Linear Encoder-Decoder-Controller Design over Channels with Packet Loss and Quantization Noise

S. Dey, A. Chiuso and L. Schenato

Abstract— **In this paper we consider the problem of designing coding and decoding schemes for linear control design of a scalar unstable stochastic linear system in the presence of a wireless communication channel between the sensor and the estimator. In particular, we consider a communication channel which is prone to packet loss and includes quantization noise due to its limited capacity. We first study the case of perfect channel feedback, where the transmitter is aware of the quantization noise and the packet loss history of the channel. We show that in this case, the optimal strategy among all possible linear encoders corresponds to the transmission of the Kalman filter innovation (the difference between the filtered state estimate at the transmitter and the predicted state estimate at the receiver) similarly to the differential pulse-code modulation (DPCM). Although the critical Signal-to-Quantization Noise Ratio (SQNR) required for stabilizing the system is the same for innovation forwarding as well as measurement forwarding at the transmitter, the latter is strictly suboptimal in terms of control performance. For the case of imperfect feedback, we assume that the channel feedback or acknowledgement is randomly lost with a certain erasure probability, rendering the transmitter ignorant of the control action taken by the receiver and subsequently applied to the plant. We propose several heuristic strategies for a suboptimal Kalman filter design at the transmitter based on estimation of the channel feedback status and compare their performances via numerical simulation studies.**

I. Introduction

The interplay between control stability and communication channels non-idealities has attracted considerable attention in past decade, mainly driven by the success of wireless communication and its penetration into automation and control applications. From a theoretical perspective, we have witnessed the convergence of control theory, communication theory and information theory which have obtained remarkable and interesting results in terms of the ultimate performance limitations which take into account both the dynamical systems characteristic, typically their unstable eigenvalues and non-minimum phase zeros, and the channel characteristic, typically its capacity [1], [2], [3], [4], [5], [6], [7], [8], [9], [10].

Nonetheless, there are still important open questions that need to be answered. For example most of the results are obtained for scenarios in which the transmitter has full knowledge of what happened over the channel in the past (perfect channel feedback), which guarantees several separation principles both in terms of controller/observer design and of source/channel coding, see e.g. [11] for early references. Differently if the channel feedback is not present or it is "imperfect", very few results are available and mainly based on heuristics [12], [13], [14], [15], [16]. Another difficulty raises when both packet loss and quantization are considered simultaneously. In fact, if these two limitations are considered separately, the resulting optimal strategies can be quite different. For example, in a scenario with packet loss only, Gupta et al. [17] showed that the optimal strategy is to sent the estimate of the state over the channel, which does not even require channel feedback to the transmitter and it is therefore quite attractive. Differently, if only rate limitation, i.e. the maximum number of symbols that can be transmitted per unit of time, is considered, then differential coding results to be the optimal strategy as in differential pulse-code modulation [18]. Obviously, this strategy is rather different from the one that transmits the complete state estimate. Recently, in [19] we have shown that a strategy inspired by DPCM, is optimal in the context of remote estimation when considering both packet loss and quantization limitations for a scalar stable system when full channel feedback is present. However, the optimal strategy in the presence of imperfect channel feedback remains elusive and only sensible heuristic have been proposed in [19]. In the context of closed loop unstable control system, simultaneous analysis of packet loss and quantization has been studied in [20] assuming that the transmitter simply forwards a quantized version of the raw measurement.

In this work we extend the results of [19] and [20] by considering the possibility to pre-process the raw measurement at the transmitter. We show that the optimal strategy when full channel feedback is available at the transmitter is to send the difference from the estimated state at the transmitter and the predicted state at the receiver as in [19] and to build a Kalman filter and a state feedback with constant gain at the receiver as in [20]. However, although the performance is improved as compared to strategy proposed in [20], the stability region is the same. In the imperfect channel scenario, we propose a number of heuristics similarly to those proposed in [19]. However, differently from remote estimation, in closed loop systems the effect of packet loss between the transmitter and the controller appears implicitly in the measurements observed at the transmitter even in the absence of any channel feedback. As so, we propose an on-line strategy to estimate whether a packet has been

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lost or received in the same spirit of [16] and we use it to send a differential signal, which is observed to provide better performance over the other strategies in extensive simulations.

II. System Model and Problem Formulation

In this section, we provide the system model of the networked control system under consideration along with the LQG problem formulation.

A. Plant Model

We consider a discrete-time linear scalar time-invariant *unstable* plant as given below:

$$\begin{aligned} x_{t+1} &= a x_t + b u_t + w_t \\ y_t &= c x_t + v_t \end{aligned} \tag{1}$$

where x_t is the scalar state, u_t is the scalar control input, and y_t is the scalar measurement or plant output. w_t, v_t are also scalar process and measurement noise processes respectively, independent and identically Gaussian distributed with zero mean and variances σ_w^2, and σ_v^2, respectively. We make the usual assumption that w_t, v_t, x_0 are mutually independent. Finally, a, b and c are the state, input and output coefficients, respectively. In our networked control system model, a sensor senses the plant output y_t and transmits a suitable signal s_t after some pre-processing to a remote estimator/controller over a communication channel. We restrict s_t to be a causal linear function of the measurements $\mathcal{Y}_t \triangleq \{y_0, y_1, \ldots, y_t\}$. The remote estimator/controller block constructs a state estimate and design a linear feedback controller which generates the control input u_t to be fed back to the plant, see Figure 1. The objective of the control design problem is to find optimal $\{s_t\}$ and $\{u_t\}$ such that the average cost $J = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \mathbb{E}[y_t^2]$ is (finite and) minimized.

B. Communication channel

We assume that the channel between the sensor (which senses the plant output) and the remote estimator/controller is subject to both bandwidth constraints and packet loss. Thus, the signal transmitted from the sensor is quantized by a quantizer of fixed but sufficiently high rate to produce a quantized signal s_t^q. Under a fine quantization assumption, it has been established by multiple authors including our previous work [19] that the error due to quantization can be represented as an additive white Gaussian noise (AWGN) which is also independent of the input signal, with a variance that is proportional to the variance of the input signal s_t. Thus, the quantized signal can be written as $s_t^q = s_t + n_t$, where n_t is an additive white Gaussian noise (AWGN) with zero mean and variance $\frac{\mathbb{E}[s_t^2]}{\Lambda}$ (note also that $\mathbb{E}[s_t] = 0$), where Λ is known as the Signal-to-Quantization Noise Ratio (SQNR).

Remark 1: The validity of the additive quantization noise model for high rate uniform scalar quantization has been rigorously shown in [21] for continuous input densities, and see also [22] for similar studies. It has been however shown in these papers as well as many other recent literature such as in [23] that although in principle only high rate quantization theory justifies such an additive white quantization noise model, in practice this model holds as a very good approximation for moderate rate quantization. If fact, it was shown in [19] via numerical simulations that, a uniform scalar quantizer with only 3-4 bits of quantization per sample used to quantize the signal s_t provides results that are sufficiently close to the theoretical values based on the additive noise model proposed in this work. Note that in a wireless local area network (WLAN) with orders of megabits per second data rates (even when shared amongst multiple links), it is not unreasonable to expect 3-4 bits per sample with a sampling rate of say 0.1 MHz which is likely to be sufficient for most physical dynamical systems. Thus, this additive white quantization noise model is also suitable for use in practical implementation of estimation over lossy wireless links.

The communication channel is also subject to a packet loss process $\gamma_t \in \{0, 1\}$, which is modelled as a independent and identically distributed (*i.i.d.*) Bernoulli process with $P(\gamma_t = 0) = \epsilon_\gamma$, which is known as the packet loss probability. When $\gamma_t = 1$, the receiver receives s_t^q perfectly. But when $\gamma_t = 0$, the information is lost and the receiver does not receive anything. We assume that the receiver sends a packet acknowledgement signal ACK/NACK back to the transmitter to indicate whether it has received the packet. In the case of full channel feedback, the transmitter has exact knowledge of the packet loss sequence $\{\gamma_t\}$, whereas in the case of imperfect channel feedback, the ACK/NACK packet can also be lost randomly according to another i.i.d. Bernoulli erasure process ν_t, which is independent of γ_t. When $\nu_t = 1$, the transmitter knows the exact value of γ_t, whereas when $\nu_t = 0$, the transmitter does not know γ_t. We also denote $P(\nu_t = 0) = \epsilon_\nu$. It is assumed that when the ACK/NACK packet is received, it is decoded correctly, as a 1 bit information can be easily coded with strong error correcting codes for reliable decoding.

The feedback control channel between the receiver (also called the remote estimator/controller) is assumed to be perfect with no delay, packet loss or quantization related losses.

C. Remote estimator/controller

The remote estimator receives the intermittent sequence $\mathcal{Z}_t \triangleq \{z_0, z_1, \ldots, z_t\}$, where $z_t = \gamma_t(s_t + n_t)$ and has to produce a control input u_t. We shall restrict ourselves to linear strategies. In the forthcoming section we shall see that under perfect channel feedback (i.e. when the transmitter knows the loss sequence $\Gamma_t \triangleq \{\gamma_0, \gamma_1, \ldots, \gamma_t\}$), the optimal strategy will be the cascade of a constant gain state estimator followed by a constant gain (estimated) state feedback controller. This motivates us to consider, also when no channel feedback is present, a similar constant gain "estimator-controller" structure.

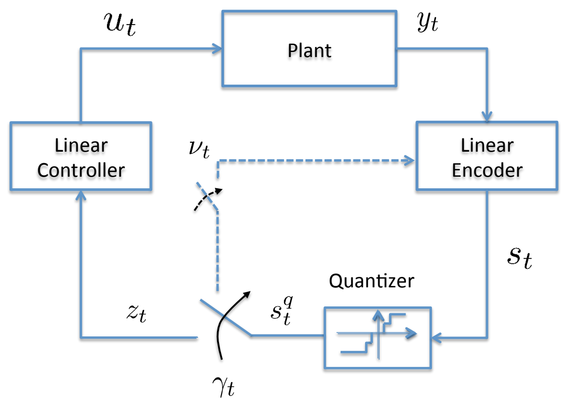

Fig. 1. General scheme of the control system under communication constraints. The sequence γ_t is fed back to the coder only when channel feedback is available.

III. Optimality of innovation forwarding with full channel feedback

We now consider a general linear coding-decoding-controller scheme as follows:

1) a linear coding mechanism produces the signal

$$s_t := \mathcal{L}_\gamma\left(\mathcal{Y}_t, \mathcal{U}_{t-1}, \mathcal{Z}_{t-1}, \Gamma_{t-1}\right) \tag{2}$$

where $\mathcal{L}_\gamma\left(\mathcal{Y}_t, \mathcal{U}_{t-1}, \mathcal{Z}_{t-1}, \Gamma_{t-1}\right)$ is, conditionally on the packet loss sequence $\gamma_{t-1}, ..., \gamma_0$, a linear operator of its arguments $y_t, y_{t-1}, .., y_0$ (the samples to be encoded), $u_{t-1}, .., u_0$ (the past control signal, which can be reconstructed at the transmitter side if channel feedback is available) and $z_{t-1}, .., z_0$ (the past received signals).

2) The signal s_t is sent through a lossy and noisy channel (see Section II-B) and produces the received signal z_t which can be modeled as

$$z_t = \gamma_t(s_t + n_t) \tag{3}$$

where n_t is a white noise signal with variance proportional to the variance of s_t, i.e. $Var\{n_t\} = \frac{Var\{s_t\}}{\Lambda}$ and γ_t is a binary random variable which models the loss events.

3) The controller uses the received signals z_t to build the control action at time t as a (conditionally on the packet loss sequence $\gamma_t, ..., \gamma_0$) linear function $\mathcal{C}_\gamma$ of the past received signals z_τ, $\tau \in [0, t]$ as well as past control signals $u_\tau, \tau \in [0, t)$

$$u_t = \mathcal{C}_\gamma\left(\mathcal{Z}_t, \mathcal{U}_{t-1}, \Gamma_t\right) \tag{4}$$

Note that, in principle, the conditional linear mappings $\mathcal{L}_\gamma$ and $\mathcal{C}_\gamma$ are time-varying. The result of this section is summarized in the next theorem. The remaining part of the section proves the result. For convenience of notation we define

$$\mathcal{I}_t^{Tx} := \{\mathcal{Y}_t, \mathcal{Z}_t, \mathcal{U}_{t-1}, \Gamma_{t-1}\} \quad \mathcal{I}_t^{Rx} := \{\mathcal{Z}_t, \mathcal{U}_{t-1}, \Gamma_t\}$$

which are, respectively, the information set at the transmitter ($\mathcal{I}_t^{Tx}$) and receiver ($\mathcal{I}_t^{Rx}$). Denote the state estimates at the transmitter and receiver as $\hat{x}^{tx}_{t|t} = \mathbb{E}[x_t|\mathcal{I}_t^{Tx}]$ and $\bar{x}_{t|t} = \mathbb{E}[x_t|\mathcal{I}_t^{Rx}]$.

Theorem 1: Consider the linear model (1) controlled through a lossy and SNR limited channel (3) using a linear encoding as in (2) and linear controller as in (4). The optimal linear quadratic strategy

$$\left(\mathcal{C}_\gamma^*, \mathcal{L}_\gamma^*\right) := \arg\min_{\mathcal{C}_\gamma, \mathcal{L}_\gamma} \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} \mathbb{E}\left[y_t^2\right] \tag{5}$$

satisfies the following:

$$s_t = \mathcal{L}_\gamma^*\left(\mathcal{I}_t^{Tx}\right) = \hat{x}^{tx}_{t|t} - \bar{x}_{t|t-1} \tag{6}$$

and

$$u_t = \mathcal{C}_\gamma^*\left(\mathcal{I}_t^{Rx}\right) = -\frac{a}{b}\bar{x}_{t|t} \tag{7}$$

The state estimator at the receiver is given by

$$\begin{array}{rcl} \bar{x}_{t+1|k} & = & a\bar{x}_{t|t} + bu_t = 0 \\ \bar{x}_{t+1|t+1} & = & \bar{x}_{t+1|t} + kz_{t+1} = kz_{t+1} \end{array} \tag{8}$$

with $k = \frac{1}{1+\frac{1}{\Lambda}}$.

Proof:

First of all we consider a finite horizon version of the optimal control problem (5)

$$\begin{array}{c} J_T^{opt}(x_0) = \min\limits_{\mathcal{C}_\gamma, \mathcal{L}_\gamma} \; J_T(x_0) \\ J_T(x_0) := \sum_{t=0}^{T_x} \mathbb{E}_\gamma[y_t^2] \end{array} \tag{9}$$

Let us define $W := c^2$ and the cost-to-go

$$\begin{array}{c} V_t(x_t) := \min\limits_{u_t = \mathcal{C}_\gamma(\mathcal{I}_t^{Rx})} G_t(x_t, u_t), \\ G_t(x_t, u_t) = x_t^2 W + \mathbb{E}_\gamma\left[V_{t+1}(x_{t+1}) \mid \mathcal{I}_t^{Rx}\right] \end{array} \tag{10}$$

where $\mathbb{E}_\gamma$ denotes the expectation with respect to the sequence $\{\gamma_t\}$. It is clear that $V_0(x_0) = J_0^{opt}(x_0)$. We shall now assume that the encoder strategy $\mathcal{L}_\gamma$ is fixed and is equal to (6), i.e. $s_t = \mathcal{L}_\gamma^*(\mathcal{I}_t^{Tx}) = \hat{x}^{tx}_{t|t} - \bar{x}_{t|t-1}$. It has been shown in [19] that this is the optimal encoding strategy to the purpose of estimating the state at the receiver, i.e. it minimizes, among all possible linear encodings, the variance of the state estimation error at the receiver:

$$p_{t|t} := Var\{x_t - \bar{x}_{t|t}\}$$

Let us denote with $p^*_{t|t}$ the optimal value of the estimation error variance achieved with the optimal encoding (6). Similarly we shall use the superscript * whenever the estimation strategy has been fixed equal to (6). It is possible to show that $p^*_{t|t}$ does not depend on the input sequence u_t. This is a straightforward extension of the computations in [19] (see e.g. the equation below (15) in [19]) to account for the presence of measurable inputs[1] in the state update equation (1). Having now fixed this encoding strategy $\mathcal{L}^*{}_\gamma$, we define

$$\begin{array}{c} V_t^*(x_t) := \min\limits_{u_t = \mathcal{C}_\gamma(\mathcal{I}_t^{Rx})} G_t^*(x_t, u_t), \\ G_t^*(x_t, u_t) = x_t^2 W + \mathbb{E}\left[V_{t+1}^*(x_{t+1}) \mid \mathcal{I}_t^{Rx}\right] \end{array} \tag{11}$$

[1]Note that, under the assumption that channel feedback is available, the input u_t is known both at the receiver as well as at the transmitter side.

so that

$$V_0^*(x_0) = J_T^*(x_0) = \min_{\mathcal{C}_\gamma} \ J_T(x_0) \quad s.t. \ \mathcal{L} = \mathcal{L}^* \tag{12}$$

Using Lemma 5.1 in [5] one can prove that

$$\begin{aligned} G_t^*(x_t, u_t) &= x_t^2 W + \mathbb{E}\left[V_{t+1}^*(x_{t+1}) \mid \mathcal{I}_t^{Rx}\right] \\ &= x_t^2 W + WQ + \\ &\quad \mathbb{E}[c_{t+1}^* \mid \mathcal{I}_t^{Rx}] + \\ &\quad +(a\bar{x}_{t|t} + bu_t)^2 W^2 + a^2 W p_{t|t}^* \end{aligned}$$

where

$$c_T^* = 0 \quad c_t^* = W(p_{t|t}^* + Q) + \mathbb{E}[c_{t+1}^* \mid \mathcal{I}_t^{Rx}]$$

Clearly, using the fact that $p_{t|t}^*$ does not depend on $\{u_\tau\}_{\tau \in [0,T]}$, the minimum of $G_t^*(x_t, u_t)$ is achieved for $u_t = -\frac{a}{b}\bar{x}_{t|t} = l\bar{x}_{t|t}$, which takes the form of a dead beat controller on the estimated state. The cost-to-go takes the form:

$$V_t^*(x_t) = x_t^2 W + c_t^* \tag{13}$$

We shall now show that when a different encoding strategy is used the value of the cost-to-go is always greater or equal than $V_t^*(x_t)$ in (13). In order to do so we proceed by induction. First observe that

$$V_T(x_T) = V_T^*(x_T) = x_T^2 W$$

and

$$\begin{aligned} G_{T-1}(x_{T-1}) &= x_{T-1}^2 W + WQ + a^2 W p_{T-1|T-1} \\ &\quad +(a\bar{x}_{T-1|T-1} + bu_{T-1})^2 W^2 \end{aligned}$$

Since, regardless of the choice of the input signal $\{u_\tau\}_{\tau \in [0,T]}$, $p_{T-1|T-1} \le p_{T-1|T-1}^*$, it follows that

$$G_{T-1}^*(x_{T-1}, u_{T-1}) \le G_{T-1}(x_{T-1}, u_{T-1})$$

and thus

$$V_{T-1}^*(x_{T-1}) \le V_{T-1}(x_{T-1})$$

Now, assume $V_{t+1}^*(x_{t+1}) \le V_{t+1}(x_{t+1})$ holds true for $t < T-1$. From (10) and (11) it follows that

$$G_t^*(x_t, u_t) \le G_t(x_t, u_t)$$

and thus, minimizing over u_k we have

$$V_t^*(x_t) \le V_t(x_t)$$

This concludes the inductive argument and proves that

$$V_0^*(x_0) \le V_0(x_0) \quad \forall \, \mathcal{L}_\gamma$$

thus proving that $\mathcal{L}_\gamma^*$ is the optimal strategy.

We show now that the state estimator at the receiver $\bar{x}_{t|t}^{rx}$ can be found via a constant gain recursion of the form

$$\bar{x}_{t+1|t+1} = a\bar{x}_{t|t} + bu_t + kz_{t+1} \tag{14}$$

Following the derivation in [19] (see in particular equation (12) in [19]) it is simple to show that $\bar{x}_{k|k}$ satisfies:

$$\begin{aligned} \bar{x}_{t+t|t+t} &= \mathbb{E}_\gamma[x_{t+1}|\mathcal{I}_{t+1}^{Rx}] \\ &= \mathbb{E}_\gamma[x_{t+1}|\mathcal{I}_t^{Rx}, u_t] + \mathbb{E}_\gamma[x_{t+1}|z_{t+1}] \\ &= \bar{x}_{t+1|t} + \mathbb{E}_\gamma[x_{t+1}|z_{t+1}] \\ &= a\bar{x}_{t|t} + bu_t + \frac{\mathbb{E}_\gamma[x_{t+1}s_{t+1}]}{\mathbb{E}_\gamma[s_{t+1}^2](1+\frac{1}{\Lambda})} z_{t+1} \\ &= a\bar{x}_{t|t} + bu_t + \frac{1}{1+\frac{1}{\Lambda}} z_{t+1} \\ &= a\bar{x}_{t|t} + bu_t + kz_{t+1} \end{aligned} \tag{15}$$

which shows that satisfies a recursion of the form (14) with a constant gain $k = \frac{1}{1+\frac{1}{\Lambda}}$.

It follows that the optimal encoding-decoding and control strategies in the infinite horizon case ($T \to \infty$) has exactly the same structure with $l = -\frac{a}{b}$ and $k = \frac{1}{1+\frac{1}{\Lambda}}$. This concludes the proof.

■

IV. Optimal LQG cost with full channel feedback: innovation forwarding vs. measurement forwarding

As we have seen in the previous section the optimal linear encoding/decoding/control strategy in the full channel feedback case is obtained by sending the state innovation $s_t := \mathbb{E}[x_t|\mathcal{I}_t^{Tx}] - \mathbb{E}[x_t|\mathcal{I}_{t-1}^{Rx}] = \hat{x}_{t|t}^{tx} - \bar{x}_{t|t-1}$ and performing a constant gain state feedback control $u_t = -\frac{a}{b}\bar{x}_{t|t}$. The estimation error $\tilde{x}_{t|t}^{tx} = x_t - \hat{x}_{t|t}^{tx}$ is Gaussian distributed with zero-mean and a steady-state variance p_∞^{tx}. It can be shown from standard Kalman filter steady state analysis that $p_\infty^{tx} = \frac{\sigma_v^2 \bar{p}_\infty^{tx}}{\bar{p}_\infty^{tx} + \sigma_v^2}$ where $\bar{p}_\infty^{tx}$ is the steady state prediction error covariance at the transmitter and can be computed as a solution to an algebraic Riccati equation as

$$\begin{aligned} \bar{p}_\infty^{tx} = \frac{1}{2}(&\sigma_v^2(a^2-1) + \sigma_w^2 \\ &+ \sqrt{(\sigma_v^2(a^2-1) + \sigma_w^2)^2 + 4\sigma_v^2\sigma_w^2}) \end{aligned} \tag{16}$$

For ease of analysis, we henceforth assume that the transmitter Kalman filter has reached its steady state. When the transmitter has full channel feedback, it can also reconstruct a copy of the receiver constant gain Kalman filter (after receiving γ_t via the ACK/NACK packet), which is given by (8). When the optimal control $u_t = -\frac{a}{b}\bar{x}_{t|t}$ is used the state update takes the form:

$$\begin{aligned} \bar{x}_{t|t} &= k\gamma_t(\hat{x}_{t|t}^{tx} - \bar{x}_{t|t-1} + n_t) \\ \bar{x}_{t+1|t} &= a\bar{x}_{t|t} + bu_t = 0 \end{aligned} \tag{17}$$

where $\bar{x}_{t+1|t}, \bar{x}_{t|t}$ are the receiver predicted and filtered state estimates, respectively, and k is the constant gain of the optimal linear filter employed at the receiver. Note that the transmitter sends the innovation signal $s_t = \hat{x}_{t|t}^{tx} - \bar{x}_{t|t-1} = \tilde{x}_t - \tilde{x}_{t|t}^{tx}$ after quantization, where $\tilde{x}_t = x_t - \bar{x}_{t|t-1}$ is the receiver prediction error. As explained earlier, the effect of quantization is modelled (under high rate quantization) as an additive zero-mean white Gaussian noise n_t with variance $\sigma_n^2 = \mathbb{E}[s_t^2]/\Lambda$. It can be shown following the analysis in [19] that $\mathbb{E}[s_t^2] = \mathbb{E}[\tilde{x}_t^2] - p_\infty^{tx}$. For the subsequent analysis, without loss of generality, we assume $b = 1, c = 1$ to simplify the expressions. The receiver prediction error follows the dynamics

$$\tilde{x}_{t+1} = a(1-k\gamma_t)\tilde{x}_t + ka\gamma_t\tilde{x}^{tx}_{t|t} + (w_t - ka\gamma_t n_t) \quad (18)$$

Denoting $\mathbb{E}[\tilde{x}_t^2] = p$ one can easily see that $\sigma_n^2 = \frac{(p-p_\infty^{tx})}{\Lambda}$. Note that here the expectation is taken over the noise processes w_t, n_t and the packet loss sequence Γ_t. Since $y_t = x_t + v_t$ and $\tilde{x}_t := x_t - \bar{x}_{t|t-1} = x_t$, it follows also that the optimal control cost is given by $J = \lim_{t\to\infty} \mathbb{E}[(\tilde{x}_t)^2] = p + \sigma_v^2$.

Next, we can derive the modified algebraic Riccati equation (MARE) satisfied by p, given as follows:

$$\begin{aligned} p &= (1-\epsilon_\gamma)a^2(1-k)^2p + \epsilon_\gamma a^2 p + \sigma_w^2 \\ &+ (1-\epsilon_\gamma)k^2(p_\infty^{tx} + \sigma_n^2)a^2 \\ &+ 2(1-\epsilon_\gamma)k(1-k)a^2p_\infty^{tx} \end{aligned} \quad (19)$$

where we have used the fact that $\mathbb{E}[\tilde{x}^{tx}_{t|t}\tilde{x}_t] = p_\infty^{tx}$. By equating the corresponding matrix elements of the left hand side with the right hand side of (19), we obtain (after some algebraic manipulations)

$$p = \frac{\sigma_w^2 + a^2k(1-\epsilon_\gamma)p_\infty^{tx}\left(2-k(1+\frac{1}{\Lambda})\right)}{\left[1-\epsilon_\gamma a^2 - a^2(1-\epsilon_\gamma)\left((1-k)^2+\frac{k^2}{\Lambda}\right)\right]} \quad (20)$$

We can now compute the optimal cost for the full channel feedback case.

Theorem 2: With perfect causal knowledge of packet loss sequence Γ_t at the transmitter, the optimal controller gain and the optimal filter gain at the receiver are given by $l^{opt} = -a$ (dead-beat control) and $k^{opt} = \frac{1}{1+1/\Lambda}$ The corresponding steady state control cost $\lim_{t\to\infty}\mathbb{E}[y_t^2]$ is given by $J_{if}^{opt} = p^{opt} + \sigma_v^2$, where

$$p^{opt} = \frac{a^2(1-\epsilon_\gamma)\frac{p_\infty^{tx}}{1+1/\Lambda} + \sigma_w^2}{(1-\epsilon_\gamma a^2) - \frac{a^2(1-\epsilon_\gamma)}{\Lambda+1}} \quad (21)$$

which is finite under the assumption that SQNR Λ is larger than a critical threshold $\Lambda^{opt} = \frac{a^2-1}{1-\epsilon_\gamma a^2}$ with $\epsilon_\gamma < \frac{1}{a^2}$.

It is instructive to compare this result against the corresponding analysis for measurement forwarding performed in [24], where the sensor simply forwards each measurement y_t as a packet, and hence does not need to know packet acknowledgement process. In this case, it was shown in [24] that the optimal controller gain is the sane dead-beat control $l^{opt} = -a$, and the critical SQNR threshold for stabilization is also $\Lambda^{opt} = \frac{a^2-1}{1-\epsilon_\gamma a^2}$ with $\epsilon_\gamma < \frac{1}{a^2}$., but the optimal control cost is given by $J_{mf}^{opt} = \bar{p} + \sigma_v^2$ where $\bar{p}$ satisfies the following quadratic equation

$$\bar{p} = a^2\bar{p} + \sigma_w^2 - \frac{(1-\epsilon_\gamma)}{1+1/\Lambda}\frac{\bar{p}}{\bar{p}+\sigma_v^2} \quad (22)$$

In this case, the optimal constant gain for the receiver filter is given by $\bar{k} = \frac{1}{1+1/\Lambda}\frac{\bar{p}}{\bar{p}+\sigma_v^2}$. Further details can be found in [24]. One can therefore draw the conclusion that with innovation forwarding, one cannot gain in terms of the critical SQNR threshold, in that it does not enlarge the stability margin. However, the optimal constant gain for the receiver filter is given by a simple expression, which is also the same for innovation forwarding with full channel feedback for remote estimation as studied in [19]. This is expected because the optimal control design with innovation forwarding also employs the same constant gain receiver filter as remote estimation with innovation forwarding, as explained in Section III, where we established the optimality of innovation forwarding at the transmitter when restricted to linear encoding.

The optimality of innovation forwarding is reflected not in the enlargement of stability margin, but in terms of performance, that is, with a reduced control cost J_{if}^{opt} which is smaller than J_{mf}^{opt}, as guaranteed by Theorem 1.

In the remaining part of the Section we provide an analysis of the difference $J_{mf}^{opt} - J_{if}^{opt}$ which also reveals that it is and increasing function of the packet loss probability ϵ_γ.

Let us first introduce a few notations. Define $b_p = \sigma_v^2(a^2-1) + \sigma_w^2$, $c_p = \sigma_w^2\sigma_v^2$, and $\delta_p = \frac{a^2(1-\epsilon_\gamma)}{1+\frac{1}{\Lambda}} - (a^2-1)$. Note that when $\Lambda > \Lambda^{opt} = \frac{a^2-1}{1-\epsilon_\gamma a^2}$ with $\epsilon_\gamma < \frac{1}{a^2}$., $\delta_p > 0$. Based on the above notation, it is easy to show that $\bar{p}$, as a solution of (22) satisfies $\bar{p} = \frac{1}{2\delta_p}(b_p + \sqrt{b_p^2 + 4c_p\delta_p})$, whereas p^{opt} from (21) satisfies

$$p^{opt} = \frac{1}{\delta_p}\left((\delta_p + a^2 - 1)\frac{\sigma_v^2\bar{p}_\infty^{tx}}{\sigma_v^2 + \bar{p}_\infty^{tx}} + \sigma_w^2\right) \quad (23)$$

where recall that $\bar{p}_\infty^{tx}$ is the steady-state transmitter prediction error variance, given by $\bar{p}_\infty^{tx} = \frac{1}{2}(b_p + \sqrt{(b_p^2 + 4c_p)})$. We also have the difference between the optimal control cost with measurement forwarding and innovation forwarding as $J_{mf}^{opt} - J_{if}^{opt}$, given by $\bar{p} - p^{opt} = \tilde{p}_{diff}$. Using the expressions for $\bar{p}$ and p^{opt} above, one can show the following:

$$\begin{aligned} \frac{d\tilde{p}_{diff}}{d\delta_p} &= \frac{1}{2\delta_p^2}\Bigg[-b_p + 2(\sigma_w^2 + (a^2-1)p_\infty^{tx} \\ &- \frac{b_p^2 + 2c_p\delta_p}{\sqrt{b_p^2 + 4c_p\delta_p}}\Bigg] \\ &< \frac{1}{2\delta_p^2}\Bigg[-b_p + 2(\sigma_w^2 + (a^2-1)\sigma_v^2 \\ &- \frac{b_p^2 + 2c_p\delta_p}{\sqrt{b_p^2 + 4c_p\delta_p}}\Bigg] \\ &= \frac{1}{2\delta_p^2}\left[b_p - \frac{b_p^2 + 2c_p\delta_p}{\sqrt{b_p^2 + 4c_p\delta_p}}\right] < 0 \end{aligned} \quad (24)$$

where the first inequality follows from the fact that $p_\infty^{tx} = \frac{\sigma_v^2\bar{p}_\infty^{tx}}{\sigma_v^2 + \bar{p}_\infty^{tx}} < \sigma_v^2$, and the second inequality follows from simple algebra, and we make use of the definition of b_p. Recall that $\delta_p > 0$ when $\Lambda > \Lambda^{opt} = \frac{a^2-1}{1-\epsilon_\gamma a^2}$ with $\epsilon_\gamma < \frac{1}{a^2}$. It is also obvious that δ_p is maximum when $\epsilon_\gamma = 0$ (no packet loss) and $\Lambda = \infty$ (no quantization noise). Therefore the maximum value of $\delta_p = 1$. It follows from (24) that

$\tilde{p}_{diff}$ is a decreasing function of δ_p, and attains its minimum value when $\delta_p = 1$. This minimum value is given by (from simple substitution in the expressions of $\bar{p}$ and p^{opt} above) $\bar{p}_\infty^{tx} - (a^2 p_\infty^{tx} + \sigma_w^2) = 0$, since $\bar{p}_\infty^{tx}, p_\infty^{tx}$ are the transmitter steady-state prediction error and filtering error variances, respectively. Thus we have the following result.

Theorem 3: The difference between the optimal control cost with measurement forwarding and innovation forwarding with perfect acknowledgements, $J_{mf}^{opt} - J_{if}^{opt}$, is an increasing function of $\delta_p = \frac{a^2(1-\epsilon_\gamma)}{1+\frac{1}{\Lambda}} - (a^2 - 1)$ as long as $\Lambda > \frac{a^2-1}{1-\epsilon_\gamma a^2}$ with $\epsilon_\gamma < \frac{1}{a^2}$.. If the SQNR $\Lambda > \frac{a^2-1}{1-\epsilon_\gamma a^2}$ is kept fixed, then $J_{mf}^{opt} - J_{if}^{opt}$ is an increasing function of the packet loss probability ϵ_γ. In addition, $J_{mf}^{opt} = J_{if}^{opt}$ precisely when $\delta_p = 1$, or when there is no packet loss and the SQNR is infinity so that there is no quantization noise.

The above theorem clearly quantifies the difference in the control performance between the measurement forwarding and the innovation forwarding strategies.

V. Encoder-Decoder Design with Imperfect Channel Feedback

In this section, we discuss a few suboptimal strategies for encoder-decoder design when the packet acknowledgements are erased randomly according to a Bernoulli process $\nu_t \in \{0, 1\}$ where $P(\nu_t = 0) = \epsilon_\nu$. Hence, the transmitter receives $\hat{\gamma}_t = \gamma_t$ when $\nu_t = 1$, but does not know γ_t when $\nu_t = 0$. We focus on the innovation forwarding scenario since this is the optimal strategy with perfect channel feedback. However, since the transmitter does not know γ_t whenever $\nu_t = 0$, it also does not know the control input u_t generated by the controller at the receiver end. Therefore, we focus on three different strategies for estimating γ_t at the transmitter whenever $\nu_t = 0$, see Figure 2. Below, we describe these three strategies without providing a detailed theoretical analysis, which is the focus of ongoing work. The first two strategies do not use the output y_t to estimate γ_t, while the third one explicitly use it, resulting in a nonlinear scheme. In all of these strategies, we only design the encoder. The decoder and the controller are left as the constant gain filter with k^{opt} and l^{opt}, respectively. In what follows, we assume $b = c = 1$ for simplicity, as before.

Randomized policy

In this case, the transmitter replaces γ_t by $\hat{\gamma}_t$ whenever $\nu_t = 0$ by the following rule: $P(\hat{\gamma}_t = 1 | \nu_t = 0) = \beta$ and $P(\hat{\gamma}_t = 0 | \nu_t = 0) = 1 - \beta$, where $0 \leq \beta \leq 1$. The value of β is optimized by exhaustive search such that $J = \lim_{t\to\infty} \mathbb{E}(y_t^2)$ is minimized. Of course, $\hat{\gamma}_t = \gamma_t$ when $\nu_t = 1$.

Soft estimation

In this case, whenever $\nu_t = 0$, γ_t is replaced by $\mathbb{E}[\gamma_t] = \epsilon_\gamma$ as suggested in [15], and when $\nu_t = 1$, $\hat{\gamma}_t = \gamma_t$, as before. Note that for both the *Randomized policy* and *Soft estimation* policy, the control cost can be obtained via an average covariance analysis of the matrix $\bar{P} = \mathbb{E}[\xi_t \xi_t']$, where

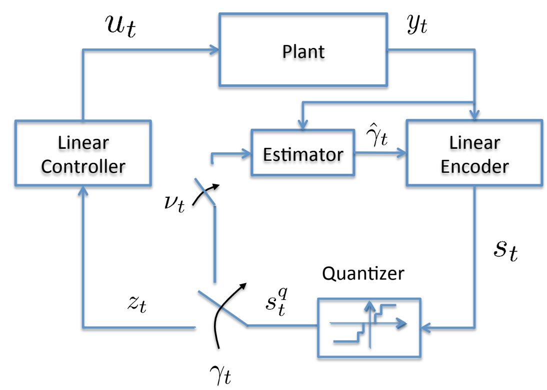

Fig. 2. Control scheme with imperfect feedback: when $\nu_t = 0$ an estimate of the loss sequence is produced by the "Estimator' block. Three strategies are considered: (1) Randomized policy $P[\hat{\gamma}_t = 1 | \nu_t = 0] = \beta$, (2) Soft estimation: $\hat{\gamma}_t = \mathbb{E}[\gamma_t] = \epsilon_\gamma$, (3) Non-linear estimator: $\hat{\gamma}_t = \arg\min_{\gamma\in\{0,1\}} (y_{t+1} - \hat{y}_{t+1}(\gamma))^2$.

$\xi_t = [x_t \hat{x}_{t|t-1}^{tx}]'$, and the control cost is given by $J = \mathbb{E}[x_t^2] + \sigma_v^2 = \bar{p}_{11} + \sigma_v^2$, with $\bar{p}_{11}$ being the first diagonal element of $\bar{P}$. We leave this analysis and corresponding stability margin calculations for future work.

Nonlinear encoder design

In this case, we estimate γ_t from the observed measurement y_{t+1} at the transmitter by a simple distance-based rule, whenever $\nu_t = 0$. Due to the dead beat controller, we have the control input $u_t = -a\bar{x}_{t|t} = -ak^{opt}\gamma_t(\hat{x}_{t|t}^{tx} + n_t)$. Then one can predict a value of y_{t+1} as a function of γ_t as $y_{t+1}(\gamma_t) = a\hat{x}_{t|t}^{tx} - ak^{opt}\gamma_t(\hat{x}_{t|t}^{tx} + n_t)$. We estimate γ_t as $\hat{\gamma}_t = \arg\min_{\gamma\in\{0,1\}} (y_{t+1} - \hat{y}_{t+1}(\gamma))^2$, similarly to [16]. Clearly, in this case, since $\hat{\gamma}_t$ is a nonlinear function of y_{t+1}, the encoder implements a nonlinear filter at the transmitter, implying that a steady state analysis of the feedback control performance will be difficult and indeed we leave it for ongoing work.

VI. Numerical Results

In this section we present some numerical results on a comparative study of the various encoder-decoder design strategies discussed in the earlier sections. In particular, we consider a linear dynamical system with $a = 1.25, b = c = 1$ and $\sigma_w^2 = 0.02, \sigma_v^2 = 0.2$ and feedback erasure probability $\epsilon_\nu = 0.2$. We vary the forward channel (between sensor and decoder) packet loss probability between 0.1 and 0.4, while the SQNR is chosen to be $\Lambda(\epsilon_\gamma) = 1.2\Lambda^{opt}$ where $\Lambda^{opt} = \frac{a^2-1}{1-\epsilon_\gamma a^2}$. Figure 3 illustrates the control cost performance of the various encoding-decoding strategies discussed above for perfect and imperfect channel feedback with measurement and innovations forwarding. Note that for measurement forwarding at the transmitter, the channel feedback is irrelevant to the transmitter, whereas the strategy of innovation forwarding is optimal when there is perfect channel feedback. It is clearly seen from the graph that innovation forwarding does substantially better than measurement forwarding for all

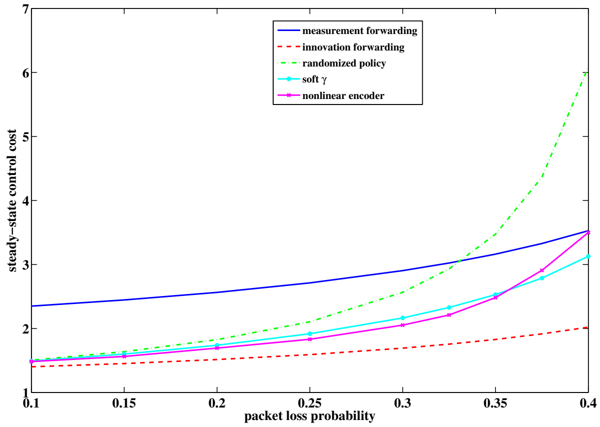

Fig. 3. Control cost comparison for various encoding-decoding strategies with measurement and innovation forwarding

packet loss probabilities. Amongst the suboptimal strategies for encoder design with imperfect channel feedback, it can be seen that the randomized policy performs the worst, whereas the soft γ estimation and the nonlinear encoding strategies perform much better, with the nonlinear encoding strategy performing the best at lower packet loss probabilities. When the packet loss probability becomes reasonably high, the suboptimal schemes can perform poorly. It can also be seen that the randomized policy performs even poorer than measurement forwarding, implying that the randomized policy is not very useful when the packet loss probability becomes higher.

VII. Conclusions

In this paper, we considered an unstable scalar linear dynamical system that operates over a channel that is prone to packet loss as well as quantization noise. We show that when there is perfect packet acknowledgement available at the transmitter, the optimal linear encoding strategy at the transmitter is to encode the innovations defined as the difference between the transmitter pre-processing Kalman filtered estimate and the receiver filter predicted estimate. For this strategy, we also find the optimal controller and the optimal constant gain receiver filter, which allows us to characterize the steady-state control cost. We also provide an analysis that quantitatively characterizes the difference between the control costs obtained via measurement forwarding at the transmitter and the optimal innovation forwarding strategy described above. For the case of imperfect channel feedback, we propose three suboptimal encoding strategies the performances of which are investigated via simulation studies. Future work will investigate the extension of these studies to the case of unstable vector systems and consider further theoretical performance analysis of the suboptimal strategies for the case of imperfect channel feedback.

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