On rendezvous control with randomly switching communication graphs

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Abstract

In this paper we analyze randomized coordination control strategies for the rendezvous problem of multiple agents with unknown initial positions. The performance of these control strategies is measured in terms of three metrics: average relative agents' distance, total input energy consumption, and number of packets per unit time that each agent can receive from the other agents. By considering an LQ-like performance index, we show that a-priori knowledge about the first and second order statistics of agents' initial position can greatly improve performance as compared to rendezvous control strategies based only on relative distance feedback. Moreover, we show that randomly switching communication topologies, as compared to static communication topologies, require very little information exchange to achieve high performance even when the number of agents grows very large.

Index Terms

Rendezvous control, networked control system, consensus, agreement, optimal control, randomized communication

I. Introduction and Previous Work

The need for coordination of multiple mobile vehicles appears in many applications such as search-and-rescue missions and pursuit evasion games [1][2][3][4]. Coordination among vehicles requires exchange of information between them. However, the amount of information that can be exchanged is limited by many factors such as channel bandwidth, limited communication range and interference, therefore it is desirable to devise coordination strategies that require the transmission of a limited number of messages among the agents [5][6]. However, limiting information exchange among agents negatively impacts the performance of the vehicles as a group in terms of other metrics such as energy consumption and time required to accomplish a task. The goal of this paper is to analyze the tradeoffs between these aspects within the framework of rendezvous control, i.e. convergence of all agents to a common location not necessarily specified.

Recently the rendezvous control has been approached by reformulating it as a consensus problem. The consensus problem has been widely studied in terms convergence properties of Markov chains [7] [8], and it has been recently proposed as an effective approach to flocking of mobile agents [9]. Since then many results have been obtained, as summarized in the survey paper [10]. In particular, some researchers studied convergence rate for fixed communication topologies [11][12][13]. If the communication graph is time-varying, then convergence to a common location is not guaranteed, therefore another line of research focused on finding sufficient convergence conditions [14][15][16][17][18]. Other authors studied instead convergence rates for fixed communication topologies chosen at random from an underlying communication graph in order to reduce communication load while still maintaining fast convergence rates [19][13]. Some other researchers studied stochastic communication switching of the communication topology where only one agent at a time can communicate and they looked explicitly at first and second order statistics to estimate convergence rate [20][21][22]. Finally, another direction of research focused on deterministic state-dependent communication graphs which result from limited communication range of vehicles by using control strategies based on potential functions to find convergence criteria and performance [23] or by using strategies that guarantee connectivity and convergence to a common location [24].

Despite all these remarkable results, there are still interesting questions to be answered. For example, by recasting the rendezvous problem of mobile robots as a consensus problem, only convergence rate has been studied so far. However, energy expenditure is also important and cannot be neglected. Moreover, the effect of any prior information about statistical distribution of agents on the overall performance has not been studied in the literature. Finally, it is also relevant to analyze the impact of the number of messages exchanged between the agents per unit period in terms of overall performance, and how this performance scales with the number of agents. The goal of this paper is to address some of these aspects.

In particular, in this paper we assume that each agent has a GPS-like sensor which provides its position with respect to some absolute coordinate frame. Also we consider a time-varying random communication topology, where every agent exchanges messages with a small set of other agents which are selected at random among all agents. The rationale behind this communication scheme is that the random selection of communicating agents enhances information diffusion rates, very similarly to the well known "small world effect" initially proposed by Watts and Strogatz [25] and recently extended to the consensus problem by several authors [26][27][28]. The difference between the previous works and this one is that in the former

This research is partially supported by the European Community Research Information Society Technologies under Grant No. SENSNET MIRG-6-CT-2005-014815, and by the Italian Ministry of Education, University and Research (MIUR)

the random rewiring between agents in the network is applied only once, while here it is applied at every communication step, thus increasing even further the degree of randomness in the communication topology. In spirit, this approach is also similar to the one proposed in [21][22]. However, differently from those works, the specific structure adopted here allows us to explicitly compute the overall system performance and the convergence rate from a two-dimensional optimization problem independently of the number of agents. Moreover, the protocol which implements the randomized communication strategy is very simple, and the performance does not degrade beyond a certain value as the number of agents increases even when only one message per time step is exchanged. Differently, most popular fixed communication strategies are either easy to build but have poor performance, like the symmetric communication graphs [13], or have good performance but are difficult to construct, like the Ramanujan graphs [29].

Moreover, we show that prior information about agents initial position distribution can greatly improve performance and reduce communication load among agents. In particular, the performance improvement depends on the ratio between the spreading (variance) of the initial agents distribution and the variance of the agents center of mass with respect to its a-priori estimate. In fact, if this ratio is small, then applying feedback on the a-priori expected agents center of mass dramatically reduces rendezvous cost, even in the extreme scenario when agents do not communicate at all. Differently, if the ratio is large then we recover that the optimal rendezvous strategy is indeed a consensus strategy under which each agent uses only relative distance information from the other agents.

The paper is organized as follows. In the next section we provide the mathematical formulation for the rendezvous control problem as a LQ-like optimization and we describe the rationale behind the proposed control feedback. In Section III we explicitly compute the performance of the overall systems for fixed control gains and we show that it can be reduced to a two-dimensional nonlinear control problem. Section IV provides the numerical tools to find the optimal control parameters and also discuss some relevant analytical results in terms of systems performance and design parameters. Section V presents a numerical example based an a specific rendezvous control scenario and shows how the tools developed in the previous section can be used. In Section VI we considered a GPS-free scenario where agents can only measure relative position with respect to the other agents and we compare this performance with the GPS-based performance. Finally, in Section VII we summarizes the results.

II. Problem formulation

Consider N identical agents whose dynamics are described by 2D linear discrete time integrators:

$$x_i(t+1) = x_i(t) + u_i(t), \quad i = 1, \dots, N$$

where $x_i, u_i \in \mathbb{C}$, whose real and imaginary parts correspond to the coordinates in the two-dimensional plane. We assume that each agent has access to its own position through a GPS-like sensor, and that process disturbance and measurement noises are absent. More compactly we can describe the agents dynamics in vector form as follows:

$$x(t+1) = x(t) + u(t) \tag{1}$$

where $x = (x_1, x_2, \dots, x_N)^T$ and $u = (u_1, u_2, \dots, u_N)^T$. Finally, to keep the mathematical analysis simple, we assume that each robot is a point agent, i.e. we neglect vehicle size.

The objective of rendezvous control is to devise a coordination scheme that forces the agents to converge to a common location, or, equivalently, that forces the relative distances among all agents to be null. A natural way to enforce this objective is to penalize relative distances among agents using a quadratic cost $c_x(x) : \mathbb{C}^N \to \mathbb{R}^+$ defined as follows:

$$\begin{aligned} c_x(x) &= \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} |x_i - x_j|^2 = x^* \Pi x = \sum_{i=1}^{N} |x_i - x_{cm}|^2 = ||x - x_{cm}\mathbf{1}||^2, \\ \Pi &= I - \frac{1}{N} \mathbf{1}\mathbf{1}^*, \end{aligned}$$

where $\mathbf{1} = (1, 1, \dots, 1)^T \in \mathbb{R}^N$, I is the identity matrix, $||x||^2 = x^* x = \sum_i |x_i|^2$ is the Euclidian norm of a vector, the superscript * indicates the complex conjugate transpose operator, and x_{cm} is the instantaneous center of mass of all agents:

$$x_{cm}(t) = \frac{1}{N} \mathbf{1}^* x(t).$$

Note that

$$c_x(x) = 0 \iff x_i = x_j, \ \forall (i,j) = 1, ..., N \iff x = \bar{x}\mathbf{1}$$

where $\bar{x} \in \mathbb{C}$, i.e. the cost $c_x(x)$ is null if and only if the agents are in the same location. Note that minimizing this cost does not require that the agents move to any predefined location, in fact the cost would be null even if $x(t) = \bar{x}(t)\mathbf{1}$, where $\bar{x}(t) \in \mathbb{C}$. This property is rather important since it is possible to independently design leader-following and rendezvous control algorithms. We also want to penalize the total agents' energy expenditure in achieving rendezvousing by using a quadratic cost $c_u(x) : \mathbb{C}^N \to \mathbb{R}^+$, where:

$$c_u(u) = r||u||^2$$

where $r \in \mathbb{R}^+$. The goal is to obtain a (possibly time-varying) feedback control

$$u(t) = K(t)x(t) \tag{2}$$

where $K(t) \in \mathbb{R}^{N\times N}$, which minimizes the expected total cost given by:

$$J_T = J_T\big(u(0), ..., u(T-1)\big) = \mathbb{E}_{x_0}\left[x^*(T)\Pi x(T) + \sum_{t=0}^{T-1}\Big(x^*(t)\Pi x(t) + r||u(t)||^2\Big)\right]. \tag{3}$$

The expectation is performed with respect to the initial position distribution since each $x(t)$ inside the square bracket is a function of the control input sequence $\{u(t)\}_{t=0}^{T-1}$ and of the initial position x_0 . If we substitute Equation (2) into Equation (1), we get the closed loop dynamics given by:

$$x(t+1) = (I + K(t))x(t). \tag{4}$$

Despite its simple formulation, the previous problem is rather challenging since the communication graph among agents imposes some constraints on the choice of the matrix $K(t)$. In particular if at time t the agent j cannot transmit its position to agent i, then the ij-th entry of the matrix K must be null, i.e. $K_{ij}(t) = 0$, since $x_j(t)$ is not available to agent i. This can happen for different reasons such as unreliable communications links, interference, packet collision, limited communication range or simply because only a maximum number of packets can be transmitted per unit time. Therefore, it is useful to define the adjacency matrix $E \in \{0, 1\}^{N\times N}$ as follows:

$$E_{ij} = \begin{cases} 1 & \text{if agent } i \text{ receives packet from agent } j, \\ 0 & \text{otherwise.} \end{cases}$$

The rendezvous control problem can be summarized as follows:

$$\begin{aligned} &\min_{\{K(t)\}_{t=0}^{T-1}} J_T \\ &s.t. \quad x(t+1) = (I + K(t))x(t), \\ &\qquad\quad K_{ij}(t) = 0 \text{ if } E_{ij}(x(t), t) = 0 \end{aligned} \tag{5}$$

where $E : \mathbb{R}^N \times \mathbb{R} \to \{0, 1\}^{N\times N}$ is a function associating an adjacency matrix (and hence a directed graph) to each state x and each time t. The second constraint makes the problem highly non-convex and time-varying, in general. Note also that under the previous formulation the communication graph can depend on agents' position, or can be time-varying and asymmetric, i.e. it is possible that node j can transmit its position to node i but not vice versa. Solving the previous problem in full generality is hopeless. Most of recent work on rendezvous control has concentrated on optimizing the rate of convergence with fixed communication topologies, i.e. $K(x, t) = K$, where most of the off diagonal entries are null [11][13][30]. In particular, the goal was to analytically determine the rate of convergence based on some a-priori constrains on the structure of K and to optimally design classes of communication topologies with limited communication requirements. These sets of problems are rather difficult and often lead to combinatorial optimization problems. Differently, in [23] the authors considered a feedback matrix dependent on agents location $K(x, t) = K(x)$; in particular they assumed that the agents can communicate only with agents which are within a fixed communication range, i.e. $E(x, t) = E(x(t))$. This strategy reduces communication burden but cannot guarantee convergence of all agents to a common location. With this respect, in [9][18] it was shown that the agents communication topology needs to form a sufficiently connected graph within an arbitrary large but finite time interval in order for the agents to converge to a common location. In other words this means that there must exist a time interval $T \in \mathbb{N}$, such that the graph resulting from the concatenation of all adjacency matrices $\{E(t)\}_{t=t_1}^{t_2}$, where $t_2 - t_1 \geq T$, is strongly rooted, i.e. it is a graph where there exists a node that can communicate with all other nodes with a single hop. Inspired by this result, we propose to consider a stochastic communication topology, i.e. a time-varying control feedback $K(t)$ where most of the off-diagonal entries are zeros, i.e. $K_{ij}(t) = 0$ for most of the indexes i, j, but *on average* they are not, i.e. $\mathbb{E}[K_{ij}(t)] \neq 0, \forall i, j$. Moreover, we assume that the agents can transmit their current position to some other agents, independently of their relative distance, i.e. we assume they have infinite power antennas. This last assumption is rather unrealistic, but will allow us to derive close form solutions for the agents performance. We will come back to this assumption in the Conclusions section. The randomized communication strategy gives rise on average to a fully connected graph (actually a complete graph). Our strategy does not satisfy the condition stated in [9][18] as there is always a small probability that the communication topology graph is not connected for any arbitrary but finite time interval T. In other terms, the approach used in [9][18] is a *worst-case* approach while the one adopted in this work is *probabilistic*. In fact, the probabilistic analysis emphasizes the advantages of time-varying strategies with respect to static ones, while the worst-case analysis, though ensuring exponential convergence, provides only conservative bounds on the convergence rate. Accordingly, we adopt the following probabilistic definition for stability:

Definition 1: The closed loop system (4) is *rendezvous asymptotically mean square stable* if:

$$\lim_{t\to\infty} \mathbb{E}[|x_i(t) - x_j(t)|^2] = 0,\ \forall(i,j) = 1,\ldots,N.$$

The study of mean square behavior is important because it is possible to show, by using concentration theorems, that the agents reach a consensus almost surely if they are rendezvous asymptotically mean square stable and their rate of convergence is close to the one predicted by the mean square analysis [22]. The following lemma links this definition of rendezvous-stability to the performance cost J_T.

Lemma 1: The closed loop system (4) is rendezvous asymptotically mean square stable if the following limit is finite:

$$J_\infty = \lim_{T\to+\infty} J_T = \sum_{t=0}^{\infty} \mathbb{E}[x^*(t)\Pi x(t) + r||u(t)||^2].$$

Proof: If J_∞ exists, then also $\sum_{t=0}^{\infty} \mathbb{E}[x^*(t)\Pi x(t)] \leq J_\infty$ must exist. Since the sum includes infinite terms, then each term must decrease to zero for the sum to be finite. In particular we must have $\lim_{t\to\infty} \mathbb{E}[x^*(t)\Pi x(t)] = 0$ otherwise we can find an infinite number of terms in the sum which are greater than a non-negative number $\epsilon > 0$ and the sum would therefore diverge. Since $0 \leq \mathbb{E}[|x_i(t) - x_j(t)|^2] \leq \sum_i \sum_j \mathbb{E}[|x_i(t) - x_j(t)|^2] = 2N\mathbb{E}[x^*(t)\Pi x(t)]$ for all (i,j,t), then this implies that $\lim_{t\to\infty} \mathbb{E}[|x_i(t) - x_j(t)|^2] = 0$. ∎

Note that the expectation in the previous definition is performed over all random variables, possibly including any stochasticity of the gain $K(t)$. In our framework we quantify information exchange as number of messages received by each agent at any time step, which corresponds to the non-zero off-diagonal entries of each row in the adjacency matrix $E(t)$. Therefore our objective is to analyze performance of rendezvous control as a function of the number of messages exchanged among agents. To get some more insight about structure of rendezvous control feedback we study two interesting limiting cases. In the first scenario let us assume that each agent receives messages from all other agents, i.e. $E(t) = \mathbf{1}\mathbf{1}^*$. Therefore the optimization problem of Equation (5) for the infinite horizon , i.e. $T \to +\infty$, becomes:

$$\begin{array}{l} \min_{\{K(t)\}_{t=0}^{\infty}} \mathbb{E}_{x_0}\left[\sum_{t=0}^{\infty}\left(x^*(t)\Pi x(t) + r||u(t)||^2\right)\right] \\ s.t. \quad x(t+1) = (I + K(t))x(t) \end{array} \tag{6}$$

which is the classic LQ optimal control problem. It is well known that the optimal feedback gains $\{K_{opt}(t)\}_{t=0}^{\infty}$ are static, i.e. $K_{opt}(t) = K_\infty$, and K_{opt} can be obtained from the solution of the following algebraic Ricatti equation:

$$\begin{array}{rcl} P &=& P + \Pi - P(P + rI)^{-1}P, \quad P \geq 0 \\ K_{opt} &=& -(P + rI)^{-1}P. \end{array}$$

After some simple matrix manipulations it is possible to show that the previous equations can be written as

$$K_{opt} = -\bar{\kappa}\Pi, \quad P = p\Pi$$

where $\bar{\kappa}, p \in \mathbb{R}$ satisfy the following scalar Riccati equation:

$$p = p + 1 - \frac{p^2}{p + r}, \quad p \geq 0 \tag{7}$$

$$\bar{\kappa} = \frac{p}{p + r}. \tag{8}$$

The feedback control given by Equation (2) can be written as:

$$u = -\bar{\kappa}(I - \frac{1}{N}\mathbf{1}\mathbf{1}^*)x(t) \Longrightarrow u_i = -\bar{\kappa}(x_i - x_{cm}) = -\bar{\kappa}\frac{1}{N}\sum_{j=1}^{N}(x_i - x_j).$$

This means that the optimal strategy of each agent is to move towards the instantaneous center of mass. Equivalently, the optimal control is proportional to the sum of the relative distances with the other agents. Note that this control feedback is independent of the reference frame, and that $x(t) \to x_{cm}(0)\mathbf{1}$, i.e. all agents converge to the initial agents' center of mass.

In the second scenario, we assume that no communication is allowed among the agents, i.e. $E = I$ which gives rise to the following optimization problem:

$$\begin{array}{l} \min_{\{K(t)\}_{t=0}^{\infty}} \mathbb{E}_{x_0}\left[\sum_{t=0}^{\infty}\left(x^*(t)\Pi x(t) + r||u(t)||^2\right)\right] \\ s.t. \quad x(t+1) = (I + K(t))x(t) \\ \qquad\ \ K(t) = \mathrm{diag}(k_1(t),\ldots,k_N(t)) \end{array} \tag{9}$$

where the last constraint enforces that agents do not communicate. Using symmetry arguments it follows that the optimal feedback gains $\{K_{opt}(t)\}_{t=0}^{\infty}$ are constant and time independent, i.e. $K_{opt}(t) = k_{opt}I$, therefore the control feedback can be written as:

$$u = -K_{opt}x = -k_{opt}x \Longrightarrow u_i = -k_{opt}x_i$$

which means that the optimal input for each agent is a linear feedback of its own position with respect to the reference frame. Differently from the fully connected scenario, here $x(t) \to 0$, i.e. all agents converge to the system origin.

III. LQ-like rendezvous control with mixed feedback

Based on these two scenarios and the discussion regarding the randomized communication topology with limited number of communication messages per unit time, we propose a rendezvous control strategy where at any time step each agent receives the current location of some other $\nu \in \{0, 1, \ldots, N-1\}$ distinct agents chosen uniformly at random. The control scheme is a linear feedback with constant gains of its own position with respect to a predefined fixed location $\bar{x}$ and the relative distance with the other visible agents:

$$u_i = -k(x_i - \bar{x}) - h \sum_{j=1}^{N} E_{ij}(t)(x_i - x_j) \tag{10}$$

where $k, h \in \mathbb{R}$, $\bar{x} \in \mathbb{C}$, $E_{ij} \in \{0,1\}$, $E_{ii} = 0$, and $\sum_{j=1}^{N} E_{ij} = \nu$. The non-zero $E_{ij}(t)$ correspond to the incoming communication links to agent i from the other agents at time step t. The control feedback is the sum of two terms: the first depends only on the system origin and requires no communication, while the second requires communication but it is independent of the system origin. Therefore, by appropriately choosing k and h, it is possible to place more weight on one term or on the other. More compactly, this control scheme can be written as:

$$u(t) \quad = \quad (hE(t) - (k + \nu h)I)\,(x(t) - \bar{x}\mathbf{1}) \tag{11}$$

where $E(t) \sim \mathcal{U}(\mathbb{E})$, i.e. the matrix E is uniformly sampled from a set of matrices $\mathbf{E}$ defined as follows:

$$\mathbf{E} = \{E \in \{0,1\}^{N \times N} \,|\, E\mathbf{1} = \nu\mathbf{1}, E_{ii} = 0\}. \tag{12}$$

It is important to remark that, although it is not possible to prove that the randomized control strategy is the optimal among all possible strategies having constraints on the maximum number of messages exchanged among agents, in the two extreme scenarios where $\nu = 0$ or $\nu = N-1$, the previous control strategy does give the optimal solution for the infinite horizon scenario.

Before continuing let us define the matrix $\Pi_\perp$ as follows:

$$\Pi_\perp \triangleq \frac{1}{N}\mathbf{1}\mathbf{1}^* \tag{13}$$

which has the following properties:

$$\begin{array}{l} \Pi = \Pi^* \geq 0, \quad \Pi_\perp = \Pi_\perp^* \geq 0, \quad \Pi = \Pi^2, \quad \Pi_\perp = \Pi_\perp^2, \\ \Pi + \Pi_\perp = I, \quad \Pi\,\Pi_\perp = \Pi_\perp\,\Pi = 0. \end{array} \tag{14}$$

Moreover, the matrix $E(t)$ satisfies some properties:

$$\begin{aligned} \mathbb{E}[E(t)] &= \nu\Pi_\perp - \tfrac{\nu}{N-1}\Pi, \\ \mathbb{E}[E^*(t)E(t)] &= \nu^2\Pi_\perp + \tfrac{\nu(N-\nu)}{N-1}\Pi, \\ \mathbb{E}[E^*(t)\Pi E(t)] &= \nu\left(1 - \nu\tfrac{N-2}{(N-1)^2}\right)\Pi, \\ \mathbb{E}[E^*(t)\Pi_\perp E(t)] &= \nu^2\Pi_\perp + \tfrac{\nu(N-\nu-1)}{(N-1)^2}\Pi. \end{aligned} \tag{15}$$

The derivations of these properties can be found in Lemma 4 present in the Appendix.

Let us consider the total cost $J_T = J_T(k, h, \bar{x})$ for fixed control parameters $(k, h, \bar{x})$ and finite horizon T, defined in Equation (3), where the parameter $r \in [0, +\infty)$ tunes the tradeoff between small agents relative distances (r small) and small input control energy (r large). We can now compute explicitly the cost function $J_T(k, h, \bar{x})$ using the standard dynamic programming approach based on the cost-to-go function $V_t(x)$ recursively defined as follows:

$$V_T(x_T) \quad \triangleq \quad \mathbb{E}[x_T^* \Pi x_T], \tag{16}$$

$$V_t(x_t) \quad \triangleq \quad \mathbb{E}[x_t^* \Pi x_t + r||u_t||^2 + V_{t+1}(x_{t+1})] \tag{17}$$

where we used $x_t = x(t)$ to simplify notation.

Theorem 1: Let us consider the cost-to-go function defined in Equations (16) and (17). Then it can be written as

$$V_t(x_t) = s_t \mathbb{E}[x_t^* \Pi x_t] + s_t^\perp \mathbb{E}[(x_t - \bar{x}\mathbf{1})^* \Pi_\perp (x_t - \bar{x}\mathbf{1})] \tag{18}$$

where s_t and $s_t^\perp$ and nonnegative scalars that can be obtained iteratively for $t = T, \ldots, 0$ as follows:

$$s_T \quad = \quad 1, \quad s_T^\perp = 0, \tag{19}$$

$$\begin{bmatrix} s_t \\ s_t^\perp \end{bmatrix} \quad = \quad A \begin{bmatrix} s_{t+1} \\ s_{t+1}^\perp \end{bmatrix} + b \tag{20}$$

where the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and the vector $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ are given by:

$$A = \begin{bmatrix} (1-k-\alpha_1 h)^2+\alpha_2 h^2 & \frac{1}{N-1}\alpha_2 h^2 \\ 0 & (1-k)^2 \end{bmatrix};\ b = \begin{bmatrix} 1+\big((k+\alpha_1 h)^2+\frac{N}{N-1}\alpha_2 h^2\big)r \\ k^2 r \end{bmatrix} \tag{21}$$

and the coefficients (α_1, α_2) are functions of the number of agents N and the number of received messages ν:

$$\alpha_1 = \frac{\nu N}{N-1}, \tag{22}$$

$$\alpha_2 = \frac{\nu(N-\nu-1)}{N-1}. \tag{23}$$

Proof: See Appendix. ∎

The previous equations fully determine the cost function in terms of the initial position distribution of agents x_0, feedback gains (k, h), and feedback reference position $\bar{x}$. Note that the sequences $\{s_t\}_{t=T}^0$ and $\{s_t^\perp\}_{t=T}^0$ are monotonically increasing since all a_{ij} coefficients are positive, therefore the limits $\lim_{T\to\infty} s_0$ and $\lim_{T\to\infty} s_0^\perp$ are finite if and only if $a_{11}(k,h) < 1, a_{22}(k) < 1$ or $a_{11}(k,h) < 1, k = 0^*$. If these limits exist, then also the infinite horizon cost exists. We can summarize the results obtained so far in the following lemma:

Lemma 2: Let us consider the closed loop system defined by dynamics given by Equation (1) and the randomly switching control feedback specified in Equation (11). Then, the LQ performance criterion defined by Equation (3), can be written as

$$J_T(k,h,\bar{x}) = s_0\mathbb{E}[x_0^*\Pi x_0] + s_0^\perp \mathbb{E}[(x_0-\bar{x}\mathbf{1})^*\Pi_\perp(x_0-\bar{x}\mathbf{1})] \tag{24}$$

where s_0 and $s_0^\perp$ are nonnegative scalars that can be computed iteratively as indicated in Equations (19) and (20). If $a_{11}(k,h) < 1$ and $a_{22}(k) < 1$ then the infinite horizon cost $J_\infty = \lim_{T\to\infty} J_T$ exists and it is given by:

$$J_\infty(k,h,\bar{x};\nu,N) = c^T(I-A)^{-1}b \tag{25}$$

where $c = [c_1\ c_2]^T$, $c_1 = \mathbb{E}[x_0^*\Pi x_0]$, $c_2 = \mathbb{E}[(x_0-\bar{x}\mathbf{1})^*\Pi_\perp(x_0-\bar{x}\mathbf{1})]$, and A and b as defined in Equation (21).

Proof: Equation (24) follows directly from Theorem 1 and the fact that $J_T = V_0(x_0)$. If $a_{11}(k,h) < 1$ and $a_{22}(k) < 1$, then it is easy to see that the sequences s_t and $s_t^\perp$ are monotonically increasing and bounded from above, therefore the limits $\lim_{T\to\infty} s_0 = s_\infty$ and $\lim_{T\to\infty} s_0^\perp = s_\infty^\perp$ exist and are finite. If we define the new vector $\mathbf{s} = [s_\infty\ s_\infty^\perp]^T$, then the infinite horizon cost can be written as $J_\infty(k,h,\bar{x}) = c^T\mathbf{s}$, where $\mathbf{s}$ must satisfy the equation $\mathbf{s} = A\mathbf{s}+b$. Finally, since the last equality is equivalent to $\mathbf{s} = (I-A)^{-1}b$, then a simple substitution gives Equation (25). ∎

It is interesting to note that, although the system is N-dimensional, the performance for fixed control parameters can be obtained by solving two dimensional linear problems. Also note that the cost $J_\infty(k,h,\bar{x})$ is a quadratic function in $\bar{x}$ and rational in the gains h, k. This rational dependence on h and k does not allow for a closed form minimization of the cost J_∞, as will be shown in the next section.

The total cost is a function of both the number of messages exchanged ν and the total number of agents N. However, for large numbers of agents the previous problem simplifies to:

Lemma 3: Let us consider the optimization problem defined in Theorem 1. If $(1-k-\nu h)^2+\nu h^2 < 1$ and $(1-k)^2 < 1$, then $\lim_{N\to\infty} J_\infty(k,h,\bar{x};\nu,N) = \overline{J}_\infty(k,h,\bar{x};\nu) = c^T(I-A_\infty)b_\infty$ where

$$A_\infty = \begin{bmatrix} (1-k-\nu h)^2+\nu h^2 & 0 \\ 0 & (1-k)^2 \end{bmatrix}; \quad b_\infty = \begin{bmatrix} 1+\big((k+\nu h)^2+\nu h^2\big)r \\ k^2 r \end{bmatrix}.$$

Proof: The proof follows from the previous Lemma by noting that for fixed ν then $\lim_{N\to\infty}\alpha_1 = \nu$ and $\lim_{N\to\infty}\alpha_2 = \nu$. ∎

The previous lemma implies that for large number of agents, the optimal gains h and k are independent of the number of agents N for fixed ν, as long as the ratio between $\mathbb{E}[x_0^*\Pi x_0]$ and $\mathbb{E}[(x_0-\bar{x}\mathbf{1})^*\Pi_\perp(x_0-\bar{x}\mathbf{1})]$ remains constant.

IV. Optimal Infinite Horizon for LQ-like rendezvous control

In this section, we want to find the optimal values for the control gains k, h and reference location $\bar{x}$ that minimize the infinite horizon cost J_∞, i.e. we want to solve the following optimization problem:

$$J_\infty^{opt}(\nu,N) = \min_{k,h,\bar{x}} c^T(I-A)^{-1}b = \quad \min_{k,h,\bar{x}} \quad J_\infty(k,h,\bar{x};\nu,N) = c^T\mathbf{s}, \tag{26}$$

$$\text{subject to} \quad \mathbf{s} = A\mathbf{s}+b. \tag{27}$$

*Note that for $k=0$, we necessarily have $s_t^\perp = 0, \forall t$. This case corresponds to the GPS-free scenario considered in Section VI.

First, note that Equation (27) is independent of $\bar{x}$, therefore the optimal choice for $\bar{x}$ is the value that minimizes $c_2 = \mathbb{E}[(x_0 - \bar{x}\mathbf{1})^* \Pi_\perp (x_0 - \bar{x}\mathbf{1})]$, i.e.:

$$\begin{aligned}
\bar{x}_{opt} &= \operatorname*{argmin}_{\bar{x}} \mathbb{E}[(x_0 - \bar{x}\mathbf{1})^* \Pi_\perp (x_0 - \bar{x}\mathbf{1})] \\
&= \operatorname*{argmin}_{\bar{x}} \mathbb{E}[x_0^* \Pi_\perp x_0] - 2\bar{x}^* \mathbf{1}^* \Pi_\perp \mathbb{E}[x_0] + |\bar{x}|^2 \mathbf{1}^* \Pi_\perp \mathbf{1} \\
&= \operatorname*{argmin}_{\bar{x}} \mathbb{E}[x_0^* \Pi_\perp x_0] - \frac{1}{N}|\mathbb{E}[\mathbf{1}^* x_0]|^2 + N \left| \bar{x} - \frac{1}{N}\mathbb{E}[\mathbf{1}^* x_0] \right|^2 \\
&= \mathbb{E}\left[\frac{1}{N}\mathbf{1}^* x_0\right] = \mathbb{E}[x_{cm}(0)]
\end{aligned}$$

where $x_{cm}(0)$ is the center of mass of agents at time $t = 0$. This means that the optimal reference location to be used in feedback (11) coincides with the a-priori expected center of mass of all agents.

After this optimization step, the optimization problem defined in Equations (26) and (27) is equivalent to

$$J_\infty^{opt}(\nu, N) = \quad \min_{k,h} \quad \overline{c}^T \mathbf{s} \tag{28}$$

$$\text{subject to} \quad \mathbf{s} = A\mathbf{s} + b \tag{29}$$

where we used the facts $\Pi x_0 = x_0 - x_{cm}(0)\mathbf{1}$ and $\Pi_\perp x_0 = x_{cm}(0)\mathbf{1}$, and the components of the cost vector $\overline{c} = [\overline{c}_1 \ \overline{c}_2]$ are given by:

$$\overline{c}_1 = \mathbb{E}[||x_0 - x_{cm}(0)\mathbf{1}||^2] = \sum_{i=1}^{N} \mathbb{E}[|x_i(0) - x_{cm}(0)|^2], \tag{30}$$

$$\overline{c}_2 = N\big(\mathbb{E}[|x_{cm}(0)|^2] - |\mathbb{E}[x_{cm}(0)]|^2\big) = N\text{var}\big(x_{cm}(0)\big). \tag{31}$$

It is interesting to note that, differently from the classic LQ control, the optimal gain k, h depend on the second order statistics of initial agents location. In particular the term $\overline{c}_1$ is proportional to the expected spreading of agents initial position with respect to their initial center of mass, while the second term $\overline{c}_2$ is proportional to the expected distance between the actual agents' center of mass, $x_{cm}(0)$, and its a-priori value, $\mathbb{E}[x_{cm}(0)]$.

The previous optimization problem is highly nonlinear and cannot be solved in closed form. However, some interesting results can still be inferred, which are summarized in the following theorem:

Theorem 2: Consider the optimization problem defined by Equations (28) and (29). Also, consider the corresponding optimal gains $h^{opt}(\nu, N)$ and $k^{opt}(\nu, N)$ as a function of ν and N. Also consider the positive scalar $\overline{c}_1$ and $\overline{c}_2$ defined in Equations (30) and (31). Finally let p and $\bar{\kappa}$ be defined in Equations (7) and (8). Then the following statements are true:

(a) $J_\infty^{opt}(\nu + 1, N) \le J_\infty^{opt}(\nu, N)$,
(b) $J_\infty^{opt}(0, N) \le J_\infty(\bar{\kappa}, 0, \bar{x}^{opt}; 0, N) \le \overline{c}_1 p + \overline{c}_2 \frac{pr^2}{p+2r}$,
(c) $J_\infty^{opt}(N-1, N) = J_\infty(0, \frac{\bar{\kappa}}{N}, \bar{x}^{opt}; N-1, N) = \overline{c}_1 p$,
(d) $\overline{c}_1 p \le J_\infty^{opt}(\nu, N) \le \overline{c}_1 p + \overline{c}_2 \frac{pr^2}{p+2r}, \quad \forall \nu, N$,
(e) $\frac{\overline{c}_2}{\overline{c}_1} \to \infty \Longrightarrow k^{opt}(\nu, N) \to 0, h^{opt}(\nu, N) \to h_{k=0}^{opt}(\nu, N), \ \forall \nu > 0$,
(f) $\frac{\overline{c}_2}{\overline{c}_1} \to 0 \Longrightarrow k^{opt}(\nu, N) \to \bar{\kappa}, h^{opt}(\nu, N) \to 0, \ \forall \nu$.

Proof: See Appendix. ∎

Let us comment upon the previous theorem. Statement (a) claims that if the number of the exchanged messages increases, then the performance improves, as expected since more information is available. Statement (b) states that, even when no communication among agents is available, the total cost is finite and can be achieved by applying a scalar LQ optimal feedback on the expected center of mass $\bar{x}^{opt} = \mathbb{E}[x_{cm}(0)]$. Statement (c) shows that in the other extreme situation when full communication is available, the optimal strategy is to do feedback only on the relative distances with the other agents and the knowledge of any a-priori statistics is irrelevant. Statement (d) provides bounds on the performance of the proposed strategy, and it clearly shows that communication among agents can truly improve performance only if $\overline{c}_2 \gg \overline{c}_1$, i.e. when the uncertainty about the a-priori location of agents' center of mass is much larger than the spreading of agents' initial positions. This implies that a-priori knowledge about agents' statistics can greatly improve the performance of the rendezvous control. Therefore, the common belief that the algorithms which use only relative distance consensus feedback are more energetically efficient as compared to a fixed location feedback is not true in general, since performance depends on statistical distribution of the initial agents' positions. Such statistics are often available and it would be suboptimal not to use them. The last two statements confirm the previous consideration by showing that as $\overline{c}_2$ grows much larger than $\overline{c}_1$, i.e. $\frac{\overline{c}_2}{\overline{c}_1} \to \infty$, then all weight should be placed on the relative distance term h (Statement (e)). Conversely, when $\frac{\overline{c}_2}{\overline{c}_1} \to 0$, then all weight should be placed on the fixed a-priori expectation agents' center of mass (Statement (f)). Moreover, in the limit $\frac{\overline{c}_2}{\overline{c}_1} \to \infty$, i.e. when the uncertainty about the location of agents' center of mass is large, we recover that the optimal strategy coincides with the strategy that uses only relative distance feedback, even when communication is infrequent, i.e. $\nu \ll N$. The numerical example in the next section will help to clarify the statements of the previous theorem.

V. Numerical example

In this section we explore the previous results for a specific choice of agents initial position distribution. We assume that the initial agents' positions are i.i.d. and drawn from the following probability distribution:

$$\begin{aligned} x_i(0) &\sim f(x;\bar{\mu},\sigma_1,\sigma_2) = \int_\mu f_1(x|\mu) f_2(\mu) d\mu, \quad i=1,\ldots,N, \\ f_1(x|\mu) &= \mathcal{N}(\mu,\sigma_1), \\ f_2(\mu) &= \mathcal{N}(\bar{\mu},\sigma_2) \end{aligned}$$

where $\bar{\mu},\sigma_1,\sigma_2$ are known parameters, and $\mathcal{N}$ indicates the scalar gaussian distribution. In particular, $\bar{\mu}$ represents the a-priori expected value for the agents center of mass, σ_2 is the uncertainty we have about true agents' center of mass μ, and σ_1 quantifies the spreading of the agents' position with respect to the true agents' center of mass. Note that if $\sigma_2 = 0$, then we recover the usual gaussian distribution of agents' position, i.e. $x_i(0) \sim \mathcal{N}(\bar{\mu},\sigma_1)$. The distribution above is more general than the usual gaussian distribution, and allows for the modeling of a larger class of scenarios. This is a suitable model, for example, if agents are dropped from an airplane. The agents will land in slightly different positions, however it is generally possible to estimate their spreading represented by the parameter σ_1. Wind, however, can shift the actual agents' center of mass from the expected a-priori value $\bar{\mu}$, but the spreading of the agents among each other, quantified by σ_1, is almost unaffected. The effect of the wind can be taken into account by associating some uncertainty on the exact position of the agents' center of mass through the parameter σ_2.

Besides being useful for some applications, this distribution allows us to derive explicit values for the parameters $\overline{c}_1,\overline{c}_2,\bar{x}_{opt}$ in the previous section. In fact, the optimal choice for the parameter $\bar{x}_{opt}$ is given by:

$$\bar{x}_{opt} = \mathbb{E}\left[\frac{1}{N}\mathbf{1}^* x_0\right] = \mathbb{E}\left[x_i(0)\right] = \mathbb{E}_\mu\left[\ \mathbb{E}\left[x_i(0)|\mu\right]\ \right] = \mathbb{E}_\mu[\mu] = \bar{\mu}.$$

As expected, the optimal choice for this parameter is exactly the expected a-priori value of the agents' center of mass. As for the other two parameters, we have:

$$\begin{aligned} \overline{c}_1 &= \mathbb{E}[||x_0 - x_{cm}(0)\mathbf{1}||^2] = \mathbb{E}[||(I-\frac{1}{N}\mathbf{1}\mathbf{1}^*)x_0||^2] = \text{trace}(\mathbb{E}[\Pi x_0 x_0^*]) \\ &= \text{trace}(\mathbb{E}_\mu \Pi[\mathbb{E}[x_0x_0^*|\mu]]) = \text{trace}(\mathbb{E}_\mu[\Pi(\mathbf{1}\mathbf{1}^*\mu^2+\sigma_1^2 I)]) = \text{trace}(\Pi\sigma_1^2) \\ &= (N-1)\sigma_1^2, \\ \overline{c}_2 &= N\big(\mathbb{E}[|x_{cm}(0)|^2] - |\mathbb{E}[x_{cm}(0)]|^2\big) = \mathbb{E}[x_0^*\Pi_\perp x_0] - N\bar{\mu}^2 = \text{trace}(\Pi_\perp \mathbb{E}[x_0x_0^*]) - N\bar{\mu}^2 \\ &= \text{trace}(\mathbb{E}_\mu[\Pi_\perp(\mathbf{1}\mathbf{1}^*\mu^2+\sigma_1^2 I)])\sigma_1^2) - N\bar{\mu}^2 = \text{trace}(\Pi_\perp(N\mathbb{E}_\mu[\mu^2]+\sigma_1^2)) - N\bar{\mu}^2 \\ &= N(\bar{\mu}^2+\sigma_2^2)+\sigma_1^2 - N\bar{\mu}^2 = N\sigma_2^2+\sigma_1^2 \end{aligned}$$

As expected, the parameter $\overline{c}_1$, which is related to agents' spreading, depends only on σ_1, while $\overline{c}_2$, which quantifies the uncertainty about the a-posteriori agents' center of mass, mainly depends on σ_2. Figure 1 presents an intermediate scenario where $\overline{c}_1 = \overline{c}_2$, which shows that even a small exchange of information can substantially improve performance (left panel) and that more weight is placed on the relative distance feedback as the number of exchanged massages increases (right panel).

Moreover, according to the analysis in the previous section, if $\overline{c}_1 \gg \overline{c}_2$, which in this case is equivalent to stating that $\sigma_1 \gg \sigma_2$, then the performance obtained by just applying feedback on the expected a-priori center of mass $\bar{\mu}$ with no communication among agents, given by $J^{opt}(0,N) \leq \overline{c}_1 p + \frac{pr^2}{p+2r}\overline{c}_2$ (Theorem 2(b)), is not much worse then the ideal case when full communication graph is available, given by $J^{opt}(N-1,N) = \overline{c}_1 p$ (Theorem 2(c)). In other words, this states that if we have a good estimate of the initial agents' center of mass $x_{cm}(0)$, it is not necessary to communicate. This case corresponds to the left side of Figure 2, where $k^{opt} \approx \bar{\kappa}$ and $h^{opt} \approx 0$ (Theorem 2(e)). On the other hand, if $\sigma_1 \ll \sigma_2$, which implies $\bar{c}_1 \ll \bar{c}_2$, then, according to Theorem 2(f), more weight should be placed on the feedback gain h (which depends on the exchanged information) than the gain k (which depends on the expected a-priori center of mass), even for low numbers of exchanged messages ν. This is confirmed in the right side of Figure 2, which corresponds to a scenario for which there is no a-priori information about the agents' center of mass.

VI. GPS-free rendezvous under randomly switching communication topologies

It is important to remark that the considerations in the previous section follow from the assumption that each agent has access to its own position with respect to a common system origin. This is the case, for example, for mobile agents provided with GPS-like sensors. Therefore, it might be unfair to compare them with strategies that use only relative distance information. In fact, there are scenarios where GPS-like sensors are not available or are unreliable such as in indoor or urban environments. Moreover, we have not yet highlighted the performance improvement of the proposed randomly switching communication topology with respect to static communication topologies. Therefore, in this section we analyze the LQ-like total cost defined

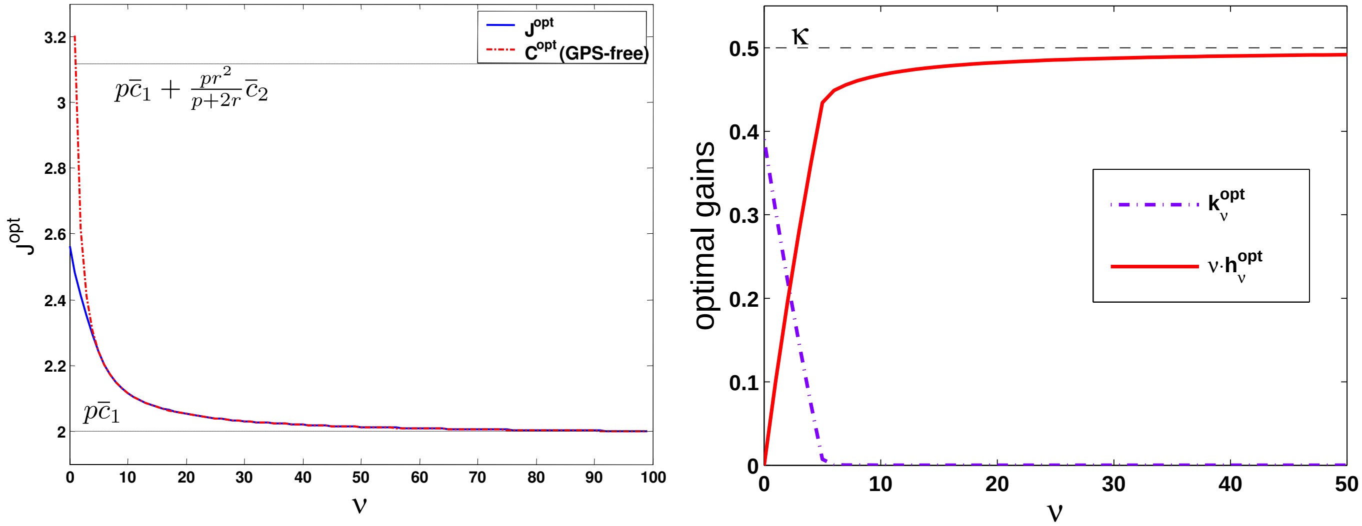

Fig. 1. Optimal cost *(left)* and optimal gains *(right)* as a function of exchanged messages ν for $N = 10000$, $r = 2$, $\bar{c}_1 = \bar{c}_2 = 1$. For this choice of parameters $p = 2$ and $\bar{\kappa} = 0.5$.

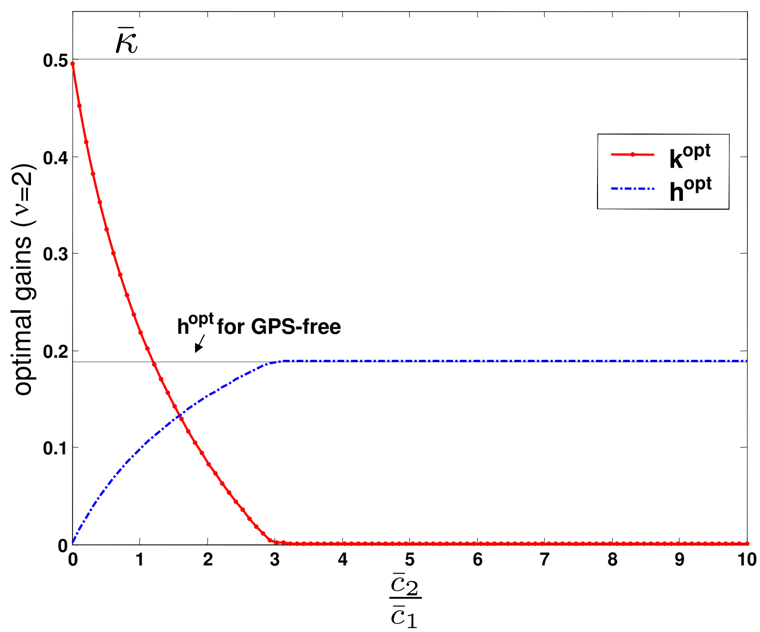

Fig. 2. Optimal gains k^{opt} and h^{opt} as a function of the ratio $\frac{\bar{c}_2}{\bar{c}_1}$, where $\lim_{N\to\infty} \frac{\bar{c}_2}{\bar{c}_1} = \frac{\sigma_2^2}{\sigma_1^2}$. For $\frac{\bar{c}_2}{\bar{c}_1} \to \infty$, then $k^{opt} \to 0$ and $h^{opt} \to h^{opt}_{GPS-free}$, while for $\frac{\bar{c}_2}{\bar{c}_1} \to 0$ then $k^{opt} \to \bar{\kappa}$ and $h^{opt} \to 0$.

in Equation (3) and we set the gain $k = 0$ in the control feedback of Equation (10) obtaining the following purely distributed feedback:

$$u(t) \quad = \quad h\,(E(t) - I)\,x(t). \tag{32}$$

By using arguments based on the cost-to-go functions similar to Section III, it is easy to obtain the following theorem.

Theorem 3: Let us consider the closed loop system defined by dynamics given by Equation (1) and the randomly switching control feedback specified in Equation (32). Then, the LQ performance criterion $C_T(h;\nu,N) = J_T(0,h,\bar{x};\nu,N)$ defined by Equation (3), can be written as

$$C_T(h;\nu,N) = s_0\mathbb{E}[x_0^*\Pi x_0] \tag{33}$$

where s_0 is a nonnegative scalar that can be computed iteratively as follows:

$$s_t = a_{11}(0,h)s_{t+1} + b_1(0,h), \quad s_T = 1 \tag{34}$$

where a_{11}, b_1 are defined in Equations (20) and (21). If $a_{11}(0,h) < 1$, then the infinite horizon cost $C_\infty(h;\nu,N) = \lim_{T\to\infty} J_T(0,h,\bar{x};\nu,N)$ exists and it is given by:

$$C_\infty(h;\nu,N) = \frac{b_1(0,h)}{1-a_{11}(0,h)}\mathbb{E}[x_0^*\Pi x_0]. \tag{35}$$

Let $C_\infty^{opt}(\nu,N) = \min_h C_\infty(h;\nu,N)$, and $h^{opt}(\nu,N)$ the corresponding minimizer, where we made explicit that these are functions of the number of the exchanged messages, $\nu \geq 1$, and number of agents, N. Then we have:

$$C_\infty^{opt}(\nu,N) = s_\infty^{opt}\mathbb{E}[x_0^*\Pi x_0], \tag{36}$$

$$s_\infty^{opt} = 1 + s_\infty^{opt} - \frac{(\alpha_1 s_\infty^{opt})^2}{(\alpha_1^2+\alpha_2)s_\infty^{opt} + (\alpha_1^2 + \frac{N-1}{N}\alpha_2)r}, \tag{37}$$

$$h^{opt} = \frac{\alpha_1 s_\infty^{opt}}{(\alpha_1^2+\alpha_2)s_\infty^{opt} + (\alpha_1^2 + \frac{N-1}{N}\alpha_2)r}. \tag{38}$$

For $N \to \infty$ the previous equations simplifies to:

$$s_\infty^{opt} = 1 + s_\infty^{opt} - \frac{\nu}{\nu+1}\frac{(s_\infty^{opt})^2}{s_\infty^{opt}+r}, \tag{39}$$

$$h^{opt} = \frac{1}{\nu+1}\frac{s_\infty^{opt}}{s_\infty^{opt}+r}. \tag{40}$$

Proof: The first part of the proof follows directly from Theorem 1. In fact, if we set $k=0$, then $s_{t+1}^\perp = s_t^\perp$. Since $s_T^\perp = 0$, it follows that $s_t^\perp = 0, \forall t$. Therefore the Equations (20) and (24) simplify to Equations (34) and (33), respectively. If $T \to \infty$, since s_t is a nonnegative increasing sequence, its limit is finite if and only if $a_{11}(0,h) < 1$, otherwise it grows unbounded. Under this condition $\lim_{T\to\infty} s_0 = s_\infty < \infty$. Moreover, it must be true that $s_\infty = a_{11}(0,h)s_\infty + b_1(0,h)$, or equivalently that $s_\infty = \frac{b_1(0,h)}{1-a_{11}(0,h)}$ from which follows Equation (35).

Now, by standard LQ optimality arguments, it is easy to show that the optimal h^{opt} must be a minimizer of the right-hand-side of Equation (34) at the optimal point, i.e. it must satisfy

$$\frac{d}{dh}\big(a_{11}(0,h)s + b_1(0,h)\big)\Big|_{s=s_\infty^{opt},h=h^{opt}} = 0.$$

From this condition follows directly Equation (38). Also by substitution of Equation (38) into $s_\infty = a_{11}(0,h)s_\infty + b_1(0,h)$ we get the Riccati-like Equation (37).

Finally, the last part of the Theorem follows from the fact that $\lim_{N\to\infty}\alpha_1 = \lim_{N\to\infty}\alpha_2 = \nu$ and $\lim_{N\to\infty}\frac{N-1}{N} = 1$. ■

The previous theorem states that it is possible to compute the optimal gain and the corresponding minimal cost by solving a scalar Riccati-like equation given by Equation (37). Similar to the optimal cost for the general case $J^{opt}(\nu,N)$, it is easy to show that $C_\infty^{opt}(\nu,N) \geq C_\infty^{opt}(\nu+1,N)$, i.e. the performance increases with the number of messages. Also it is easy to verify that for $\nu = N-1$, then $s_\infty^{opt} = p$, $h^{opt} = \frac{\bar{\kappa}}{N}$, and $J_\infty^{opt}(N-1,N) = C_\infty^{opt}(N-1,N)$, i.e. the optimal strategy for the two scenarios is the same when the communication graph is fully connected. The performance of the GPS-free scenario cannot be better than the performance obtained by the mixed feedback in Section IV, i.e. $J_\infty^{opt}(\nu,N) \leq C_\infty^{opt}(\nu,N)$, as graphically illustrated on the left panel of Figure 1. However, this gap rapidly decreases as the number of exchanged massages increases. This is the result of the randomized strategy that allows for fast spreading of information among all agents.

The rapid increase of performance as a function on the number of exchanged messages is well illustrated by the special case where $r=0$ corresponding to the scenario which gives the fastest convergence rate. In this case we have that:

$$r=0,\ N\to\infty \implies s_\infty^{opt} = \frac{\nu+1}{\nu},\ h^{opt} = \frac{1}{\nu+1},\ a_{11}(h^{opt},0) = \frac{1}{\nu+1}.$$

In particular, the rate of convergence, given by the coefficient a_{11}, is independent of the number of agents, and even with a single message exchanged per time step, i.e. $\nu = 1$, the convergence rate is very rapid, since $\mathbb{E}[||x_i(t) - x_j(t)||^2] \leq d\left(a_{11}(h^{opt},0)\right)^t = d\left(\frac{1}{2}\right)^t$, where d is some positive scalar that depends on the initial conditions. It can be shown that the exponential convergence is independent of the number of agents for all values of r and ν, but the proof is not reported here in the interest of space. This is rather remarkable since, in general, the convergence rate decreases as the number of agents increases in many classes of fixed communication graphs, like the circulant graphs [13] or random geometric graphs [21].

VII. Conclusions

In this paper, we studied linear feedback strategies for rendezvous control based on randomized communication protocols. In particular we showed that prior information about agents initial position distributions can greatly improve performance and reduce the need of information exchange among agents. In particular, this is the case when the expected distance from agents' center of mass and its a-priori estimate is small as compared to the agents spreading distance. In fact, in this scenario moving toward the expected center of mass is very close to the optimal strategy.

Another relevant result of this work was to show that randomized communication requires only a small number of exchanged to achieve high performance. In fact, even with a single incoming communication message per time step, the convergence rate does not reduces beyond a certain minimum value even when the number of agents grows to infinity. This is another example of the so called "small world effect", where the average graph distance of any agent pair is small. This suggests that whenever possible, a randomized communication strategy should be applied to achieve fast traveling of information through the network.

Many problems still remain unsolved. The most important one is that communication radius is limited in real mobile robots, while in this work we assumed infinite communication radius. In fact, it is well known that most rendezvous strategies where each agent moves toward the center of mass of its neighbors easily lead to connectivity loss and robots disconnect into separate clusters [23]. One avenue of research in this respect would be the integration of randomized communication strategies with the use of motion strategies that preserve communication connectivity among agents similarly to [24].

APPENDIX

Lemma 4: Let the matrix $E(t)$ uniformly randomly chosen from the set $\mathbf{E}$ defined in Equation (12). Then it has the following properties:

1) $\mathbb{E}[E(t)] = \nu\Pi_\perp - \frac{\nu}{N-1}\Pi$,
2) $\mathbb{E}[E^*(t)E(t)] = \nu^2\Pi_\perp + \frac{\nu(N-\nu)}{N-1}\Pi$,
3) $\mathbb{E}[E^*(t)\Pi_\perp E(t)] = \nu^2\Pi_\perp + \frac{\nu(N-\nu-1)}{(N-1)^2}\Pi$,
4) $\mathbb{E}[E^*(t)\Pi E(t)] = \nu\left(1-\nu\frac{N-2}{(N-1)^2}\right)\Pi$.

Proof:

1) We start by indicating the $i-j$ element of the matrix E with the letter E_{ij}. The matrix E has the property that:

$$\mathbb{P}[E_{ij}=1] = \begin{cases} 0, & i=j, \\ \frac{\nu}{N-1}, & i\neq j. \end{cases}$$

Therefore we have:

$$\mathbb{E}[E] = \mathbb{P}[E_{jj}=1]I + \mathbb{P}[E_{ij}=1, i\neq j](\mathbf{1}\mathbf{1}^* - I) = \frac{\nu}{N-1}(\mathbf{1}\mathbf{1}^* - I) = \nu\Pi_\perp - \frac{\nu}{N-1}\Pi$$

where we use the facts $\mathbf{1}\mathbf{1}^* = N\Pi_\perp$ and $I = \Pi + \Pi_\perp$.

2) Observe that, since each row of $E(t)$ is i.i.d. by construction, then the matrix $\mathbb{E}[E^*(t)E(t)]$ must have the same value on the diagonal terms and the same value on the off-diagonal terms, therefore it can be written as $\mathbb{E}[E^*(t)E(t)] = \alpha\Pi_\perp + \beta\Pi$, where α and β are scalars. Using this representation we can also show that:

$$\alpha\Pi_\perp = \mathbb{E}[E^*E]\Pi_\perp = \mathbb{E}[E^*E\frac{1}{N}\mathbf{1}\mathbf{1}^*] = \mathbb{E}[E^*\frac{\nu}{N}\mathbf{1}\mathbf{1}^*] = \mathbb{E}[E^*]\nu\Pi_\perp = \left(\nu\Pi_\perp - \frac{\nu}{N-1}\Pi\right)\nu\Pi_\perp = \nu^2\Pi_\perp$$

where we used the fact that $\Pi_\perp^2 = \Pi_\perp$ and $\Pi_\perp\Pi = 0$. Therefore $\alpha = \nu^2$. Also note that a generic term on the diagonal can be written as:

$$\mathbb{E}[E^*E]_{ii} = [\alpha\Pi_\perp + \beta\Pi]_{ii} = \frac{\alpha}{N} + \frac{\beta(N-1)}{N}$$

from which it follows:

$$\beta = \frac{N}{N-1}\mathbb{E}[E^*E]_{ii} - \frac{1}{N-1}\alpha = \frac{N}{N-1}\mathbb{E}[E^*E]_{ii} - \frac{\nu^2}{N-1}.$$

We now compute a generic term on the diagonal:

$$\mathbb{E}[E^*E]_{ii} = \mathbb{E}[\sum_{j=1}^N E_{ji}^2] = \sum_{j=2}^N \mathbb{E}[E_{ji}^2] = \sum_{j=2}^N \mathbb{E}[E_{ji}] = \sum_{j=2}^N \mathbb{P}[e_{j1}=1] = (N-1)\frac{\nu}{N-1} = \nu.$$

If we substitute this value into the previous equation we find $\beta = \frac{\nu(N-\nu)}{N-1}$.

3) The computation of $\mathbb{E}[E^*\Pi_\perp E]$ follows along the same lines of the previous point. First of all it is easy to check that also in this case $\alpha = \nu^2$. Then the generic diagonal term is given by:

$$\begin{aligned}\mathbb{E}[E^*\Pi_\perp E]_{ii} &= \mathbb{E}[E^*\tfrac{1}{N}\mathbf{1}\mathbf{1}^*E]_{ii} = \tfrac{1}{N}\mathbb{E}[\textstyle\sum_{j,k} E_{ji}E_{ki}] = \tfrac{1}{N}\left(\textstyle\sum_{j\neq k}\mathbb{E}[E_{ji}]\mathbb{E}[E_{ki}] + \textstyle\sum_j \mathbb{E}[E_{jj}]\right) \\ &= \tfrac{1}{N}\left((N^2-3N+2)\tfrac{\nu^2}{(N-1)^2} + (N-1)\tfrac{\nu}{N-1}\right) = \tfrac{1}{N}\left(\nu^2\tfrac{N-2}{N-1} + \nu\right).\end{aligned}$$

Similar to the previous point, it can be shown that $\beta = \frac{N}{N-1}\mathbb{E}[E^*\Pi_\perp E]_{ii} - \frac{\nu^2}{N-1} = \frac{\nu(N-\nu-1)}{(N-1)^2}$.

4) Finally, the last matrix $\mathbb{E}[E^*\Pi E]$ can be readily obtained by observing that, from $\Pi_\perp + \Pi = I$, it also follows that $\mathbb{E}[E^*\Pi_\perp E] + \mathbb{E}[E^*\Pi E] = \mathbb{E}[E^*E]$, from which we have:

$$\mathbb{E}[E^*\Pi E] = (\nu^2-\nu^2)\Pi_\perp + \left(\frac{\nu(N-\nu)}{N-1} - \frac{\nu(N-\nu-1)}{(N-1)^2}\right)\Pi = \nu\left(1-\nu\frac{N-2}{(N-1)^2}\right)\Pi\ .$$

■

Proof of Theorem 1. The claim is clearly true for $t = T$, where $s_T = 1$ and $s_T^\perp = 0$. We can prove our claim for all other time steps t by induction. To simplify notation we define the following change of coordinate system:

$$z = x - \bar{x}\mathbf{1}$$

therefore the dynamics of the system and the cost-to-go can be written as:

$$\begin{aligned}
u_t &= (-(k+\nu h)I + hE_t)z_t \\
z_{t+1} &= ((1-k-\nu h)I + hE_t)z_t \\
V_t(x_t) &= s_t\mathbb{E}[(z_t+\bar{x}\mathbf{1})^*\Pi(z_t+\bar{x}\mathbf{1})] + s_t^\perp\mathbb{E}[z_t^* Y_\perp z_t] \\
&= s_t\mathbb{E}[z_t^*\Pi z_t] + s_t^\perp\mathbb{E}[z_t^* Y_\perp z_t] \\
&= V_t(z_t)
\end{aligned}$$

where we used the fact that $\Pi\mathbf{1} = 0$. Let us suppose that the claim is true for $t+1$, then we want to show that the claim is true also for time t.

$$\begin{aligned}
V_t(z_t) &= \mathbb{E}[z_t^*\Pi z_z + r||u_t||^2 + V_{t+1}(z_{t+1})] \\
&= \mathbb{E}[z_t^*\Pi z_t + r||u_t||^2 + s_{t+1}z_{t+1}^*\Pi z_{t+1} + s_{t+1}^\perp z_{t+1}^*\Pi_\perp z_{t+1}] \\
&= \mathbb{E}[z_t^*\Pi z_t] + r\mathbb{E}[z_t^*(hE_t-(k+\nu h)I)^*(hE_t-(k+\nu h)I)z_t||^2] + \\
&\quad + s_{t+1}\mathbb{E}[z_t^*\big(hE_t+(1-k-\nu h)I\big)^*\Pi\big(hE_t+(1-k-\nu h)I\big)z_t] + \\
&\quad + s_{t+1}^\perp\mathbb{E}[z_t^*\big(hE_t+(1-k-\nu h)I\big)^*\Pi_\perp\big(hE_t+(1-k-\nu h)I\big)z_t] \\
&= \mathbb{E}[z_t^*\Pi z_t] + r\,\mathbb{E}\big[z_t^*\,\big(h^2\mathbb{E}[E_t^*E_t] - 2h(k+\nu h)\mathbb{E}[E_t] + (k+\nu h)^2 I\big)\,z_t\big] + \\
&\quad + s_{t+1}\,\mathbb{E}\big[z_t^*\,\big(h^2\mathbb{E}[E_t^*\Pi E_t] + 2h(1-k-\nu h)\Pi\mathbb{E}[E_t] + (1-k-\nu h)^2\Pi\big)\,z_t\big] + \\
&\quad + s_{t+1}^\perp\,\mathbb{E}\big[z_t^*\,\big(h^2\mathbb{E}[E_t^*\Pi_\perp E_t] + 2h(1-k-\nu h)\Pi_\perp\mathbb{E}[E_t] + (1-k-\nu h)^2\Pi_\perp\big)\,z_t\big] \\
&= \mathbb{E}[z_t^*\Pi z_t] + r\mathbb{E}\Big[z_t^*\Big(h^2(\nu^2\Pi_\perp + \frac{\nu(N-\nu)}{N-1}\Pi) - 2h(k+\nu h)(\nu\Pi_\perp - \frac{\nu}{N-1}\Pi) + \\
&\quad + (k+\nu h)^2(\Pi+\Pi_\perp)\Big)z_t\Big] + s_{t+1}\,\mathbb{E}\Big[z_t^*\Big(h^2\left(\nu - \nu^2\frac{N-2}{(N-1)^2}\right)\Pi + \\
&\quad + 2h(1-k-\nu h)\Pi(\nu\Pi_\perp - \frac{\nu}{N-1}\Pi) + (k+\nu h)^2\Pi\Big)z_t\Big] + \\
&\quad + s_{t+1}^\perp\,\mathbb{E}\Big[z_t^*\Big(h^2(\nu^2\Pi_\perp + \frac{\nu(N-\nu-1)}{(N-1)^2}\Pi) + \\
&\quad + 2h(1-k-\nu h)\Pi_\perp(\nu\Pi_\perp - \frac{\nu}{N-1}\Pi) + (k+\nu h)^2\Pi_\perp\Big)z_t\Big] \\
&= \mathbb{E}[z_t^*\Pi z_t]\Big(q + rh^2(\frac{\nu(N-\nu)}{N-1} + 2\frac{\nu^2}{N-1} + \nu^2) - 2rkh(\frac{\nu}{N-1} + \nu) + rk^2 + \\
&\quad + s_{t+1}h^2\big(\nu - \nu^2\frac{N-2}{(N-1)^2} + 2\frac{\nu^2}{N-1} + \nu^2\big) + 2s_{t+1}(1-k)h(\frac{\nu}{N-1} + \nu) + s_{t+1}(1-k)^2 + \\
&\quad + s_{t+1}^\perp h^2\frac{\nu(N-\nu-1)}{(N-1)^2}\Big) + \mathbb{E}[z_t^*\Pi_\perp z_t]\Big(\nu^2h^2 - 2\nu h(1-k) - 2\nu^2h^2 + (1-k)^2 + \\
&\quad + 2\nu h(1-k) + \nu^2h^2\Big)(r + s_{t+1}^\perp)
\end{aligned}$$

where we used the fact that E_t is independent of z_t. Finally, by substituting back $z = x - \bar{x}\mathbf{1}$ and using once again the fact $\Pi\mathbf{1} = 0$, we prove the theorem claim. ■

Proof of Theorem 2. To simplify the following derivations we adopt the following notation $J_\nu^{opt} = J^{opt}(\nu, N), h_\nu^{opt} = h^{opt}(\nu, N)$ and similarly for $k_\nu^{opt}, s_\nu^{opt}, s_\nu^{\perp\,opt}$.

(a) Let us consider the first element in the vector in right hand side of Equation (27) evaluated at the minimum cost, which can be written as:

$$s_\nu^{opt} = (1 - k_\nu^{opt} - b_1(\nu)h_\nu^{opt})^2 s_\nu^{opt} + (k_\nu^{opt} + b_1(\nu)h_\nu^{opt})^2 r + (s_\nu^{opt} + \frac{1}{N-1}s_\nu^{\perp\,opt} + \frac{N}{N-1}r)b_2(\nu)(h_\nu^{opt})^2$$

where we made explicit the dependence on the number of exchanged messages ν. Let us choose $k_{\nu+1} = k_\nu^{opt}$ and $h_{\nu+1} =$

$\frac{\nu}{\nu+1}h_\nu^{opt}$, then $s_{\nu+1}^\perp = s_\nu^{\perp\, opt}$ and

$$\begin{aligned}
s_\nu^{opt} &= (1-k_\nu^{opt}-b_1(\nu)h_\nu^{opt})^2 s_\nu^{opt} + (k_\nu^{opt}+b_1(\nu)h_\nu^{opt})^2 r + \\
&\quad + \Big(s_\nu^{opt} + \frac{1}{N-1}s_\nu^{\perp\, opt} + \frac{N}{N-1}r\Big) b_2(\nu)(h_\nu^{opt})^2 \\
&= (1-k_\nu-b_1(\nu+1)h_{\nu+1})^2 s_\nu^{opt} + (k_{\nu+1}+b_1(\nu+1)h_{\nu+1})^2 r + \\
&\quad \Big(s_\nu^{opt} + +\frac{1}{N-1}s_{\nu+1}^\perp + \frac{N}{N-1}r\Big)\frac{(\nu+1)(N-1-\nu)}{\nu(N-1-\nu-1)} b_2(\nu+1)h_{\nu+1}^2 \\
&\geq (1-k_\nu-b_1(\nu+1)h_{\nu+1})^2 s_\nu^{opt} + (k_{\nu+1}+b_1(\nu+1)h_{\nu+1})^2 r + \\
&\quad + \Big(s_\nu^{opt} + \frac{1}{N-1}s_{\nu+1}^\perp + \frac{N}{N-1}r\Big) b_2(\nu+1)h_{\nu+1}^2 = \phi(s_\nu^{opt})
\end{aligned}$$

where $\phi()$ is a linear monotonically increasing operator for fixed $h_{\nu+1}, k_{\nu+1}, s_{\nu+1}^\perp$, i.e. $x_1 \geq x_2 \Rightarrow \phi(x_1) \geq \phi(x_2)$. This implies that, if we set $x_0 = s_\nu^{opt}$ and define $x_{n+1} = \phi(x_n)$, then $x_{n+1} \leq x_n$ for all n. Since this sequence is monotonically decreasing and bounded from below, as $\phi(x) \geq 0$ for all $x \geq 0$, this implies that $\lim_{n\to\infty} x_n = \bar{x} \geq 0$, where $\bar{x}$ is also the unique fixed point of $\bar{x} = \phi(\bar{x})$. Note that $s_\nu^{opt} = x_0 \geq \bar{x} = s_{\nu+1}$, therefore this implies that

$$\begin{aligned}
J_\nu^{opt} &= J_\nu(k_\nu^{opt}, k_\nu^{opt}, \bar{x}^{opt}) = \overline{c}_1 s_\nu^{opt} + \overline{c}_2 s_\nu^{\perp\, opt} \\
&\geq \overline{c}_1 s_{\nu+1} + \overline{c}_2 s_{\nu+1}^\perp = J_{\nu+1}(k_\nu^{opt}, \frac{\nu}{\nu+1}h_\nu^{opt}, \bar{x}^{opt}) \\
&\geq \min_{h_{\nu+1},k_{\nu+1}} J_{\nu+1}(k_{\nu+1}, k_{\nu+1}, \bar{x}^{opt}, x_0) = J_{\nu+1}^{opt}.
\end{aligned}$$

(b) If we set $\nu = 0$ the optimization problem simplifies to:

$$\begin{aligned}
J_0^{opt} &= \min_{k,h} J_0(k,h,\bar{x}^{opt}) = s\overline{c}_1 + s^\perp \overline{c}_2 \\
\text{subject to} &\quad s = 1 + (1-k)^2 s + k^2 r, \\
&\quad s^\perp = (1-k)^2 s^\perp + k^2 r.
\end{aligned}$$

If we set $k = \bar{\kappa}$, then $s = p$. Since $s^\perp = \frac{kr^2}{2-k}$, if we substitute $k = \bar{\kappa} = \frac{p}{p+r}$, then we get $s^\perp = \frac{pr^2}{p+2r}$. This gain choice is not necessarily optimal, therefore we have

$$J_0^{opt} \leq J_0(\bar{\kappa}, 0, \bar{x}^{opt}, x_0) = \overline{c}_1 p + \overline{c}_2 \frac{pr^2}{p+2r}.$$

(c) If we set $\nu = N-1$ the optimization problem simplifies to:

$$\begin{aligned}
J_{N-1}^{opt} &= \min_{k,h} J_{N-1}(k,h,\bar{x}^{opt},x_0) = s\overline{c}_1 + s^\perp \overline{c}_2 \\
\text{subject to} &\quad s = 1 + (1-k-Nh)^2 s + (k+Nh)^2 r, \\
&\quad s^\perp = (1-k)^2 s^\perp + k^2 r.
\end{aligned}$$

If we define the new variable $\xi = k + Nh$ it is clear that the first constraint equation is independent from the second, therefore they can be minimize separately. The second constraint is minimized for $k_{N-1}^{opt} = 0$ which implies $s^\perp = 0$. Therefore, the optimization reduces to minimize s subject to $s = 1 + (1-Nh)^2 s + (Nh)^2 r$ which is obtained by setting $Nh_{N-1}^{opt} = \bar{\kappa}$ from which it follows $s = p$, which concludes the proof.

(d) This statement follows directly from statements (a),(b) and (c).

(e) Without loss of generality we can set $\overline{c}_1 = 1$ and let $\overline{c}_2 \to \infty$. We claim that $s^\perp \to 0$. In fact if this is not the case, then there exist a sequence of $\{\overline{c}_{2n}\}_{n=1}^\infty$ such that $\overline{c}_{2n} \to \infty$ and the corresponding $s^\perp \geq \epsilon > 0$. This means that also $J^{opt} \geq \overline{c}_2 \epsilon \to \infty$. This is not possible since if we choose $k = 0$, then $s^\perp = 0$, and also we have $s < \infty$ for any $\nu > 0$, which implies that the corresponding cost is finite, i.e. $J < \infty$. Since $J^{opt} \leq J < \infty$, then $s^\perp \to 0$ for $\overline{c}_2 \to \infty$. However $s^\perp \to 0$ if and only if $k \to 0$, therefore $k_\nu^{opt} \to 0$ for all ν. By continuity it follows that $h_\nu^{opt} \to h_{\nu,k=0}^{opt}$, i.e. the optimal choice of h when setting $k = 0$. This scenario is considered in Section VI.

(f) The proof is similar to the previous point. Without loss of generality we set $\overline{c}_2 = 1$ and let $\overline{c}_1 \to \infty$. We claim that $s_\nu^{opt} \to \bar{s}$, where $\bar{s} = \min_{h,k} s_\nu(h,k)$, which is achieved for $h_\nu = 0$ and $k_\nu = \bar{\kappa}$ and gives $\bar{s} = p$, $s_\nu^\perp = \frac{r^2 p}{p+2r}$ and $J_\nu(\bar{\kappa}, 0) = \overline{c}_1 p + \overline{c}_2 \frac{r^2 p}{p+2r} = \overline{c}_1 p + \frac{r^2 p}{p+2r}$. If $s_\nu^{opt} \to \bar{s}$ is not true, then there exist a sequence $\{\overline{c}_{1n}\}_{n=1}^\infty$ and a positive scalar $\epsilon > 0$, such that $s_\nu^{opt} \geq p + \epsilon$. The corresponding optimal cost is given by $J_\nu^{opt} = \overline{c}_{1n} s_\nu^{opt} + s_\nu^\perp \geq \overline{c}_{1n}(p+\epsilon)$. Then, there exists $\bar{n}$ such that $\overline{c}_{1n}\epsilon > (p-1)$ for all $n > \bar{n}$. This implies that $J_\nu^{opt} \geq \overline{c}_{1n}(p+\epsilon) > J_\nu(\bar{\kappa}, 0)$ for $n > \bar{n}$, which is a contradiction since $J_\nu^{opt} \leq J_\nu(k,h)$ for all k, h. Therefore, the claim that $s_\nu^{opt} \to \bar{s} = p$ is true. This also implies that $k_\nu^{opt} \to \bar{\kappa}$ and $h_\nu^{opt} \to 0$ for all $\nu \geq 0$, which concludes the proof. ■

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