Consensus-based Source-seeking with a Circular Formation of Agents

Lara Briñón-Arranz[1] and Luca Schenato[2]

Abstract— **This paper deals with the source-seeking problem in which a group of autonomous vehicles must locate and follow the source of some signal based on measurements of the signal strength at different positions. First, we show that the gradient of the signal strength can be approximated by a circular formation of agents via a simple weighted average of the signal strength measured by the agents. Then, we propose a distributed source-seeking algorithm based on a consensus method which is guaranteed to steer the circular formation towards the source location using the estimated gradient direction. The proposed algorithm is provided with two tunable parameters that allow for a tradeoff between speed of convergence, noise filtering and formation stability. The benefit of using consensus-based algorithms resides in a more realist discrete time control of the agents and in asynchronous communication resilient to delays which is particularly relevant for underwater applications. The analytical results are finally complemented with numerical simulations.**

I. INTRODUCTION

Detecting the source of a signal is relevant to many complex applications as environmental monitoring [8], search and rescue operations [9], odor source detection [13], sound source localization [22] and pollution sensing [11]. Source localization is also a fundamental problem in nature. Inspired by some bacteria behavior which are able to find chemical sources, the problem of seeking a maximum using autonomous vehicles is studied in [12], [14].

There are different approaches to deal with this topic in the current literature. For example, several source-seeking algorithms are based on gradient-descent methods. If it is available, the gradient of the signal strength can be used to produce a gradient-descent algorithm for a vehicle or group of vehicles [2]. However, in practice the agents are only capable of collecting measurements of the signal strength and the gradient information is usually unknown. In this situation, the gradient can be approximated using spatially distributed measurements of the signal distribution. In the literature there are two different strategies to collect distributed measurements. The first one uses a single vehicle which changes its position over time in order to measure the signal propagation in different positions [2], [3]. The other option consists in multiple vehicles collaborating to collect concentration measurements at different locations [8], [15].

The application of extremum-seeking techniques to the source localization problem has been analyzed under different constraints using a single nonholonomic vehicle [21], [5]. The idea is to add an excitatory input to the vehicle steering control in order to approximate the gradient of the signal strength and using this information to drive the vehicle towards the source. A novel stochastic approach based on the clasical extremum seeking algorithm is introduced in [12] and [18]. A recent paper [1] proposes a strategy to streer a single robot to the maximum of a scalar field based on a stochastic gradient-descent algorithm. This approach is applicable to robots with various types of dynamics. The main disadvantage of these strategies is that in order to collect sufficient information, the vehicle may have to travel over large distances. As a consequence, in this situation the vehicle convergence to the source location may be slow.

Some collaborative methods using multi-agent systems have been proposed in recent literature. In [8], [6] a group of vehicles equipped with appropriate sensors estimates the model parameters of the scalar field via collected measurements. A least-squares approximation is applied in order to steer the group of agents to the source location. Other works are based on distributed estimation of the concentration plume [17], [16]. In this case, the function signal is estimated or approximated and the source localization becomes a distributed optimization problem. These strategies rely on a prior model of the signal distribution, which might not be known a-priori if the environment is unknown.

Other alternative approaches to estimate the gradient of the signal in a cooperative way are available. For example, a collaborative control law to steer a circular formation of nonholonomic vehicles to the source of a signal distribution using only their direct signal measurements is presented in [15]. In [10] a distributed source-seeking algorithm is proposed using optimization tehcniques. The main drawback of both works is the assumption about the spatial propagation of the signal (concave function). Motivated by behaviors of fish groups seeking darker regions the authors of [19] proposed a distributed source-seeking algorithm for a group of vehicles with no explicit gradient estimation.

The present paper addresses an alternative solution to the source localization problem. In order to locate the source of a scalar field, we consider a group of vehicles equipped with sensors which measure the field of interest such as, temperature, salinity, pollutant flow. In this situation, the fleet of vehicles can be seen as a mobile sensor network. We will show that a group of vehicles uniformly distributed in a circular formation, is able to approximate the gradient direction of the measured signal. The problem is tackled in a 2-dimensional space, hence the configuration considered is a planar formation. No prior knowledge of the environment or convexity of the signal field are required. In order to keep the formation and to steer its center towards the source

[1]L. Briñón-Arranz is with the Department of Automatic Control, GIPSA-Lab, Grenoble, France. lara.brinon at gmail.com

[2]L. Schenato is with the Department of Information Engineering University of Padova, Italy. schenato at dei.unipd.it

location, we propose a distributed algorithm based on the multidimensional Newton-Raphson consensus strategy from [20]. The suggested strategy thus inherits the good properties of consensus algorithms [7], namely their simplicity, their potential implementation with asynchronous communication schemes, their ability to adapt to time-varying network topologies, and their resilience to packet loss and random delay.

The rest of the paper is organized as follows. First, Section II presents the problem statment introducing the model of the agents and some assumptions of the signal strength. In Section III, preliminary results on gradient approximation and distributed optimization are presented. Section IV exposes the main contribution, a distributed source-seeking algorithm based on Newton-Raphson consensus. Section V includes illustrating simulation results. Finally, we will present our conclusions and future directions.

II. Problem formulation

The main objective of this paper is to design a distributed algorithm to steer a group of agents to the source location of a field of interest.

A. Agents

Consider a circular formation of N agents described by a radius $D > 0$, an arbitrary rotation angle ϕ_0 and a given center point $c = [c_x\ c_y]^T \in \mathbb{R}^2$. In a real situation the given center could be an external reference corrupted by noise or could not be avaiable to all the agents, thus, we assume that each agent computes its own center $c_i \in \mathbb{R}^2$. Therefore, the position of each agent $i = 1, \ldots, N$ at time k is given by the following equation:

$$r_i(k) = c_i(k) + DR(\phi_i)\, e \qquad (1)$$

where $r_i(k) = [x_i(k)\ y_i(k)]^T \in \mathbb{R}^2$ is the position vector, $\phi_i = \phi_0 + i\frac{2\pi}{N}$ is the rotation angle, $R(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ is the rotation matrix, and $e = [1\,0]^T$. If all the agents compute the same center, i.e., $c_i(k) = c(k), \forall i$, then this previous equation describes a formation where the agents are uniformly distributed on a circle of radius D centered at $c(k)$.

B. Signal strength

Each agent represents a mobile sensor or a vehicle equiped with a sensor which is able to measure the signal strength emitted by the source. The source could be a point of chemical contamination and the signal would be the chemical's concentration in the environment, for instance. Alternatively, the source could be a radio transmitter and the signal would be a radio frequency transmission.

In mathematical terms, the signal distribution emitted by the source is a bidimensional spacial function representing the scalar field with a maximum or with a minimum in the position where the source is located. The distribution of the signal strength in the environment will be described by an unknown positive spatial mapping $\sigma(z) : \mathbb{R}^2 \to \mathbb{R}^+$, and so

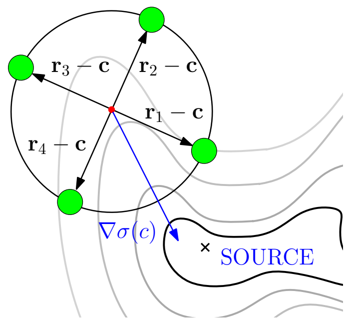

Fig. 1. Problem formulation

agent i measures the signal strength at its position $r_i(k)$, as $\sigma(r_i(k))$. We assume here that the signal is emitted by a single source such that the source is the only maximum of the scalar field. The signal distribution is assumed to decay away from the position of the source. Therefore, the following assumption is considered:

Assumption 1 *Function σ belongs to $\mathscr{C}^2$ i.e., is continuous up to the second partial derivative, its first partial derivative is zero only at the souce position z^*, i.e., $\nabla\sigma(z^*) = 0$, its second partial derivative is defined for all $z \in \mathbb{R}^2$, it is strictly negative, and bounded such that the Hessian matrix satisfies*

$$0 < m_1 \leq \|H(z)\| \leq m_2$$

Moreover each scalar component of the global maximizer z^* does not take value on the extended values $\pm\infty$.

C. Control objectives

Using a gradient-descent algorithm the group of agents can be driven to the source of the the signal distribution, see [2], [15]. Nevertheless, the gradient information is not usualy available. In that situation, we propose a cooperative approach in order to satisfy the following control objectives:

(i) estimating the gradient

(ii) keeping the circular formation of agents

(iii) steering the formation towards the source location

The gradient direction of the signal distribution can be estimated via concentration measurements collected by a circular formation of agents. This result will be analyzed in Section III-A. The estimated direction of the gradient will be the reference velocity of the formation center in order to steer the group of agents to the source location as represented in Fig. 1. In order to keep the fotmation, the agents must reach an agreement on the center position of the circular formation, therefore a consensus algorithm on centers c_i will be implemented. To acomplish the objective (iii), the common center of the circular formation will be driven towards the source using the estimated direction of the gradient. In order to achieve all the control aims, we propose a distributed algorithm based on the Newton-Rapson consensus method for distributed optimization from [20].

III. Preliminary results

A. Gradient approximation by a fixed circular formation

Consider a circular formation of agents given by (1) with $c_i(k) = c, \forall i$, taking measurements of a signal distribution $\sigma(z)$. Let $\nabla\sigma(c) = [\nabla_x\sigma(c)\,\nabla_y\sigma(c)]^T \in \mathbb{R}^2$ denote the gradient of $\sigma(z)$ at the center of the circular formation. Based on the previous result from [4] following lemma is proposed:

Lemma 1 *Let $\sigma : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function and $\sigma(r_i)$ be the measure collected by agent i where r_i is its position vector given by* (1) *with $c_i(k) = c$. Considering a fleet of $N > 2$ agents uniformly distributed along the circle centered at c, the following equation is satisfied:*

$$\frac{1}{N}\sum_{i=1}^{N}\sigma(r_i)\,(r_i - c) = \frac{D^2}{2}\nabla\sigma(c) + o(D^2) \tag{2}$$

Proof: If we use first order Taylor expansion of each measurement $\sigma(r_i)$ about the point c and recalling that $||r_i - c|| = D$, then the following equation holds for all $i = 1, \ldots, N$:

$$\sigma(r_i) - \sigma(c) = \nabla\sigma(c)^T(r_i - c) + o(D) \tag{3}$$

By multiplying the previous equation by the relative vector $(r_i - c)$ and summing over $i = 1, \ldots, N$, we get:

$$\frac{1}{N}\sum_{i=1}^{N}\sigma(r_i)\,(r_i - c) + c\frac{1}{N}\sum_{i=1}^{N}(r_i - c) =$$
$$= \frac{1}{N}\left(\sum_{i=1}^{N}(r_i - c)(r_i - c)^T\right)\nabla\sigma(c) + o(D^2)$$

Since the agents are distributed uniformly along a fixed circle, then we have $\sum_{i=1}^{N}(r_i - c) = 0$ and

$$\begin{aligned}
&\textstyle\sum_{i=1}^{N}(r_i - c)(r_i - c)^T = D^2\sum_{i=1}^{N}R(\phi_i)ee^TR(\phi_i)^T\\
&= D^2R(\phi_0)\left(\textstyle\sum_{i=1}^{N}R(i2\pi/N)ee^TR(i2\pi/N)^T\right)R(\phi_0)^T\\
&= D^2R(\phi_0)\left(\textstyle\sum_{i=1}^{N}\begin{bmatrix}\cos^2(i2\pi/N) & 0.5\sin(i4\pi/N)\\ 0.5\sin(i4\pi/N) & \sin^2(i2\pi/N)\end{bmatrix}\right)R(\phi_0)^T\\
&= D^2R(\phi_0)(\tfrac{N}{2}I_2)R(\phi_0)^T = \tfrac{ND^2}{2}I_2
\end{aligned}$$

since $\cos^2\phi = 1/2(1+\cos(2\phi))$, $\sin^2\phi = 1/2(1-\cos(2\phi))$, and $\sum_{i=1}^{N}\cos(2i\frac{2\pi}{N}) = \sum_{i=1}^{N}\sin(2i\frac{2\pi}{N}) = 0$ for $N > 2$, where $I_2 \in \mathbb{R}^{2\times2}$ represents the identity matrix. Thus, the equality of Eq. (2) is satisfied. ■

This result provides an approximation of the gradient of the signal distribution at the center $c(k)$ of a circular formation at each instant k.

B. Gradient-ascent consensus

Source localization can be appoched using optimization methods. Mathematically, the source-seeking problem is equivalent to find the maximum or minimum of a scalar function $\sigma(z) : \mathbb{R}^2 \to \mathbb{R}$. In [20], a multidimensional gradient-ascent consensus for distributed optimization is presented. In this paper, the authors assume that N agents, each endowed with a local multidimensional *strictly convex* cost function $\sigma_i(z)$, aim to collaborate in order to minimize or maximize the global cost function $\bar{\sigma}(z) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i(z)$, i.e. the average of all local cost functions. Algorithm 1 provides a strategy to move the center c_i of each agent i at the global maximum of $\bar{\sigma}(z)$ under the assumption that they can exactly compute the gradient of their own cost function. To simplify the notation, we indicate with bold letters vectors obtained by collecting local variables into a single column vector, i.e. $\mathbf{c} = [c_1^T\ c_2^T\ \ldots\ c_N^T]^T \in \mathbb{R}^{2N}$ and $\mathbf{h} = [h_1^T\ h_2^T\ \ldots\ h_N^T]^T \in \mathbb{R}^{2N}$, where $h_i \in \mathbb{R}^2$ are defined in Algorithm 1.

The communication topology of the agents is defined by means of a graph $\mathscr{G}$. Let $\mathscr{G} = (V, E)$ be an undirected communication graph. The set of nodes (agents) is denoted by $V = \{1, \ldots, N\}$ and the set of edges $(i, j) \in E$ represents the communication links. Let $P \in \mathbb{R}^{N\times N}$ denote a stochastic matrix, i.e. a matrix whose elements are non-negative and $P\mathbf{1} = \mathbf{1}$, where $\mathbf{1} := [1, \ldots, 1]^T \in \mathbb{R}^N$. It is consistent with a graph $\mathscr{G}$ if $P_{ij} > 0$ only if $(i, j) \in E$. Such matrix P is also often referred as a *consensus matrix*. A stochastic matrix P is doubly stochastic if also $\mathbf{1}^T = \mathbf{1}^TP$. The essential spectral radius of a stochastic matrix is defined as $\mathrm{esr}(P) = \max_{\lambda_i\neq1}|\lambda_i(P)|$, where $\lambda_i(P)$ indicates the eigenvalues of P. In the sequel, $\otimes$ denotes the Kronecker product.

The Algorithm 1 works as follows: Lines 6 and 7 are local computation needed to track the local quantities which are necessary to compute the approximated gradient as in Lemma 1. Line 9 is needed to compute the average of these local quantities and it is based on standard consensus communication given by the matrix P. Line 11 instead is responsable for steering the center of each agent towards the estimated center of the source $h_i(k)$. It is fundamental that the steering of the agents' center is sufficiently slow as compared to the computation of the average given by the matrix P, otherwise these centers might diverge and the estimated center of the source $h_i(k)$ can be totally wrong. Such separation of time scales is regulated by the parameter ε: the smaller it is, the slower the convergence to the source is, but at the benefit of a guaranteed stability.

Algorithm 1 Distributed gradient-ascent consensus [20]

```
1:  for i = 1,...,N do
2:      h_i(0) = g_i(0) = c_i(0) + ∇σ_i(c_i(0))
3:  end for
4:  for k = 1,2,... do
5:      for i = 1,...,N do
6:          g_i(k) = c_i(k−1) + ∇σ_i(c_i(k−1))
7:          h̃_i(k) = h_i(k−1) + g_i(k) − g_i(k−1)
8:      end for
9:      h(k) = (P ⊗ I_2) h̃(k)
10:     for i = 1,...,N do
11:         c_i(k) = (1−ε)c_i(k−1) + εh_i(k)
12:     end for
13: end for
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Lemma 2 *Let us consider Algorithm 1 where $||c_i(0) - z^*|| < r$ for some arbitrary $r > 0$, and P is a doubly stochastic matrix with essential spectral radius $\mathrm{esr}(P) < 1$. Then there* *exists $\bar{\varepsilon}_r > 0$ (possibly depending on r) such that $\forall \varepsilon \in (0, \bar{\varepsilon}_r)$*

$$\lim_{k\to\infty} c_i(k) = z^*, \quad \forall i$$

exponentially fast.

Proof: The formal proof is given in [20], but we still want to provide a sketch of the proof for the benefit of understanding. The proof is based on the separation of time-scales and basically reduces first to study the fast-dynamics and proving that it has exponential convergence to a manifold, and later to study the slow dynamics when the fast variables are constrained to live in the manifold.

Fast Dynamics: If we set $\varepsilon = 0$, then $c_i(k) = c_i(0) = c_i$ for all $k \geq 0$. This implies that $g_i(k) = c_i + \nabla\sigma_i(c_i)$ for all $k \geq 1$, and therefore $\tilde{h}_i(1) = c_i + \nabla\sigma_i(c_i)$ and $\tilde{h}_i(k) = h_i(k-1)$ for $k > 1$. As a further consequence, the dynamics of $\mathbf{h}$ becomes $\mathbf{h}(k) = (P \otimes I_2)\mathbf{h}(k-1)$ for $k \geq 1$, which implies that $\lim_{k\to\infty} h_i(k) = \frac{1}{N}\sum_{i=1}^{N}(c_i + \nabla\sigma_i(c_i)) = \bar{h}(\mathbf{c})$ exponentially fast with rate given by esr(P).

Slow Dynamics: If we insert the steady state of the fast dynamics $h_i(k) = \bar{h}(\mathbf{c})$ into the slow dynamics we get

$$c_i(k) = (1-\varepsilon)c_i(k-1) + \varepsilon\bar{h}(\mathbf{c}(k-1))$$

Since each system is driven by the same forcing term $\bar{h}(\mathbf{c}(k-1))$, then $\lim_{k\to\infty} c_i(k) - c_j(k) = 0$, therefore we can restrict our attention to the scenario where $c_i(k) = \bar{c}(k), \forall i$, which implies that $\bar{h}(\mathbf{c}(k)) = \bar{c}(k) + \frac{1}{N}\sum_{i=1}^{N}\nabla\sigma_i(\bar{c}(k)) = \bar{c}(k) + \nabla\bar{\sigma}(\bar{c}(k))$ whose dynamics are given by

$$\begin{aligned}\bar{c}(k+1) &= (1-\varepsilon)\bar{c}(k) + \varepsilon(\bar{c}(k) + \nabla\bar{\sigma}(\bar{c}(k))) \\ &= \bar{c}(k) + \varepsilon\nabla\bar{\sigma}(\bar{c}(k))\end{aligned}$$

which is the standard gradient-ascent update. If ε is sufficiently small, then the separation of time-scale holds and $\lim_{k\to\infty} c_i(k) = \lim_{k\to\infty}\bar{c}(k) = z^*$.

■

Remark 1 *Although in the theorem we considered a constant consensus matrix P, the same conclusion applies even for time-varying consensus matrices $P(k)$ with possibly random but bounded time-delay, as long as the slow dynamics is sufficiently slow as compared to the convergence rate of the product of the consensus matrices $P(k)$. Since the slow dynamics are regulated by the tunable parameter ε, this is always possible. This implies that asynchronous communication does not impair the algorithm, however $P(k)$ still need to be doubly stochastic such as in symmetric gossip consensus in order to exactly compute the exact average of the local vectors $\tilde{h}_i(k)$.*

IV. Distributed Source-seeking

As presented in previous section, if the gradient is not available, the direction on the gradient of a signal strength can be approximated by a group of agents distributed uniformly along a circular formation. If we assume all-to-all and instantaneous communication, all the agents compute the same estimated gradient direction and then this direction can be used to drive the formation towards the source following a gradient-descent method as shown in [15], [4]. However, the all-to-all communication assumption is not realistic and in several situations each agent communicates only with their neighbors.

Our main contribution is to modify previous algorithm in order to use the collaborative estimation of the gradient direction presented in Lemma 1 and thus, to achieve the source-seeking task in a distributed way. At each instant k, each agent computes its position $r_i(k)$, its center $c_i(k)$ and its estimated gradient vector $f_i(k) = \sigma(r_i)(r_i(k) - c_i(k))$. The objective for the agents is now, to reach an agreement on the centers' position c_i and to compute the vector $\bar{f}(k)$ at each time k defined by:

$$\bar{f}(k) = \frac{1}{N}\sum_{i=1}^{N}\sigma(r_i)(r_i(k) - c_i(k)) \tag{4}$$

which is a good approximation of the gradient direction of the measured signal distribution. The aim for the formation is to reach the source location, such that, $\lim_{k\to\infty} c_i(k) = z^*$.

The proposed source-seeking strategy is described in Algorithm 2, which is very similar to Algorithm 1. There are two differences: the first difference being that $\nabla\sigma_i$ in Line 6 is substituted with $\sigma(r_i)(r_i - c_i)$, while the second difference is the inclusion of a local low-pass filtering of the local signal $g_i(k)$ in order to make the algorithm more robust to measurement noise. The low-pass filter is regulated by the parameter α which tradeoffs smoothing of the signal ($\alpha \approx 0$) with responsiveness to changes of the signal $g_i(k)$ ($\alpha \approx 1$). Indeed, for $\alpha = 1$, Algorithm 2 and Algorithm 1 are substantially the same.

Algorithm 2 Distributed source-seeking algorithm

1: **for** $i = 1, \ldots, N$ **do**
2: $\quad h_i(0) = \tilde{g}_i(0) = \tilde{g}_i(-1) = c_i(0) + \sigma(r_i(0))(r_i(0) - c_i(0))$
3: **end for**
4: **for** $k = 1, 2, \ldots$ **do**
5: $\quad$ **for** $i = 1, \ldots, N$ **do**
6: $\quad\quad g_i(k) = c_i(k) + \sigma(r_i(k))(r_i(k) - c_i(k))$
7: $\quad\quad \tilde{g}_i(k) = (1-\alpha)\tilde{g}_i(k-1) + \alpha g_i(k)$
8: $\quad\quad \tilde{h}_i(k) = h_i(k-1) + \tilde{g}_i(k-1) - \tilde{g}_i(k-2)$
9: $\quad$ **end for**
10: $\quad \mathbf{h}(k) = (P \otimes I_2)\tilde{\mathbf{h}}(k)$
11: $\quad$ **for** $i = 1, \ldots, N$ **do**
12: $\quad\quad c_i(k) = (1-\varepsilon)c_i(k-1) + \varepsilon h_i(k)$
13: $\quad$ **end for**
14: **end for**

This algorithm has two tunable parameters, namely ε and α that can be used to tradeoff rate of convergence, robustness to noisy measurements and formation stability. A formal statement of the properties of this algorithm is given in the following theorem:

Theorem 1 *Let $\sigma : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function which satistifies Assumption 1 and $\sigma(r_i)$ be the measure collected by agent i where $r_i(k)$ is its position vector given by* (1). *Let* *us consider Algorithm 2 where $||c_i(0) - z^*|| < r$ for some arbitrary $r > 0$, $\alpha \in (0,1]$, and P is a doubly stochastic matrix with essential spectral radius* $\mathrm{esr}(P) < 1$. *Then there exists $\bar{\varepsilon} > 0$ (possibly depending on r and α) such that for all $\varepsilon \in (0, \bar{\varepsilon})$*

$$\lim_{k\to\infty} c_i(k) - c_j(k) = 0, \quad \forall i, j$$

exponentially fast. Moreover, all the centers c_i converge asymptotically to the neighborhood of the maximum of the signal distribution $\sigma(z)$ located at z^*.

Proof: We present here a sketch of the proof which follows the same steps that in previous Lemma 2 based on the separation of time-scales.

Fast Dynamics: If we set $\varepsilon = 0$, then $c_i(k) = c_i(0) = c_i$ for all $k \geq 0$ and thus, according to Eq. (1) the position of agent i becomes $r_i(k) = c_i + DR(\phi_i)e = r_i(0) = r_i$ for all $k \geq 0$. This implies that $g_i(k) = c_i + \sigma(r_i)(r_i - c_i)$ for all $k \geq 1$ and thus $\forall \alpha \in (0,1]$ $\tilde{g}_i(k) = c_i + \sigma(r_i)(r_i - c_i)$ for all $k \geq 1$. Therefore $\tilde{h}_i(1) = c_i + \sigma(r_i)(r_i - c_i)$ and $\tilde{h}_i(k) = h_i(k-1)$ for $k > 1$. As a further consequence, the dynamics of $\mathbf{h}$ becomes $\mathbf{h}(k) = (P \otimes I_2)\mathbf{h}(k-1)$ for $k \geq 1$, which implies that

$$\lim_{k\to\infty} h_i(k) = \frac{1}{N}\sum_{i=1}^{N}(c_i + \sigma_i(r_i)(r_i - c_i)) = \bar{h}(\mathbf{c})$$

exponentially fast with rate given by $\mathrm{esr}(P)$.

Slow Dynamics: If we insert the steady state of the fast dynamics $h_i(k) = \bar{h}(\mathbf{c})$ into the slow dynamics we get

$$c_i(k) = (1-\varepsilon)c_i(k-1) + \varepsilon\bar{h}(\mathbf{c}(k-1))$$

Since each system is driven by the same forcing term $\bar{h}(\mathbf{c}(k-1))$, then $\lim_{k\to\infty} c_i(k) - c_j(k) = 0$, which implies that the circular formation of agents is maintained and their position vectors depend on the common center, such that $r_i(\mathbf{c})$. Therefore we can restrict our attention to the scenario where $c_i(k) = \bar{c}(k), \forall i$, which implies that $\bar{h}(\mathbf{c}(k)) = \bar{c}(k) + \frac{1}{N}\sum_{i=1}^{N}\sigma(r_i(\bar{c}(k)))(r_i(\bar{c}(k)) - \bar{c}(k))$. Therefore, each agent computes the estimate of the gradient direction $\bar{f}(\bar{c}) = \sum_{i=1}^{N}\sigma(r_i)(r_i - \bar{c})$. Thanks to Lemma 1 dealing with the appoximation via a circular formation of agents of the gradient of a signal strength at the circle center, the following equation holds:

$$\bar{h}(\mathbf{c}(k)) = \bar{c}(k) + \bar{f}(\bar{c}(k)) = \bar{c}(k) + \frac{D^2}{2}\nabla\sigma(\bar{c}) + o(D^2)$$

And thus the dynamics of $\bar{c}$ are given by

$$\begin{aligned}\bar{c}(k+1) &= (1-\varepsilon)\bar{c}(k) + \varepsilon(\bar{c}(k) + \tfrac{D^2}{2}\nabla\sigma(\bar{c}) + o(D^2)) \\ &= \bar{c}(k) + \varepsilon\tfrac{D^2}{2}\nabla\sigma(\bar{c}) + \varepsilon o(D^2)\end{aligned}$$

which is the standard gradient-ascent update away from the approximation error $o(D^2)$. Thanks to the Taylor's theorem we can quantify this error. Since Assumption 1 is satisfied, the Hessian of the signal strength is bounded and thus the error of the gradient approximation is negligible with respect to $\frac{1}{2}D^3 m_1$. If ε is sufficiently small, then the separation of time-scale holds and $\lim_{k\to\infty} c_i(k) = \lim_{k\to\infty}\bar{c}(k)$ which converges to the neigborhood of z^*. ∎

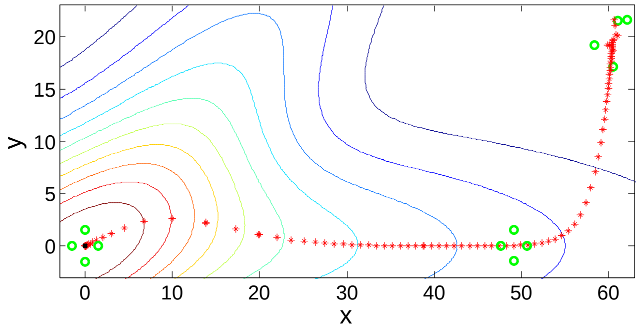

Fig. 2. Source-seeking Algorithm 2 via a circular formation of agents ($\varepsilon = 0.2$, $\alpha = 1$)

V. Simulation Results

In this section we present some simulations to show the convergence of the proposed source-seeking algorithm. For all simulations, the scalar field is a combination of two ellipsis and thus with non convex level curves given by

$$\sigma(z) = exp\left(-z^T S_1 z\right) + exp\left(-z^T R(\pi/4)^T S_2 R(\pi/4) z\right)$$

where $S_1 = \frac{1}{100}\begin{bmatrix} 1/\sqrt{30} & 0 \\ 0 & 1 \end{bmatrix}$, $S_2 = \frac{1}{100}\begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{15} \end{bmatrix}$. The maximum corresponding to the source is located at $z^* = [0 \;\; 0]^T$ represented by the black $\times$. The communication topology considered is a ring, where agents can communicate only to their left and right neighbors, and thus the symmetric circulant communication matrix is

$$P = \begin{bmatrix} 1/2 & 1/4 & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 & 0 \\ 0 & 1/4 & 1/2 & 1/4 \\ 1/4 & 0 & 1/4 & 1/2 \end{bmatrix}$$

Fig. 2 shows a simulation of four agents modeled by (1) with radius $D = 1.5$ computing Algorithm 2. The control parameters are $\varepsilon = 0.5$ and $\alpha = 1$. The void green circles represents the agents at three different iterations, the initial conditions, an intermediate state at $k = 4000$ and the final state at $k = 8500$. The red stars represent the position of each center $c_i(k)$ during the source-seeking task. Thanks to Algorithm 2, the fleet of agents reaches a consensus on the center position and the formation is steered to the source location.

Fig. 3 displays the evolution of the first component of the centers' trajectories $c_{x,i}$ at the first iteration steps for three simulations of four agents computing Algorithm 2 with $\alpha = 1$ and different values of ε. The number of iterations to reach the source position for each simulation are respectively $k = 22000$, $k = 8500$, and $k = 4500$. Reasonably, the larger is the value of ε the faster is the convergence of the algorithm and hence the circular formation reach the souce location in fewest iterations. However, as displayed in Figure 3, the centers' trajectories c_i oscillates for largest values of ε.

Fig. 4 shows a simulation of four agents modeled by (1) with radius $D = 1.5$ computing Algorithm 2. The control parameters are $\varepsilon = 0.5$ and $\alpha = 0.5$. The signal measurements are corrupted by zero-mean Gaussian noise $w(k)$. Thanks

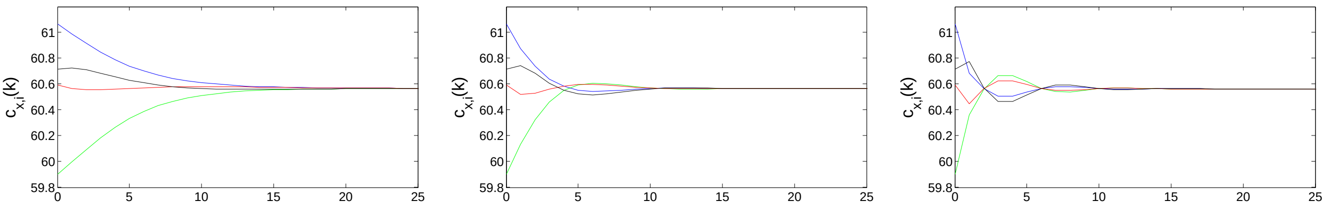

Fig. 3. Evolution of the first component of the centers $c_{x,i}$ for $\alpha = 1$ and $\varepsilon = 0.2$, $\varepsilon = 0.5$ and $\varepsilon = 1$ respectively.

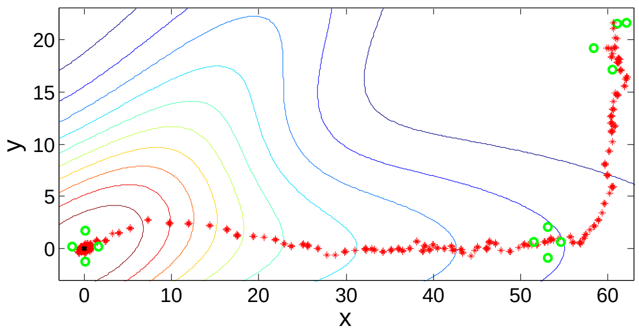

Fig. 4. Source-seeking Algorithm 2 via a circular formation of agents when the measurements are corrupted by Gaussian white noise ($\varepsilon = 0.5$, $\alpha = 0.5$)

to Algorithm 2, the fleet of agents reaches a consensus on the center position and the formation is steered to the neiborghood of the source location.

Fig. 5 displays the evolution of the first component of the centers' trajectories $c_{x,i}$ at the first iteration steps for three simulations of four agents computing Algorithm 2 with $\varepsilon = 0.5$ and different values of α. The signal measurements are corrupted by zero-mean Gaussian noise $w(k)$. The number of iterations to reach the source position for each simulation are respectively $k = 28000$, $k = 14000$ and $k = 12000$. The low-pass filter regulated by parameter α allows attenuating the measurement noise and hence the centers' trajectories are *smoother* for smaller values of α. However, the smaller is this parameter, the slower is the convergence to the source location.

VI. Conclusion and Futures Works

This paper provides a distributed solution to the 2-dimensional souce localization problem. Our cooperative approach considers a group of agents which are able to measure the signal distribution emitted by the source. No previous acknowledgment of the signal is assumed. Firstly, we show that collecting the measurements of agents uniformly distributed along a circular formation, the gradient of the signal strenght at the center of the formation can be extimated. Using this information, a distributed source-seeking algorithm is proposed in order to steer the fleet of vehicles to the source location. This result is based on a Newton-Raphson consensus algorithm for distributed optimization. Our solution allows keeping the circular formation, estimating the gradient of the signal and driving the center of the formation to the maximum of the scalar field of interest. Moreover, we included a low-pass filter in order to make the algorithm more robust to measurement noise.

Future works will be focused on extending previous algorithm to the 3-dimensional case. Another future direction is to approximate the Hessian of the signal distribution at the center of the circular formation in order to improve the convergence rate of our source-seeking algorithm.

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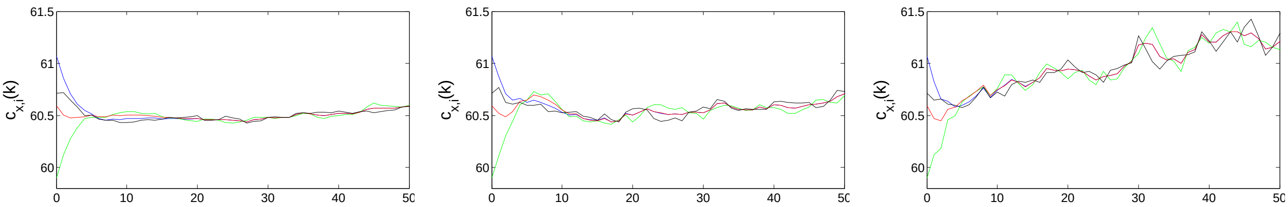

Fig. 5. Evolution of the first component of the centers $c_{x,i}(k)$ for $\varepsilon = 0.5$ and $\alpha = 0.2$, $\alpha = 0.5$ and $\alpha = 1$ respectively.

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