exists a symmetric matrix* $P \in \mathbb{R}^{n\times n}$ *satisfying, for all nonzero* $z \notin \operatorname{span}\{v\}$,

$$Pv = 0, \tag{C.2}$$

$$z^* P z > 0, \tag{C.3}$$

and

$$z^* \left(\frac{A_i^* P A_j + A_j^* P A_i}{2} - P \right) z < 0, \quad \textit{for all } i, j \in \{1, \ldots, k\}, \tag{C.4}$$

then, for all initial conditions $x(0) \in \mathbb{R}^n$ *and sequences*

$$\{A(t)\}_{t=0}^{+\infty} \subset \operatorname{Co}\{A_1, \ldots, A_k\},$$

the solution to (C.1) *satisfies*

$$\lim_{t\to+\infty} x(t) = \alpha v, \quad \alpha = \frac{1}{\|v\|^2} v^* x(0).$$

Proof: Because v is a left and right eigenvector with eigenvalue 1, we have

$$A_i v = v, \quad \text{and} \quad v^* A_i = v^*, \quad \text{for } i \in \{1, \ldots, k\}. \tag{C.5}$$

Consider the following decomposition

$$x(t) = x_{\text{ave}}(t)\, v + x_\perp(t),$$

where $x_\perp \perp v$ and where $x_{\text{ave}}(t) = \frac{1}{\|v\|^2} v^*\, x(t) \in \mathbb{R}$. Straightforward calculations show that x_{ave} satisfies the recursive relation

$$x_{\text{ave}}(t+1) = x_{\text{ave}}(t) + \frac{1}{\|v\|^2}\, v^*\, A(t) x_\perp(t) = x_{\text{ave}}(t),$$

where in the last equality we have used the facts that $v^* A(t) = v^*$ and $v^* x_\perp = 0$. Hence, $x_{\text{ave}}(t) = x_{\text{ave}}(0) = (1/\|v\|^2) v^* x(0)$, for all t. Now, let $v_1 = v$ and consider a basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathbb{R}^n$ with the orthogonality property $v_1 \perp v_i$ for all $i \in \{2, \ldots, n\}$. Let $T = [v_1, \ldots, v_n] \in \mathbb{R}^{n\times n}$. Let $\bar{x} =$

$T^{-1}x = [\tilde{x}_1, \ldots, \tilde{x}_n]^*$ and let $\tilde{x} = [\tilde{x}_2, \ldots, \tilde{x}_n]^* \in \mathbb{R}^{n-1}$. By the hypothesis (C.2) and the assumption (C.5) we have that

$$T^{-1}PT = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P} \end{bmatrix},$$

$$T^{-1}A_iT = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_i \end{bmatrix},$$

and

$$T^{-1}A_jT = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_j \end{bmatrix},$$

where $\tilde{P}$, $\tilde{A}_i$, $\tilde{A}_j \in \mathbb{R}^{(n-1)\times(n-1)}$ and, by hypothesis (C.3) $\tilde{P} > 0$. It follows that

$$\frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_i^* \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \tilde{P} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_j \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_j^* \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \tilde{P} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_i \end{bmatrix}\right) - \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2}\left(\tilde{A}_i^*\tilde{P}\tilde{A}_j + \tilde{A}_j^*\tilde{P}\tilde{A}_i\right) - \tilde{P} \end{bmatrix},$$

where, by hypothesis (C.4), $\frac{1}{2}\left(\tilde{A}_i^*\tilde{P}\tilde{A}_j + \tilde{A}_j^*\tilde{P}\tilde{A}_i\right) - \tilde{P} < 0$, for all $i, j \in \{1, \ldots, k\}$. Hence, Theorem C.1 implies $\lim_{t\to+\infty} \tilde{x}(t) = 0$ and, in turn, $\lim_{t\to+\infty} x_{\text{ave}}(t) = (1/\|v\|^2)v^*x(0)$. ■

C.2 Solvability of a Lyapunov equation

The proof of some results contained in Chapter 7 are based on the solvability of the following Lyapunov equation

$$z^*(R_1^*LR_1 - (1-\beta)L)z < 0, \qquad \forall\ z \in < [\mathbb{1}^*\ \mathbf{0}^*]^* >^{\perp} \tag{C.6}$$

where

$$L = \begin{bmatrix} I - P & 0 \\ 0 & \gamma I \end{bmatrix}, \qquad R_1 = \begin{bmatrix} I & 0 \\ 0 & \delta I \end{bmatrix}\begin{bmatrix} P & P - I \\ P - I & P - 2I \end{bmatrix}, \tag{C.7}$$

$P \in \mathbb{R}^{N\times N}$ satisfies Assumption 7.1 (pag. 153), $0 < \delta < 1$ and $\gamma > 0$.

The following lemma helps to determine for what parameters γ, β, δ the Lyapunov inequality (C.6) holds.

Lemma C.3 *Let λ_{min} and λ_{max} be the minimum and the maximum eigenvalue in $\sigma(P) \setminus \{1\}$, respectively. Define the polynomial*

$$f(\lambda, \delta, \gamma, \beta) := \gamma\beta^2 + \left((1-\lambda)^3 + \gamma\delta^2(\lambda-2)^2 - \gamma + \gamma(1-\lambda)(\gamma\delta^2 - 1 - \lambda)\right)\beta + (1-\lambda)\left(-\gamma^2\delta^2 + \gamma(1+\lambda+\delta^2(\lambda-3)) - (1-\lambda)^2\right). \tag{C.8}$$

Then (C.6) holds true if and only if

$$\beta < \min\left\{\beta^-_{min}(\delta,\gamma), \beta^-_{max}(\delta,\gamma)\right\}, \tag{C.9}$$

where $\beta^-_{min}(\delta,\gamma)$ and $\beta^-_{max}(\delta,\gamma)$ are the minimum real roots of $f(\lambda_{min}, \delta, \gamma, \beta)$ and of $f(\lambda_{max}, \delta, \gamma, \beta)$ as polynomials in β.

Proof: We start by observing that

$$R_1^* L R_1 - (1-\beta)L = \begin{bmatrix} (I-P)^2(\gamma\delta^2 I - I - P) + \beta(I-P) & (I-P)(P(P-I) - \gamma\delta^2(P-2I)) \\ (I-P)(P(P-I) - \gamma\delta^2(P-2I)) & (I-P)^3 + \gamma\delta^2(P-2I)^2 - \gamma I + \gamma\beta I \end{bmatrix}.$$

Note that

$$(R_1^* L R_1 - (1-\beta)L)\begin{bmatrix} \mathbb{1} \\ \mathbf{0} \end{bmatrix} = 0$$

and hence showing (C.6) is equivalent to show that the symmetric matrix $-R_1^* L R_1 + (1-\beta)L$ has all positive eigenvalues except one, which is zero and has multiplicity one. If we define the polynomials $q_{11}(\lambda), q_{22}(\lambda), q_{12}(\lambda)$ as follows

$$\begin{aligned} q_{11}(\lambda) &:= (1-\lambda)^2(\gamma\delta^2 - 1 - \lambda) + \beta(1-\lambda) \\ q_{22}(\lambda) &:= (1-\lambda)^3 + \gamma\delta^2(\lambda-2)^2 - \gamma + \gamma\beta \\ q_{12}(\lambda) &:= (1-\lambda)(\lambda(\lambda-1) - \gamma\delta^2(\lambda-2)) \end{aligned}$$

we can write

$$R_1^* L R_1 - (1-\beta)L = \begin{bmatrix} q_{11}(P) & q_{12}(P) \\ q_{12}(P) & q_{22}(P) \end{bmatrix},$$

To compute the eigenvalues of $-R_1^* L R_1 + (1-\beta)L$ we consider its characteristic polynomial. Using the same arguments used in the proof of Lemma 7.10 we can argue that

$$\begin{aligned} &\det(sI + R_1^* L R_1 - (1-\beta)L) = \\ &\quad = \prod_{i=0}^{N-1} \left[s^2 + (q_{11}(\lambda_i) + q_{22}(\lambda_i))s + (q_{11}(\lambda_i)q_{22}(\lambda_i) - q_{12}(\lambda_i)^2)\right] \end{aligned}$$

where $\lambda_0 = 1, \lambda_1, \ldots, \lambda_{N-1}$ denote the eigenvalues of P. Observe now that, for $i = 0$ the polynomial in the previous product is

$$s(s - \gamma\delta^2 + \gamma - \gamma\beta)$$

which gives one zero eigevalue and another eigenvalue equal to $\gamma(1-\delta^2-\beta)$. We can argue that, since this must positive, then we have this first constraint

$$\beta < 1 - \delta^2 \tag{C.10}$$

Moreover all the roots of the other polynomials for $i = 1, \ldots, N-1$ must be all positive. Observe that $s^2 + (q_{11}(\lambda_i) + q_{22}(\lambda_i))s + (q_{11}(\lambda_i)q_{22}(\lambda_i) - q_{12}(\lambda_i)^2)$ can be seen as the characteristic polynomial of the 2×2 matrix

$$\begin{bmatrix} -q_{11}(\lambda_i) & -q_{12}(\lambda_i) \\ -q_{12}(\lambda_i) & -q_{22}(\lambda_i) \end{bmatrix}$$

and, to impose that the roots of this polynomial are positive is equivalent to impose that such a matrix is positive definite and so that

$$-q_{11}(\lambda_i) > 0 \qquad\qquad q_{11}(\lambda_i)q_{22}(\lambda_i) - q_{12}^2(\lambda_i) > 0.$$

Therefore, together with condition (C.10), we have other $2N-2$ conditions. Some of these conditions are superfluous. We start from condition $-q_{11}(\lambda_i) > 0$. Observe that, since $1 - \lambda_i > 0$ for all $i \in \{1, \ldots, N-1\}$, then $q_{11}(\lambda_i) < 0$ for all $i \in \{1, \ldots, N-1\}$ if and only if $\beta + \gamma\delta^2(1-\lambda_i) - 1 + \lambda_i^2 < 0$ for all $i \in \{1, \ldots, N-1\}$ and this happens if and only if

$$\beta < 1 - \lambda_{min}^2 - \gamma\delta^2(1 - \lambda_{min}) \tag{C.11}$$

$$\beta < 1 - \lambda_{max}^2 - \gamma\delta^2(1 - \lambda_{max}) \tag{C.12}$$

Notice now that

$$q_{11}(\lambda)q_{22}(\lambda) - q_{12}^2(\lambda) = (1-\lambda)f(\lambda)$$

where $f(\lambda) = f(\lambda, \delta, \gamma, \beta)$ is defined in (C.8). Notice that

$$\frac{\partial^2 f}{\partial \lambda^2} = (1-\beta)(6\lambda - 6 - 2\gamma(1+\delta^2))$$

which is negative for $\lambda < 1$. This implies that $f(\lambda)$, is a concave function in λ, for $\lambda \in [-1, 1]$ and so $q_{11}(\lambda_i)q_{22}(\lambda_i) - q_{12}^2(\lambda_i) > 0$ for all $i = 1, \ldots, N-1$ if and only if

$$f(\lambda_{min}, \delta, \gamma, \beta) > 0 \tag{C.13}$$

$$f(\lambda_{max}, \delta, \gamma, \beta) > 0. \tag{C.14}$$

At this point we have that (C.6) holds true if and only if conditions (C.10), (C.11), (C.12), (C.13) and (C.14) hold true. Consider the condition $f(\lambda_{min}, \delta, \gamma, \beta) > 0$. Observe that, if $\beta = 1 - \lambda_{min}^2 - \gamma\delta^2(1 - \lambda_{min})$, then $q_{11} = 0$ and so

$$f(\lambda_{min}, \delta, \gamma, \beta)_{|\beta=1-\lambda_{min}^2-\gamma\delta^2(1-\lambda_{min})} = \frac{-q_{12}^2}{1-\lambda} < 0.$$

We can argue that $f(\lambda_{min}, \delta, \gamma, \beta)$ is a convex parabola in β which has always two real roots $\beta_{min}^-(\delta, \gamma)$ and $\beta_{min}^+(\delta, \gamma)$ which satisfy

$$\beta_{min}^-(\delta, \gamma) < 1 - \lambda_{min}^2 - \gamma\delta^2(1 - \lambda_{min}) < \beta_{min}^+(\delta, \gamma) \tag{C.15}$$

and morever $f(\lambda_{min}, \delta, \gamma, \beta) > 0$ if and only if

$$\beta < \beta_{min}^-(\delta, \gamma) \qquad \text{or} \quad \beta > \beta_{min}^+(\delta, \gamma). \tag{C.16}$$

This implies that conditions (C.11) and (C.13) hold if and only if $\beta < \beta_{min}^-(\delta, \gamma)$. Reasoning similarly for the condition $f(\lambda_{max}, \delta, \gamma, \beta) > 0$ we obtain that conditions (C.12) and (C.14) hold if and only if $\beta < \beta_{max}^-(\delta, \gamma)$.

We prove finally that condition (C.10) is superfluous which would give the thesis. To prove this observe that

$$\begin{aligned} &f(\lambda_{min}, \delta, \gamma, \beta)_{|\beta=0} = \\ &\quad = (1 - \lambda_{min})[-\lambda_{min}^2 + (2 + \gamma\delta^2 + \gamma)\lambda_{min} - (1 + 3\gamma\delta^2 + \gamma^2\delta^2 - \gamma)] \end{aligned}$$

and

$$\begin{aligned} &f(\lambda_{min}, \delta, \gamma, \beta)_{|\beta=1-\delta^2} = \\ &\quad = \delta^2(1 - \lambda_{min})[-\lambda_{min}^2 + (2 + \gamma\delta^2 + \gamma)\lambda_{min} - (1 + 3\gamma\delta^2 + \gamma^2\delta^2 - \gamma)] \end{aligned}$$

This implies that we can have three cases

1. We have $f(\lambda_{min}, \delta, \gamma, \beta)_{|\beta=0} = 0$. In this case we have that $\beta_{min}^-(\delta, \gamma) = 0$ and $\beta_{min}^+(\delta, \gamma) = 1 - \delta^2$.

2. We have $f(\lambda_{min}, \delta, \gamma, \beta)_{|\beta=0} < 0$. In this case we have that $\beta_{min}^-(\delta, \gamma) < 0 < 1 - \delta^2 < \beta_{min}^+(\delta, \gamma)$;

3. We have $f(\lambda_{min}, \delta, \gamma, \beta)_{|\beta=0} > 0$. In this case we may have three situations:

1a. $0 < \beta^-_{min}(\delta,\gamma) \leq \beta^+_{min}(\delta,\gamma) < 1-\delta^2$;
2b. $0 < 1-\delta^2 < \beta^-_{min}(\gamma) \leq \beta^+_{min}(\delta,\gamma)$;
3c. $\beta^-_{min}(\delta,\gamma) \leq \beta^+_{min}(\delta,\gamma) < 0 < 1-\delta^2$.

However the cases 2b and 2c cannot occur since the $\beta^-_{min}(\delta,\gamma)$ is a continuous function of γ, while in these two cases the value of this function would pass from 0 to $1-\delta^2$ in a neighbor of the γ's such that $f(\lambda_{min},\delta,\gamma,\beta)_{|\beta=0} = 0$ Notice finally that in all the cases which can occur we have that $\beta^-_{min}(\delta,\gamma) \leq 1-\delta^2$. ■

We provide now a consequence of the previous result.

Lemma C.4 *Assume the same assumptions of the previous lemma hold. Let* $\bar{\delta}$ *be defined as in* (7.13) *and* $\delta \in \mathbb{R}$ *be such that* $0 \leq \delta < \bar{\delta}$. *If we let*

$$\bar{\gamma} := \frac{1+\lambda_{min}+\delta^2(\lambda_{min}-3)}{2\delta^2}$$

Then $\bar{\gamma} > 0$ *satisfies*

$$z^*(R_1^*LR_1 - L)z < 0, \qquad \forall\ z \in < [\mathbb{1}^*\ \ \mathbf{0}^*]^* >^{\perp}, \tag{C.17}$$

Proof: Notice first that $0 \leq \delta < \bar{\delta}$ implies that $\bar{\gamma} > 0$. By the previous lemma, (C.17) holds true if and only if $\beta = 0$ is an admissible solution of (C.9) and this happens if and only if both $\beta^-_{min}(\delta,\bar{\gamma}) > 0$ and $\beta^-_{max}(\delta,\bar{\gamma}) > 0$. Notice now that $f(\lambda,\delta,\gamma,\beta)$ can be written as follows

$$f(\lambda,\delta,\gamma,\beta) = \gamma\beta^2 + [(1-\lambda)p(\lambda,\delta,\gamma) - \gamma(1-\delta^2)]\beta - (1-\lambda)p(\lambda,\delta,\gamma)$$

where

$$p(\lambda,\delta,\gamma) = \delta^2\gamma^2 - [1+\lambda+\delta^2(\lambda-3)]\gamma + (1-\lambda)^2$$

Notice moreover that, $\beta^-_{min}(\delta,\bar{\gamma}) > 0$ if and only if $(1-\lambda_{min})p(\lambda_{min},\delta,\bar{\gamma}) - \bar{\gamma}(1-\delta^2) < 0$ and $(1-\lambda_{min})p(\lambda_{min},\delta,\bar{\gamma}) < 0$ and these two conditions occurs if and only if $p(\lambda_{min},\delta,\bar{\gamma}) < 0$. Similarly we can see that $\beta^-_{max}(\delta,\bar{\gamma}) > 0$ if and only if $p(\lambda_{max},\delta,\bar{\gamma}) < 0$. Notice now that, since

$$\frac{\partial p}{\partial \lambda} = -\gamma - \gamma\delta^2 - 2(1-\lambda),$$

is negative for $\lambda < 1$, then $p(\lambda_{max},\delta,\bar{\gamma}) < 0$ is implies by $p(\lambda_{min},\delta,\bar{\gamma}) < 0$ which is the only condition we need to prove. Notice now that $\bar{\gamma}$ is the

minimizer of $p(\lambda_{min}, \delta, \gamma)$ as a function of γ. Therefore $p(\lambda_{max}, \delta, \bar{\gamma}) < 0$ if and only if the discriminant is positive, namely if and only if $(1 + \lambda_{min} + \delta^2(\lambda_{min} - 3))^2 - 4\delta^2(1 - \lambda_{min})^2 > 0$. Observe that this last inequality holds true if and only if

$$(3 - \lambda_{min})^2\delta^4 - 2(5 - 2\lambda_{min} + \lambda_{min}^2)\delta^2 + (1 + \lambda_{min})^2 > 0. \tag{C.18}$$

Consider the equation $(3 - \lambda_{min})^2x^2 - 2(5 - \lambda_{min} + \lambda_{min}^2)x + (1 + \lambda_{min})^2 = 0$. The solutions of this equation are $x_1 = 1$ and $x_2 = \left(\frac{1+\lambda_{min}}{3-\lambda_{min}}\right)^2$. Since $\lambda_{min} < 1$ we have that $x_2 < 1$. Hence, since $(3 - \lambda_{min})^2x^2 - 2(5 - \lambda_{min} + \lambda_{min}^2)x + (1 + \lambda_{min})^2 > 0$ is a convex parabola, we have that $(3 - \lambda_{min})^2x^2 - 2(5 - \lambda_{min} + \lambda_{min}^2)x + (1 + \lambda_{min})^2 > 0$ if and only if $x < x_2$ and $x > x_1$. It follows that, if $\delta^2 < \left(\frac{1+\lambda_{min}}{3-\lambda_{min}}\right)^2$, i.e., if $\delta < \bar{\delta}$, then (C.18) is satisfied.

■

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