LQG cheap control over SNR-limited lossy channels with delay

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***Abstract*— In this paper we study the effect of communication nonidealities on the control of unstable stochastic scalar linear systems. The communication protocol links the sensors to the actuators and should be studied by taking into account several limitations such as quantization errors, limited channel capacity, decoding/computational delays and packet loss. We restrict our analysis in the context of LQG cheap control subject to SNR limitations, packet loss, and delay and we derive their impact on optimal design for the controller parameters. In particular, we show that the stability of the closed loop system depends on a tradeoff among quantization, packet loss probability and delay. Through this analysis we are also able to recover, as special cases, several results already available in the literature that have treated packet loss, quantization error and delay separately.**

I. Introduction

Traditionally, control theory and communication theory have been developed independently and have reached considerable success in developing fundamental tools for designing information technology systems. The major objective of control theory has been to develop tools to stabilize unstable plants and to optimize some performance metrics in closed loop under the assumption that the communication channel between sensors and controller and between the controller and the plant were ideal, i.e. without distortion, packet loss or delay. This assumption actually holds in many control applications where the non idealities of the communication channel have negligible impact, compared to the effects of noise and uncertainty in the plants. With the advent of wireless communication, the Internet and the need for high performance control systems, however, the sharp separation between control and communication has been questioned and a growing body of literature has appeared from both the communication and the control communities trying to analyze the interaction between control and communication.

This recent branch of research is known as *Networked Control System* (NCS) and considers control systems wherein the control loops are closed through a real-time network, and feedback signals are exchanged in the form of data packets.

Recent results in this area have revealed the existence of a strict connection between the performance of the controlled plant and the Shannon capacity of the feedback channel. However, this is not sufficient to completely characterize the communication channel from a control perspective [14], [10]. For instance, it has been proved that in order to stabilize an unstable plant through a control loop, the signal-to-noise ratio (SNR) of the feedback channel must be larger than some threshold depending on the unstable eigenvalues of the plant [2], [17], [3]. Another line of research has addressed the problem of stabilizing an unstable plant in the presence of a feedback channel that is prone to random packet losses [18], [7], [8], [16], or that is rate-limited [11], [19], [5]. A subsequent step has been made to include multiple channel limitations into the model, such as packet loss and quantization [20], [9], which however results in complex optimization problems.

In this work, we address the problem of performance optimization in a NCS with a realistic feedback channel. More specifically, we consider the Linear-Quadratic-Gaussian (LQG) control problem, which consists in finding the control signal of a linear time-invariant (LTI) plant that minimizes a quadratic cost function of the system state, when both the system state and the output signal are affected by Gaussian noise. While the optimal solution to the LQG problem in LTI systems with *ideal feedback channel* is known to be achieved by a controller formed by a Kalman filter and a linear-quadratic regulator, the solution to the problem in NCS systems with *realistic feedback channels* has only been investigated for specific feedback channel models, while the general solution still remains unknown.

Our feedback channel model takes into account packet loss, code rate limitations, signal quantization and delay, while still being mathematically amenable to analysis. By using this model, we find a stability condition that depends on the packet loss probability, the signal to quantization noise ratio (SQNR) and the channel delay d. To the best of our knowledge this is the first result which takes simultaneously into account all these aspects. The LQG architecture proposed in this paper actually generalizes those considered in the previous literature; in fact we recapture several conditions available in the literature for more specific channel models as special cases of our model.

II. System model and problem formulation

In this section, we cast the LQG problem into the NCS framework. First, we introduce the LQG problem. Then, we model the feedback transmission channel that completes the NCS structure considered in this work. Finally, we formally define the LQG problem in the NCS architecture

A. LQG problem definition

We consider a plant, modeled as a discrete-time, scalar, LTI system, subject to additive white Gaussian measurement

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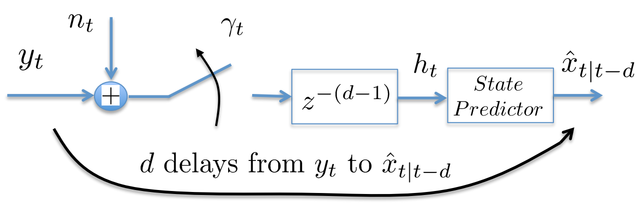

Fig. 1. Equivalent model of the feedback channel and state predictor, accounting for the presence of quantization noise, packet loss and decoding delays.

and process noise. More specifically, the state of the system at step t, denoted as x_t, evolves according to the following linear model:

$$x_{t+1} = ax_t + bu_t + w_t \quad (1)$$
$$y_t = cx_t + v_t \quad (2)$$

where u_t and y_t represent the input and output signals of the plant, respectively, whereas w_t and v_t are two independent discrete-time Gaussian white noise processes with variance σ_w^2 and σ_v^2, respectively. Finally, a, b and c are the state, input and output coefficients, respectively. Note that c can always made equal to 1 with a change of basis, which we shall do from now on. In addition we shall also assume $b = 1$ because in the cheap control scenario the static gain does not play a role.

We consider the *steady state* variance as performance index[1]

$$J = \limsup_{t\to+\infty} \mathbb{E}[y_t^2]\,. \quad (3)$$

The objective of the LQG problem is then to minimize J by means of a suitable control signal u_t, which only depends (strictly) causally on the output signal $\{y_m,\ m < t\}$, and possibly on its own previous values $\{u_m,\ m \le t-1\}$.

B. Feedback channel modeling

In the NCS framework, the plant output y_t is not directly accessible to the controller, but must be delivered by means of a suitable transmission scheme. The feedback channel will thus comprise analog to digital conversion of y_t and source coding of the corresponding bitstream into packets, channel coding and transmission over the physical channel. At the receiver, after forward error correction, typically a detection of residual errors is performed and packets that have not been correctly decoded are dropped (packet erasures). Instead, accepted packets bear correct digital values with very high probability.

We model the feedback channel as represented in Fig. 1, where n_t represents the quantization noise. It accounts for the distortion due to the quantization of the real-valued signal y_t before transmission. If quantization is fine enough, n_t can be effectively modeled as a zero-mean additive random process, with identically distributed uncorrelated samples of power $\sigma_n^2 = \mathbb{E}[n_t^2]$. The SQNR, $\alpha = \mathbb{E}[y_t^2]/\sigma_n^2$, is related to the information rate R_q of the quantized signal, and increases with it. Since the maximum information rate R_q is upper limited by the channel code rate R_c, the SQNR cannot be increased above a certain threshold α^*, which depends on R_c.

Packet erasures are instead modeled by introducing a Bernoulli process $\gamma_t \in \{0, 1\}$. Assuming that a packet is sent at each $t = 0, 1, \ldots$, we indicate a packet erasure by letting the corresponding $\gamma_t = 0$. We assume that an erasure occurs with probability ϵ at each packet transmission, independently of previous events.

Finally, we assume a transmission/processing delay of d steps between the plant output y_t and the control signal u_t. One delay is embedded[2] in the state predictor based on the measurements h_t received up to time $t-1$ (see also Figure 1). As such, it must be $d \ge 1$, and the delay block $z^{-(d-1)}$ accounts for the additional encoding/decoding delay.

The feedback channel model considered in this paper has the following input-output relationship

$$h_t = \gamma_{t-d+1}(y_{t-d+1} + n_{t-d+1}) \quad (4)$$

an it is, hence, completely characterized by three parameters, namely ϵ, d, and α^*, with

$$d \ge 1, \quad \mathbb{P}[\gamma_t = 0] = \epsilon, \quad \sigma_n^2 = \mathbb{E}[y_t^2]/\alpha\ . \quad (5)$$

These parameters are clearly related, as, for instance, reducing the erasure probability ϵ may require increasing the delay d or reducing the information rate R_q, i.e., decreasing the maximum achievable SQNR α. Therefore some trade-offs are expected in the context of feedback control, since all three terms impact the performance of the closed loop system. Unfortunately, the exact form of the relation among these parameters is not available, though some tight bounds have recently been derived in [13].

For the ease of mathematical treatment, in our analysis we will assume that these parameters can be set independently. We can thus sort out the impact of each single parameter on the system performance. Note that, the interdependencies among the channel model parameters will only shrink the design parameter space, without affecting the validity of our analysis. An extension of our approach that keeps into account this aspect is left for future work.

C. Problem statement

In order to handle the delay in a compact form, we use the standard technique of state augmentation and define

$$\xi_t := [x_{t-d+1}, \ldots., x_t]^\top. \quad (6)$$

The augmented state satisfies

$$\begin{aligned} \xi_{t+1} &= A\xi_t + Bu_t + Bw_t \\ h_t &= \gamma_{t-d+1}\left(C\xi_t + v_{t-d+1} + n_{t-d+1}\right) \end{aligned} \quad (7)$$

[1] Strictly speaking the term "steady state" should only be used when the limit is finite; this will actually hold under suitable conditions, see Theorem 2.

[2] We followed this route for ease of exposition. The delay in prediction should not be linked to the computational cost at the predictor but rather to the fact that the predictor has only access to delayed measurements.

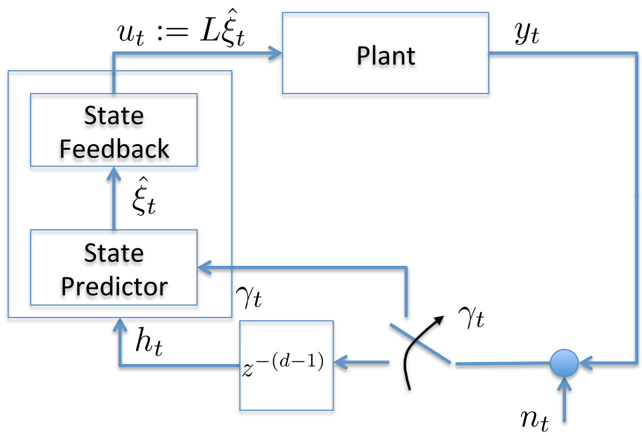

Fig. 2. NCS scheme for scalar output plants, where the plant decoder is given by the cascade of a linear state predictor and a state feedback.

where

$$A := \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & a \end{bmatrix} \quad B := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = [1 \quad 0 \quad \dots \quad 0]$$

We restrict our attention to the classical LQG structure for the plant decoder, which is given by the cascade of a linear state estimator and a state feedback, as represented in Fig. 2. The state estimator $\hat{\xi}_t$ (which uses the data up to time $t-d$) is governed by the following law

$$\hat{\xi}_{t+1} = A\hat{\xi}_t + Bu_t + \gamma_{t-d+1} G\big(h_t - C\hat{\xi}_t\big) \tag{8}$$

where G is a *constant* estimator gain, and the estimator (8) is time-varying since it depends on the sequence γ_t. In fact, if a packet is not received correctly, i.e. $\gamma_t = 0$, then the estimator updates its state using the model only, while when $\gamma_t = 1$ the estimate is adjusted by a correction term, based on the output innovation, similarly to a Kalman filter. The state feedback module, in turn, will simply return a control signal proportional to the predicted state[3] through L, i.e.,

$$u_t = L\hat{\xi}_t = [\ell_1 \; \ell_2 \; \dots \; \ell_d]\hat{\xi}_t \tag{9}$$

This scheme was first proposed in [15] and, although it does not yield the optimal time-varying Kalman filter [18], it has the advantage of being computationally simpler and allowing for the explicit computation of the performance J, as will be shown in the next section.

In this framework, the objective is to solve the following optimization problem:

$$\min_{G,L} \quad J \tag{10}$$

$$\text{s.t.} \quad \lim_{T\to+\infty} \frac{\sum_{t=0}^{T} \mathbb{E}[||y_t||^2]}{\sum_{t=0}^{T} \mathbb{E}[||n_t||^2]} \le \alpha^* \tag{11}$$

[3]Note that u_t in (9) is a function of measurements h_t up to time $t-1$, i.e. of the signal $y_s + n_s$ up to time $s = t-d$.

The constraint (11) sets an upper bound on the SQNR, which cannot exceed the maximum value α^* allowed by the channel code rate.

As a byproduct of our analysis we shall show that the optimal G and L have the following special structure (see Proposition 1):

$$\begin{aligned} G^* &= [g^* \quad ag^* \quad a^2g^* \quad \dots \quad a^{d-1}g^*] \\ L^* &= [0 \quad 0 \quad \dots \quad -a] \end{aligned}$$

Although in this study we limit our attention to the case of scalar systems, the approach can be extended to the multidimensional case. We leave this generalization to future work.

III. Analysis of the Scalar Case

As a first step, we derive the dynamical equations that govern the state as well as the error evolution for the estimator in equation (8). Inserting the control law (9) in (7) and (8) we obtain:

$$\begin{aligned} \xi_{t+1} &= A\xi_t + BL\hat{\xi}_t + Bw_t \\ &= A_L\hat{\xi}_t + A\tilde{\xi}_t + Bw_t \\ \hat{\xi}_{t+1} &= A_L\hat{\xi}_t + \gamma_{t-d+1}G\big(C\tilde{\xi}_t + v_{t-d+1} + n_{t-d+1}\big) \end{aligned} \tag{12}$$

where $\tilde{\xi}_t := \xi_t - \hat{\xi}_t$ and

$$A_L := A + BL = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ \ell_1 & \ell_2 & \dots & a+\ell_d \end{bmatrix} \tag{13}$$

Let us now define

$$\bar{A}_\gamma := \begin{bmatrix} A_L & \gamma GC \\ 0 & A - \gamma GC \end{bmatrix}$$

It follows that the equation of the feedback loop system are:

$$\begin{aligned} \begin{bmatrix} \hat{\xi}_{t+1} \\ \tilde{\xi}_{t+1} \end{bmatrix} &= \bar{A}_{\gamma_{t-d+1}} \begin{bmatrix} \hat{\xi}_t \\ \tilde{\xi}_t \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} w_t + \\ &\quad + \begin{bmatrix} \gamma_{t-d+1}G \\ -\gamma_{t-d+1}G \end{bmatrix} \begin{bmatrix} v_{t-d+1} + n_{t-d+1} \end{bmatrix} \\ y_{t-d+1} &= [C \quad C] \begin{bmatrix} \hat{\xi}_t \\ \tilde{\xi}_t \end{bmatrix} + v_{t-d+1} \end{aligned}$$

Let us now define

$$P := \mathrm{Var}\{[\hat{\xi}_t^\top, \tilde{\xi}_t^\top]^\top\} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

After some algebra, we can show that the variance P satisfies the Riccati-type equation

$$\begin{aligned} P = (1-\epsilon)\bar{A}_1 P \bar{A}_1^\top + \epsilon \bar{A}_0 P \bar{A}_0^\top + \begin{bmatrix} 0 \\ B \end{bmatrix} \sigma_w^2 \begin{bmatrix} 0 & B^\top \end{bmatrix} + \\ +(1-\epsilon) \begin{bmatrix} G \\ -G \end{bmatrix} \begin{bmatrix} \sigma_v^2 + N \end{bmatrix} \begin{bmatrix} G \\ -G \end{bmatrix}^\top \end{aligned} \tag{14}$$

where

$$N = \alpha P_y \qquad P_y = \left[\begin{array}{cc} C & C \end{array}\right] P \left[\begin{array}{c} C^\top \\ C^\top \end{array}\right] + \sigma_v^2$$

Substituting the above expression for N and P_y in (14) we obtain:

$$\begin{aligned} P = \quad & (1-\epsilon)\bar{A}_1 P \bar{A}_1^\top + \epsilon \bar{A}_0 P \bar{A}_0^\top + \\ & + \left[\begin{array}{c} 0 \\ B \end{array}\right] \sigma_w^2 \left[\begin{array}{cc} 0 & B^\top \end{array}\right] + \\ & + (1-\epsilon)(1+\alpha) \begin{bmatrix} G \\ -G \end{bmatrix} \sigma_v^2 \begin{bmatrix} G \\ -G \end{bmatrix}^\top + \\ & + \alpha(1-\epsilon)\bar{\Phi} P \bar{\Phi}^\top \end{aligned} \tag{15}$$

where

$$\bar{\Phi} := \left[\begin{array}{cc} GC & GC \\ -GC & -GC \end{array}\right]$$

For the ease of notation we define the operator on the right hand side of (15) as $\mathcal{M}(G, L, P)$, so that (15) can be written in compact form as

$$P = \mathcal{M}(G, L, P)$$

Minimization of the cost function (3) is equivalent to minimization of

$$J = \mathbb{E}\big[x_t^\top C^\top C x_t\big] = \left[\begin{array}{cc} C & C \end{array}\right] P \left[\begin{array}{c} C^\top \\ C^\top \end{array}\right] \tag{16}$$

Hence, the LQG-type cheap optimal control problem can be written as:

$$\begin{aligned} J^* := \min_{G,L} \quad & J \\ s.t. \quad & P = \mathcal{M}(G, L, P) \\ & P \geq 0 \end{aligned} \tag{17}$$

and L^*, G^* will denote the optimal gains, which can be found adapting the results in [4] as explained in the next section.

IV. Solution to the Optimal Control Problem

We now derive the solution to the LQG-type optimal control problem (17). The proof technique is borrowed from [6] and goes through the introduction of the Lagrangian

$$\begin{aligned} \mathcal{L}(P, \Lambda, L, G) := J + \mathrm{Tr}\{\Lambda\,(P - \mathcal{M}(G, L, P))\} \\ s.t. \quad P = P^\top \geq 0 \quad \Lambda = \Lambda^\top \geq 0 \end{aligned} \tag{18}$$

Accorning to the matrix maximum principle [1] the necessary conditions for optimality of G^* and L^* are

$$\frac{\partial \mathcal{L}}{\partial P} = 0 \quad \frac{\partial \mathcal{L}}{\partial \Lambda} = 0 \quad \frac{\partial \mathcal{L}}{\partial L} = 0 \quad \frac{\partial \mathcal{L}}{\partial G} = 0 \tag{19}$$

For future reference let us introduce the partition

$$\Lambda := \left[\begin{array}{cc} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{array}\right]$$

where all blocks have size $n \times n$. The following proposition summarizes the optimality conditions.

Proposition 1: The necessary conditions (19) for stationary of the Lagrangian (18) admit the unique solution P^*, Λ^*, L^*, G^* where

$$P^* := \left[\begin{array}{cc} P_{11}^* & 0 \\ 0 & P_{22}^* \end{array}\right] \qquad \Lambda^* := \left[\begin{array}{cc} \Lambda_{11}^* & \Lambda_{11}^* \\ \Lambda_{11}^* & \Lambda_{22}^* \end{array}\right]$$

and

$$\begin{aligned} G^* &= A P_{22}^* C^\top \Sigma_\alpha^{-1} \\ L^* &= -\left(B^\top \Lambda_{11}^* B\right)^{-1} B^\top \Lambda_{11}^* A \end{aligned} \tag{20}$$

while

$$\Sigma_\alpha := \left(1 + \frac{1}{\alpha}\right)\left(\sigma_v^2 + C P_{22}^* C^\top\right) + \frac{1}{\alpha} C P_{11}^* C^\top \tag{21}$$

are the optimal gains that solve the LQG-type optimal control problem (17). The matrices P_{11}^*, P_{22}^*, Λ_{11}^* and Λ_{22}^* can be found solving the following (coupled) Riccati-type equations

$$\begin{aligned} P_{11}^* &= A_{L^*} P_{11}^* A_{L^*}^\top + \\ &\quad +(1-\epsilon) A P_{22}^* C^\top \Sigma_\alpha^{-1} C P_{22}^* A^\top \\ P_{22}^* &= A P_{22}^* A^\top + \sigma_w^2 B B^\top + \\ &\quad -(1-\epsilon) A P_{22}^* C^\top \Sigma_\alpha^{-1} C P_{22}^* A^\top \\ \Lambda_{11}^* &= A_{L^*}^\top \Lambda_{11}^* A_{L^*} \\ &\quad + \tfrac{1-\epsilon}{\alpha} C^\top (G^*)^\top (\Lambda_{22}^* - \Lambda_{11}^*) G^* C + C^\top C \\ \Lambda_{22}^* &= \epsilon A^\top \Lambda_{22}^* A + C^\top C + \sigma_w^2 B B^\top \\ &\quad (1-\epsilon)(A - G^* C)^\top (\Lambda_{22}^* - \Lambda_{11}^*)(A - G^* C) + \\ &\quad +(1-\epsilon) A^\top \Lambda_{11}^* A + \\ &\quad + \tfrac{1-\epsilon}{\alpha} C^\top (G^*)^\top (\Lambda_{11}^* - \Lambda_{22}^*) G^* C \end{aligned} \tag{22}$$

where $A_{L^*} := A + BL^*$.

We now report two important results (see the Appendix for a proof) which characterize the structure as well as the existence of a stabilizing estimator-controller pair (G^*, L^*). First we show that, provided it exists, the optimal (G^*, L^*) have a special structure that guarantees the control algorithm can be implemented with memory equal to the state dimension.

Theorem 1: Assume the coupled Riccati equations (22) admits a unique solution. Then the optimal gains (G^*, L^*) satisfy

$$\begin{aligned} G^* &= [g^* \quad a g^* \quad a^2 g^* \quad \ldots \quad a^{d-1} g^*] \\ L^* &= [0 \quad 0 \quad \ldots \quad -a] \end{aligned} \tag{23}$$

so that

$$A_L^* = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \end{bmatrix}$$

is nilpotent, i.e. the optimal controller is dead-beat.

We now show that the optimal value of the cost J^* is finite only provided a certain relation between packet loss probability, SQNR and delay is satisfied. This condition neatly extends the well known condition for the zero delay case [4].

Theorem 2: Consider the optimal control problem (10) under constraint (11) (or equivalently (17)). The optimal value J^* of the cost is finite if and only if

$$\delta := \frac{1-\epsilon}{1+\frac{a^{2d}}{\alpha}} > 1 - \frac{1}{a^2} \tag{24}$$

Under this condition J^* can be expressed as

$$J^* = a^{2d} p_{22}^* + \sum_{i=1}^{d-1} a^{2i}\sigma_w^2 + \sigma_v^2$$

where p_{22}^* is unique positive solution of the scalar Modified Algebraic Riccati Equation (MARE)

$$p_{22}^* = a^2 p_{22}^* + \sigma_w^2 - \delta \frac{a^2 (p_{22}^*)^2}{p_{22}^* + \bar{r}(d)} \tag{25}$$

where

$$\bar{r}(d) := \left(1 + \frac{a^{2d}}{\alpha}\right)^{-1} \left(\left(1 + \frac{1}{\alpha}\right) R + \frac{1}{\alpha} \sum_{i=1}^{d-1} a^{2i}\sigma_w^2 \right)$$

Although the theorem has been formally derived for $d \geq 1$, it provides the correct solution also for $d = 0$, i.e. in the zero delay case that was derived in our previous work [12].

The previous theorem recovers some of the results available in the literature as special cases. In fact if we set $\alpha = \infty$, which is equivalent to consider a channel with infinite capacity, we obtain:

$$\epsilon < \frac{1}{a^2}$$

which is the same stability condition in the lossy network literature [18]. Also, it shows that in the infinite capacity scenario, the stability is independent of the delay d, as shown in [16].

Alternatively, if we assume no packet loss in the channel, i.e. $\epsilon = 0$, and no delay, i.e. $d = 0$, then the stability condition can be rewritten as

$$1 - \frac{1}{a^2} < \frac{1}{1+\frac{1}{\alpha}} = 1 - \frac{1}{1+\alpha}$$

which lead to

$$\alpha > a^2 - 1$$

which is the same stability condition presented in the context of SNR-limited control system in [2].

Finally, the quantity δ defined in Eqn. (24), which is directly related to the stability of the closed loop system, is the first expression that brings together all three channel limitations which have been considered separately in the literature. Such quantity will be useful to compare different communication protocols. In fact, by using a corse quantizer it is possible to reduce the transmission rate R_q, thus allowing more redundant channel coding schemes and consequently a smaller packet loss probability ϵ. On the other hand a corser quantizer gives a smaller α and consequently a higher α. Finally, a more complex coding scheme with higher delay d can reduce the packet loss probability ϵ. Therefore, α, ϵ and d are all coupled and cannot be designed separately.

V. Conclusions and future work

We have considered an LQG control problem which accounts for code rate limitations, as well as for packet drops and delays arising from a communication channel between the sensor and the controller. We have argued in fact that there is a tight connection between the actual rate at which one can transmit information, the decoding delay (due to long block coding) and the packet-drop probability.

We have restricted our attention to a specific control architecture in which the plant outputs are transmitted via a rate limited channel and then processed through the cascade of a state estimator followed by a linear (state) feedback controller. We have considered a scalar model, with feedback channel subject to delay, packet losses, and limited transmit rate, and found that the optimal controller has a dead-beat structure and the optimal estimator is a Kalman-like constant gain estimator (which accounts for the packet drop probability). Conditions for stability are derived in terms of a modified algebraic Riccati equation and recapture results from the literature as special cases.

Appendix

A. Proof of Theorem 1

First of all recall from [16] that the solution P_{11}^*, P_{22}^*, Λ_{11}^*, Λ_{22}^* can be obtained as fixed points of the iterates:

$$\begin{aligned}
G^i &= A P_{22}^i C^\top \Sigma_\alpha^{-1} \\
L^i &= -\left(B^\top \Lambda_{11}^i B\right)^{-1} B^\top \Lambda_{11}^i A \\
P_{11}^{i+1} &= A_{L^i} P_{11}^i A_{L^i}^\top + \\
&\quad +(1-\epsilon) A P_{22}^i C^\top \Sigma_\alpha^{-1} C P_{22}^i A^\top \\
P_{22}^{i+1} &= A P_{22}^i A^\top + \sigma_w^2 B B^\top + \\
&\quad -(1-\epsilon) A P_{22}^i C^\top \Sigma_\alpha^{-1} C P_{22}^i A^\top \\
\Lambda_{11}^{i+1} &= A_{L^i}^\top \Lambda_{11}^i A_{L^i} \\
&\quad + \tfrac{1-\epsilon}{\alpha} C^\top (G^i)^\top (\Lambda_{22}^i - \Lambda_{11}^i) G^i C + C^\top C \\
\Lambda_{22}^{i+1} &= \epsilon A^\top \Lambda_{22}^i A + C^\top C + \sigma_w^2 B B^\top \\
&\quad (1-\epsilon)(A - G^i C)^\top (\Lambda_{22}^i - \Lambda_{11}^i)(A - G^i C) + \\
&\quad +(1-\epsilon) A^\top \Lambda_{11}^i A + \\
&\quad + \tfrac{1-\epsilon}{\alpha} C^\top (G^i)^\top (\Lambda_{11}^i - \Lambda_{22}^i) G^i C
\end{aligned} \tag{26}$$

with initial conditions $P_{11}^0 = P_{22}^0 = \Lambda_{11}^0 = \Lambda_{22}^0 = I$. Now, observe that $B^\top = [0 \ldots 0 \;\; 1]$. This implies that $B^\top \Lambda_{11}^0 = [0 \ldots 0 \;\; \Lambda_{11}^0(d,d)]$ so that $L^0 = [0 \ldots 0 \;\; -a]$. Since Λ_{11}^0 is diagonal, also $A_{L^0}^\top \Lambda_{11}^0 A_{L^0}$ is diagonal and Λ_{11}^1 is still diagonal. As such $B^\top \Lambda_{11}^1 = [0 \ldots 0 \;\; \Lambda_{11}^0(d,d)]$ and therefore $L^1 = [0 \ldots 0 \;\; -a]$. The same argument can be iterated showing that, provided $L^i = [0 \ldots 0 \;\; -a]$ and Λ_{11}^i is diagonal, then Λ_{11}^{i+1} is diagonal and $L^{i+1} = [0 \ldots 0 \;\; -a]$. Therefore, by induction, $L^i = [0 \ldots 0 \;\; -a], \quad \forall i$, which implies that

$$L^* = \lim_{i\to\infty} L^i = [0 \ldots 0 \;\; -a]$$

This completes the proof as far as L^* is concerned. Let us now consider the optimal gain G^*. From Proposition 1 we know that $\hat{\xi}_t$ and $\tilde{\xi}_t$ are uncorrelated. Therefore $\hat{\xi}_t$ can be interpreted as the projection of ξ_t on a certain stationary subspace Ξ_{t-d} of (the space spanned by the components of) z^-_{t-d+1}, i.e.

$$\hat{\xi}_t = \hat{E}[\xi_t|\Xi_{t-d}] \quad \hat{\xi}_{t+1} = \hat{E}[\xi_{t+1}|\Xi_{t-d+1}]$$

where $\hat{E}[\cdot|\cdot]$ denote the orthogonal projection (linear minimum variance estimator). Recalling (6), we now compute the projection components of $\xi_{t+1} := [x_{t-d+2}\ x_{t-d+3}\dots x_{t+1}]$ assuming $\gamma_{t-d+1} = 1$. Using the standard Kalman measurements update and (8) if follows that

$$\begin{aligned}\hat{E}[x_{t-d+2}|\Xi_{t-d+1}] &= \hat{E}[x_{t-d+2}|\Xi_{t-d}]+ \\ &\quad +g^*\left(h_t - C\hat{\xi}_t\right)\end{aligned}$$

for a suitable gain g^*. Similarly

$$\begin{aligned}\hat{E}[x_{t-d+3}|\Xi_{t-d+1}] &= a\hat{E}[x_{t-d+2}|\Xi_{t-d+1}] + bu_{t-d+2} \\ &= a\hat{E}[x_{t-d+2}|\Xi_{t-d}] + bu_{t-d+2} \\ &\quad +ag^*\left(h_t - C\hat{\xi}_t\right) \\ &= \hat{E}[x_{t-d+3}|\Xi_{t-d}]+ \\ &\quad +ag^*\left(h_t - C\hat{\xi}_t\right) \\ &= \hat{E}[x_{t-d+2}|\Xi_{t-d+1}]+ \\ &\quad +ag^*\left(h_t - C\hat{\xi}_t\right)\end{aligned}$$

where the third equality has been obtained using the identity

$$x_{t-d+3} = ax_{t-d+2} + bu_{t-d+2} + w_{t-d+2}$$

and the fact that w_{t-d+2} is orthogonal to Ξ_{t-d}. Iterating we obtain, $\forall k \geq 3$:

$$\begin{aligned}\hat{E}[x_{t-d+k}|\Xi_{t-d+1}] &= a\hat{E}[x_{t-d+k-1}|\Xi_{t-d+1}]+ \\ &\quad +bu_{t-d+k} \\ &= \hat{E}[x_{t-d+k}|\Xi_{t-d+1}]+ \\ &\quad +a^k g^*\left(h_t - C\hat{\xi}_t\right)\end{aligned}$$

which shows that G^* has the structure

$$G^* = [g^*\ ag^* \dots a^{d-1}g^*].$$

B. Proof of Theorem 2

First of all let us serve that, using (12), the state update equation can be written in the form

$$\xi_{t+1} = A_{L^*}\hat{\xi}_t + A\tilde{\xi}_t + Bw_t$$

As shown in Proposition 1 when using the optimal gains L^* and G^* the estimate $\hat{\xi}_t$ and the error $\hat{\xi}_t$ are uncorrelated. Therefore, at steady state,

$$\begin{aligned}\Sigma^* &:= Var\{\xi_{t+1}\} = P^*_{11} + P^*_{22} \\ &= A_{L^*}P^*_{11}A^\top_{L^*} + AP^*_{22}A^\top + \sigma^2_w BB^\top\end{aligned} \quad (27)$$

Note also that Σ^* is the Toeplitz matrix build with the covariance function of x_{t-d+i} and, as such, it is constant along the diagonal. Therefore

$$C\Sigma^*C^\top = H\Sigma^*H^\top \quad H := [0\ 0\ \dots\ 0\ 1]$$

Note also that $HA_{L^*} = [0\ \dots\ 0]$ so that, using (27) and $CB = 0$

$$\begin{aligned}C\Sigma^*C^\top &= H\Sigma^*H^\top = HAP^*_{22}A^\top H^\top \\ &= [0\dots0\ a]P^*_{22}[0\dots0\ a]^\top \\ &= a^2 P^*_{22}(d,d)\end{aligned}$$

where $P^*_{22}(d,d)$ is the diagonal element in position (d,d) of the matrix P^*_{22} (the south-east corner). Now, using the fact the $P^*_{22}(i,i)$ is the $i-$steps ahead state prediction error, it is easy to see that

$$P^*_{22}(d,d) = a^{2d-2}P^*_{22}(1,1) + \sum_{i=0}^{d-2} a^{2i}\sigma^2_w \quad (28)$$

so that

$$\begin{aligned}C\left(P^*_{11} + P^*_{22}\right)C^\top &= C\Sigma^*C^\top \\ &= a^2 P^*_{22}(d,d) \\ &= a^{2d}P^*_{22}(1,1) + \textstyle\sum_{i=1}^{d-1} a^{2i}\sigma^2_w \\ &= a^{2d}CP^*_{22}C + \bar{q}(d)\end{aligned} \quad (29)$$

where $\bar{q}(d) := \sum_{i=1}^{d-1} a^{2i}\sigma^2_w$. We can use this last condition to manipulate Σ_α in (21) as follows:

$$\begin{aligned}\Sigma_\alpha &= \left(1+\tfrac{1}{\alpha}\right)\left(\sigma^2_v + CP^*_{22}C^\top\right) + \tfrac{1}{\alpha}CP^*_{11}C^\top \\ &= \left(1+\tfrac{1}{\alpha}\right)\sigma^2_v + CP^*_{22}C^\top + \tfrac{1}{\alpha}C\left(P^*_{11} + P^*_{22}\right)C^\top \\ &= \left(1+\tfrac{a^{2d}}{\alpha}\right)CP^*_{22}C^\top + \left(1+\tfrac{1}{\alpha}\right)R + \tfrac{1}{\alpha}\bar{q}(d) \\ &= \left(1+\tfrac{a^{2d}}{\alpha}\right)\left(CP^*_{22}C^\top + \bar{r}(d)\right)\end{aligned}$$

where the last equation defines $\bar{r}(d)$. Therefore the equation for P^*_{22} in (22) takes the form of a Modified Algebraic Riccati Equation (MARE) [18]

$$\begin{aligned}P^*_{22} &= AP^*_{22}A^\top + \sigma^2_w BB^\top \\ &\quad -\delta AP^*_{22}C^\top\left(CP^*_{22}C^\top + \bar{r}(d)\right)^{-1}CP^*_{22}A^\top\end{aligned} \quad (30)$$

where

$$\delta := \frac{1-\epsilon}{1+\frac{a^{2d}}{\alpha}}$$

Note also that $HPC^\top = \mathbb{E}[\tilde{x}_{t-d+1}\tilde{x}_t] = a^{d-1}\mathbb{E}[\tilde{x}^2_{t-d+1}] = a^{d-1}P^*_{22}(1,1)$. Now using the fact that $HA = [0\dots0\ a]$ and multiplying (30) by H and $H^\top$ from left and right respectively, we obtain

$$\begin{aligned}HP^*_{22}H^\top &= a^2 HP^*_{22}H^\top + \sigma^2_w \\ &\quad -\delta\tfrac{a^2 HP^*_{22}C^\top CP^*_{22}H^\top}{CP^*_{22}C^\top + \bar{r}(d)}\end{aligned} \quad (31)$$

Defining $p^*_{22} := CP^*_{22}C^\top$ and using (28), so that $HP^*_{22}H^\top = a^{2d-2}p^*_{22} + \sum_{i=0}^{d-2} a^{2i}\sigma^2_w$ equation (32) can be manipulated to yield:

$$p^*_{22} = a^2 p^*_{22} + \sigma^2_w - \delta\frac{\left(a^2 p^*_{22}\right)^2}{p^*_{22} + \bar{r}(d)} \quad (32)$$

It is well known (see [16]) that (32) admits a solution if and only if

$$\delta = \frac{1-\epsilon}{1+\frac{a^{2d}}{\alpha}} > 1 - \frac{1}{a^2}$$

A simple algebraic manipulation shows that this is equivalent to (24). Using now (29) we immediately obtain an expression for the optimal cost:

$$\begin{aligned} J^* &= C(P_{11}^* + P_{22}^*)C^\top + \sigma_v^2 \\ &= CP_{22}^*C + \bar{q}(d) + \sigma_v^2 \\ &= a^{2d}p_{22}^* + \sum_{i=1}^{d-1} a^{2i}\sigma_w^2 + \sigma_v^2 \end{aligned}$$

This concludes the proof.

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