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Reactive Power Pricing Strategy

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Abstract

The aim of this project is to develop a reactive power pricing method exploiting the theory of duality and the Lagrange multipliers in their sense of economic shadow prices. Our work is based on the results presented in a recent article by a research team of Padua University [1] where it has been considered the problem of optimal reactive power compensation in order to minimize the power distribution losses in a smart microgrid. As a matter of fact, we start precisely from the cost function of the article $J(q)$ which denotes the total reactive power losses of the grid and thanks to it we define the Lagrangian function associated with our problem. The Lagrange multipliers are attained by using two different methods: the first one applies the duality theory directly (centralized system) while the other is based on the gradient estimation (distributed system). However, having neglected the voltage and capacity limits of the grid, the information we obtain in relation to the calculated lagrange multipliers is not sufficient to let us interpret them as shadow prices. Owing to the impossibility of operating a standard pricing, we consider other ways to obtain useful economical information related to reactive power. Hence, we define a storage strategy which basically indicates if it is worth storing active power or whether it is preferable to inject less reactive power than that required.

Summary

- In **section 1**, we present smart grids: the reasons for their introduction, their structure and modelization using a graph. Moreover, we focus on the problem of reactive power compensation in order to minimize power losses.

- In **section 2**, we introduce the theory of duality and derive the Lagrange multiplier λ useful to formulate the cost function of the problem. Finally, we provide a graphic calculation of λ that will constitute a reference for all the next calculations.

- In **section 3**, to avoid some problems of approximation characterizing the previous method, we calculate the Lagrange multiplier towards a gradient estimation; at first we present an estimation which uses the PCC voltage and then a distribution-like way of estimation in which two nodes of the network exchange information with each other.

- In **section 4**, we provide some considerations about the communication strategies relating to the gradient estimation method parent-child.

- In **section 5**, we show some validation tests for the calculations exhibited formerly.

- In **section 6**, we start presenting the well-known economic theory of the Lagrange multipliers and we demonstrate that this theory is not applicable to our case of study. Then we discuss the storage strategy in the reactive power management as well as its limits of application.

- Finally, in **section 7**, we take stock of the different results obtained and discuss possible future developments.

Contents

1 Introduction

The current global energy system shows the preponderance of a few large-scale generating stations far from load centers. This complex and expensive infrastructure, which significantly affects final electricity price, is characterized by a certain rigidity essentially due to an unidirectional flow. In this context, the final user has a passive role which consists in being only a consumer. Recently, a push for change has come from the emergence of new factors [2], [3]:

- energy market liberalization
- improvement of the the grid efficiency and stability
- exploitation of renewable sources.

The best answer to all these points can be found in a distributed generation approach that employs small-scale technologies to produce electricity close to the end users of power. In many cases, distributed generators can provide lower-cost electricity and higher power reliability and security with fewer environmental consequences than can traditional power generators. Particularly, the introduction of new energy sources suggests that in the future much more people will become energy producers thanks to the increasing scientific interest in this field. The main goal of this new energy dispatch concept is to pass from the centralized electricity system which primarily delivers electricity in a one-way flow from generator to outlet, to the bidirectional flow of both electricity and information. This has two important consequences. It allows consumers to become active part of the energy supply process, giving them access to the demand response mechanism, and fosters the diversification of energy sources. The introduction of modern technology in utility electricity delivery systems is accompanied by the idea of dividing the electric grid into smaller portions called microgrids. Like the bulk power grid, smart microgrids generate, distribute, and regulate the flow of electricity to consumers, but do so locally. Together with the loads connected to the microgrid, we find microgeneration devices. These are connected to the microgrid via electronic interfaces inverters which not only enable the injection of the produced power into the microgrid but also perform different other tasks denoted as ancillary services [4], [5]: reactive power compensation, harmonic compensation, voltage support. As a result, the network appears to be parcelled into many autonomous subunits each one aiming at the enhancement of the global electricity service quality.

1.1 A smart grid model

The microgrid has been modelled using a directed graph $G = (\mathcal{V}, \mathcal{E}, \sigma, \tau)$ where $\mathcal{V}$ represents the set of nodes with $n = |\mathcal{V}|$, $\mathcal{E}$ corresponds to the set of edges with $r = |\mathcal{E}|$ and finally $\sigma,\ \tau\ : \mathcal{E} \mapsto \mathcal{V}$ are two functions which associate to each edge e its source node $\sigma(e)$ and its terminal one $\tau(e)$.

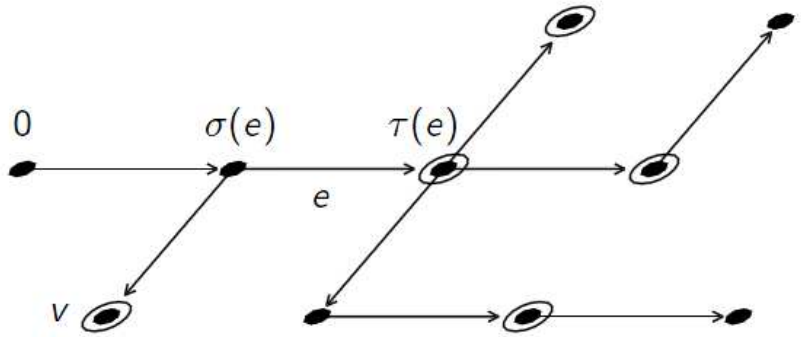

Figure 1 *Graph model: circled nodes represent microgenerators and the edge orientation is consistent with the sign of the power line currents*

A practical way to describe the topology of the grid is given by the incidence matrix associated with the graph which is defined as

$$[A]_{ev} := \begin{cases} -1 & \text{if } v = \sigma(e) \\ 1 & \text{if } v = \tau(e) \\ 0 & \text{otherwise} \end{cases}$$

From a physical perspective, each edge corresponds to a power line while each node represents both loads and generators connected with the microgrid. These include residential and industrial consumers, microgenerators and the PCC which is the point of connection of the microgrid to the transmission grid. Assuming a steady state operating system, we have that all voltages and currents are sinusoidal signals at the same frequency ω_0. Therefore, each of them can be represented by a complex number such as $y(t) = |y|\sqrt{2}\ sin(\omega_0 t + \angle y)$ where $|y|$ and $\angle y$ are the absolute value and phase of y at $t = 0$. With regard to the system variables they are respectively:

- $u \in \mathbb{C}^n$, where u_v is the voltage at node v;
- $i \in \mathbb{C}^n$, where i_v is the current injected by node v into the grid;
- $\xi \in \mathbb{C}^r$, where ξ_e is the current flowing on the edge e into the grid;

Using these together with the incidence matrix, we can write the Kirchhoff's current (KLC) and voltage (KVL) laws in a compact way obtaining the following constraints

$$\begin{aligned} A^T\xi + i &= 0 \\ Au + Z\xi &= 0 \end{aligned} \tag{1}$$

where $\mathbf{Z} = diag(z_e, e \in \mathcal{E})$ is the diagonal matrix of line inpedances.

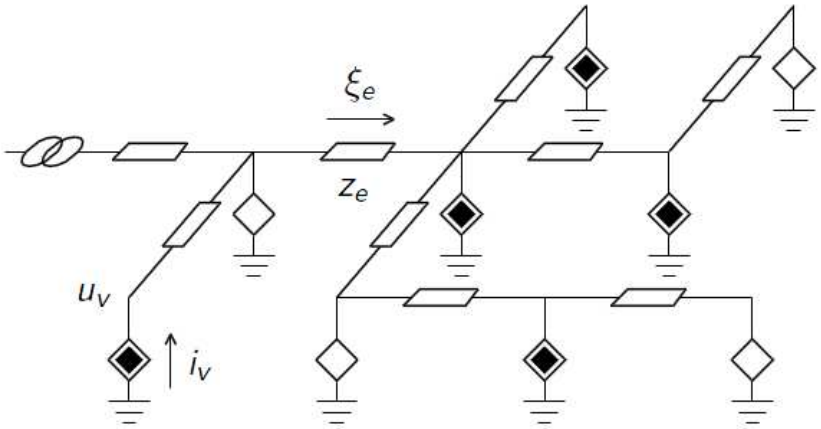

Figure 2 *Electric network where black diamonds are microgenerators, white diamonds are loads and the left-most element of the circuit represents the PCC*

We choose to model the PCC node as a costant voltage generator, i.e.

$$u_0 = U_0 \tag{2}$$

while the other ones satisfy the following law

$$u_v \bar{i}_v = s_v \left| \frac{u_v}{U_0} \right|^{\eta_v}, \ \forall v \in \mathcal{V} \backslash \{0\} \tag{3}$$

with s_v the nominal complex power and η_v the characteristic parameter associated with the respective node. This model, widely adopted in the literature of power flow analysis [6] as a generalization of the ZIP one, is called *exponential model* [7] and according to the value assumed by the parameter η_v it represents the behaviour of:

- constant power devices, for $\eta_v = 0$
- constant current devices, for $\eta_v = 1$
- constant impedance devices, for $\eta_v = 2$

Instead of solving the system relative to equations (1), (2), (3), we introduce two important matrices $\mathbf{L}$, $\mathbf{X}$ which will be used in order to obtain an approximate model

for the microgrid state. In particular, matrix $\mathbf{X}$ will come in handy when deriving the reactive power compensation distributed algorithm. Back to $\mathbf{L}$, this is the *Laplacian matrix* and is defined as

$$\mathbf{L} = A^T \mathbf{Z}^{-1} A$$

which substituted in (1) gives a solution for the current vector of the microgrid, i.e.

$$i = \mathbf{L}u \tag{4}$$

The task of deriving a similar expression for the voltages can be handled by finding a matrix which allows to express the vector of voltages as a function of that relating to currents. Since $\mathbf{L}$ is not invertible, we resort to matrix $\mathbf{X}$, called *Green matrix* [8]. This is defined by the following lemma whose proof is to be found in [1].

Lemma Let $\mathbf{L}$ be the complex valued Laplacian $\mathbf{L} = A^T \mathbf{Z}^{-1} A$. There exists a unique symmetric matrix $\mathbf{X} \in \mathbb{C}^{n \times n}$ such that

$$\begin{cases} \mathbf{X}\mathbf{L} = I - \mathbf{1}\mathbf{1}_0^T \\ \mathbf{X}\mathbf{1}_0 = 0 \end{cases} \tag{5}$$

where I is the identity matrix and $[\mathbf{1}_0]_v = 1$ for $v = 0$ and 0 otherwise. Thanks to $\mathbf{X}$, it is now possible to express the voltage u as a linear function of the current i, i.e.

$$u = \mathbf{X}i + U_0 \mathbf{1} \tag{6}$$

The last equation found allows us to write the system in the following form

$$\begin{cases} u = \mathbf{X}i + U_0 \mathbf{1} \\ \mathbf{1}^T i = 0 \\ u_v \bar{i}_v = s_v \left| \frac{u_v}{U_0} \right|^{\eta_v}, \quad \forall v \in \mathcal{V} \backslash \{0\} \end{cases} \tag{7}$$

Finally, using the Taylor approximation of $i(U_0)$ for large U_0, we obtain a useful expression for currents which is used to formulate the power losses minimization problem and whose proof can be found in [1]

$$i_v(U_0) = (\bar{s}_v + \delta_v(U_0)) \frac{1}{\overline{U}_0} \tag{8}$$

where $\delta_v(U_0) \to 0$ when $U_0 \to \infty$

1.2 Reactive power compensation problem

Reactive power contributes to no useful work, however it is required to maintain the voltage to deliver active power through transmission lines. A shortage of this kind of power results actually in a voltage drop which makes impossible to push the power demanded by loads through the lines. From a physical perspective, reactive power is present when voltage and current are not in phase and therefore it occurs in AC electric systems. Anyway, it becomes particularly relevant when inductive or capacitive loads are connected to the grid. As a consequence, we have an extra current which increases Joule heating of the transmission lines. All this compels the electric companies to install heavier wires capable of tolerating the excess current and the result is extra charges for the users. It is evident that reactive power flows across the lines concur to power losses and induce voltage drop making the grid at risk of instability. Thus, reactive power flows need to be minimized by producing it as close as possible to the users that require it [9].

As previously mentioned, since our work is based on the results presented in the recently released article "A distributed control for optimal reactive power compensation in smart microgrids" [1] which considers the problem of optimal reactive power compensation in a smart microgrid, we are going to outline the main points of the solving algorithm designed by the researchers. Anyhow, it must again be stressed that our contribution starts with this algorithm which, though, is applicable to any kind of scenarios. Indeed, the estimation of reactive power prices does not necessarily depend on the presence of a control algorithm in action, nor on the type of algorithm used.

First of all the original problem needs to be approximated to a *convex quadratic optimization* one. Choosing total active power losses as a metric for optimality of reactive power flows and exploiting the Taylor expansion of i (see expression 8), the cost function relating to the active power losses can be written as

$$\begin{aligned} J' &= \bar{i}^T Re(\mathbf{X}) i = \frac{1}{|U_0|^2} \bar{s}^T Re(\mathbf{X}) s + \frac{1}{|U_0|^2} \tilde{J}(U_0, s) \\ &= \frac{1}{|U_0|^2} p^T Re(\mathbf{X}) p + \frac{1}{|U_0|^2} q^T Re(\mathbf{X}) q + \frac{1}{|U_0|^2} \tilde{J}(U_0, s) \end{aligned} \tag{9}$$

where $\tilde{J}(U_0, s)$ is infinitesimal for large U_0 and $s = Re(s) + Im(s) = p + q$. The found expression is of notable interest as discloses two important aspects:

- power losses can be approximated as a quadratic function of the injected power s
- the problem of optimal power flows can be divided into the problem of optimal active and reactive power injection

On the basis of this uncoupling phenomenon, it is possible to formulate the problem of optimal reactive power injection as a quadratic linearly constrained problem. Let $\mathcal{C} \subset \mathcal{V}$ be the subset of the nodes (with cardinality $m := |\mathcal{C}|$) of the microgrid which can be commanded. Moreover, suppose that for each agent is only allowed to control the amount of reactive power injected into the grid as the decision on the amount of active power abides by economic rules. Thus, the problem of optimal reactive power injection becomes

$$\begin{aligned} \min_{q} \quad & J(q), \qquad \text{where} \quad J(q) = \frac{1}{2} q^T Re(\mathbf{X}) q, \\ \text{subject to} \quad & \mathbf{1}^T q = 0 \\ & q_v = Im(s_v) \quad \forall v \in \mathcal{V} \backslash \mathcal{C} \end{aligned} \tag{10}$$

$q_v = Im(s_v) \ \ \forall v \in \mathcal{V}\backslash\mathcal{C}$ being the nominal amount of reactive power injected that cannot be commanded. The distributed approach, chosen by the researchers, is based on a neighbour to neighbour exchange of information and essentially consists in decomposing the optimization problem into smaller, tractable subproblems which are assigned to small groups of agents. Consider the family of subsets of $\mathcal{C}$, $\{\mathcal{C}_1, ..., \mathcal{C}_l\}$ such that $\bigcup_{i=1}^{l} \mathcal{C}_i = \mathcal{C}$. In the light of this it is convenient to partition q as

$$q = \begin{bmatrix} q_{\mathcal{C}} \\ q_{\mathcal{V}\backslash\mathcal{C}} \end{bmatrix} \tag{11}$$

where $q_{\mathcal{C}} \in \Re^m$ and $q_{\mathcal{V}\backslash\mathcal{C}} \in \Re^{n-m}$. As a consequence, $Re(\mathbf{X})$ results to be partitioned in the following way

$$Re(\mathbf{X}) = \begin{bmatrix} M & N \\ N^T & Q \end{bmatrix} \tag{12}$$

the *i-th* optimization subproblem is

$$\min_{\Delta q} J(q + \Delta q) \quad \text{subject to} \quad \Delta q \in \mathcal{S}_i \tag{13}$$

where

$$\mathcal{S}_i := \left\{ q_{\mathcal{C}} \in \Re^m : \sum_{j \in \mathcal{C}_i} [q_{\mathcal{C}}]_j = 0, \ [q_{\mathcal{C}}]_j = 0 \ \forall j \notin \mathcal{C}_i \right\}$$

Every time a cluster $\mathcal{C}_i$ is randomly chosen, the optimization algorithm works executing the following sequence of steps:

1. agents not belonging to $\mathcal{C}_i$ keep their amount of reactive power constant while those belonging to $\mathcal{C}_i$ sense the network and calculate an estimate of the gradient

$$\nu_k^{(i)} = \left[\left(e^{-j\theta} \frac{1}{|\mathcal{C}_i|} \sum_{v \in \mathcal{C}_i} \bar{u}_v \right) u_k \right]$$

2. on the basis of the computed gradient, the agents belonging to the cluster determine a feasible update step which minimizes the given cost function, coordinating their actions, and finally update the respective state q_v

2 Duality results for reactive power pricing

Having seen the motivations which make reactive power a fundamental issue, it becomes of primary importance to establish a mechanism which allows a financial compensation of this service, that is the problem of reactive power pricing.

2.1 Theory of duality and Lagrange multipliers

We start our analysis looking for the parameters which allow us to theorize a correct pricing and so we consider the original problem of reactive power compensation, trying to solve the optimization problem using the theory of duality [10] and the Lagrange multipliers method. We consider the optimization problem in the standard form:

$$\begin{aligned} &\min \ f_0(x) \\ &\text{s.t} \ f_i(x) \leq 0, \quad h_i(x) = 0 \end{aligned} \tag{14}$$

and denote the optimal value of (14) by p^*. The basic idea in Lagrangian duality is to take the constraints into account by augmenting the objective function with a weighted sum of the constraints fuctions. In this way, we define the Lagrangian function associated with the problem as follows:

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{n} \lambda_i f_i(x) + \sum_{i=1}^{n} \nu_i f_i(x) \tag{15}$$

where we refer to λ_i and to ν_i as the Lagrange multipliers.
We define the Lagrange dual function as

$$g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu) \tag{16}$$

For each pair (λ, ν), the Lagrange dual function gives us a lower bound on the optimal value p^* of the optimization problem. In order to obtain the best one from the dual function, we formulate the following new problem

$$\begin{aligned} &\max \ g(\lambda, \nu) \\ &\text{s.t} \ \lambda \geq 0 \end{aligned} \tag{17}$$

and we refer to λ^* and ν^* as the Lagrange optimal multipliers if they are optimal for the problem above.
It is proved [11] that if x^* and λ^* are the primal and dual optimal points for a problem, they form a saddle point for the Lagrangian; the optimal dual variables so provide very useful information about the sensitivity of the optimal value with respect to the constraints perturbation. Suppose that $p^*(u, v)$ is differentiable at $u = 0$ and $v = 0$, then the optimal dual variables are related to the gradient of p^* at $u = 0$ and $v = 0$ as:

$$\lambda_i^* = -\frac{\partial p^*(0,0)}{\partial u_i} \qquad \nu_i^* = -\frac{\partial p^*(0,0)}{\partial v_i} \tag{18}$$

This property, where $-\lambda^*$ is the slope of p^* near $u = 0$ is explained and demonstrated in the appendix. Thus, we can conclude that Lagrange multipliers can be seen as the local sensitivities of the optimal value with respect to the constraints perturbation.

2.2 Load and compensators constraints analysis

At this point, we are ready to apply the illustrated theory to our initial problem. Let us consider the following partitions:

$$q = \begin{bmatrix} q_{PCC} \\ q_c \\ q_l \end{bmatrix}$$

where q_c and q_l are the reactive powers pertaining to compensators (except the PCC) and loads respectively. In order to find the solution to the problem, we first consider the Lagrangian function:

$$L(q, \lambda_l) = \frac{q^T Re(\mathbf{X}) q}{|u_0^2|} + \lambda_l^T (q - Q_l) \tag{19}$$

where $\lambda_l^T(q - Q_l)$ represents the equality constraints on load reactive power and Q_l is the load limit.

$$\lambda_l = \begin{bmatrix} 0_m \\ \lambda_l^{'} \end{bmatrix} \qquad\qquad Q_l = \begin{bmatrix} 0_m \\ Q_l^{'} \end{bmatrix}$$

Note that, thanks to this partitioning, only the load part of vector q appears to be constrained.

Deriving with respect to vector q:

$$\frac{\partial L(q,\lambda_l)}{\partial q} = \begin{bmatrix} \frac{\partial L(q,\lambda_l)}{\partial q_1} \\ \frac{\partial L(q,\lambda_l)}{\partial q_2} \\ \vdots \\ \frac{\partial L(q,\lambda_l)}{\partial q_n} \end{bmatrix} = \begin{bmatrix} \frac{2e_1^T Re(\mathbf{X})q}{|u_0^2|} + \lambda_{l_1} \\ \frac{2e_2^T Re(\mathbf{X})q}{|u_0^2|} + \lambda_{l_2} \\ \vdots \\ \frac{2e_n^T Re(\mathbf{X})q}{|u_0^2|} + \lambda_{l_n} \end{bmatrix} = \frac{2Re(\mathbf{X})q}{|u_0^2|} + \lambda_l \tag{20}$$

we obtain:

$$q^*(\lambda_l) = \frac{|u_0^2|}{2} Re(\mathbf{X})^{-1} \lambda_l \tag{21}$$

and, substituting q into the Lagrangian, we obtain the following dual function:

$$\begin{aligned} g(\lambda_l) &= -\frac{|u_0^2|}{4} \lambda_l^T Re(\mathbf{X})^{-1} \lambda_l - \lambda_l^T Q_l \\ &= -\frac{|u_0^2|}{4} \begin{bmatrix} 0_m & \lambda_l^{'T} \end{bmatrix} Re(\mathbf{X})^{-1} \begin{bmatrix} 0_m \\ \lambda_l^{'} \end{bmatrix} - \begin{bmatrix} 0_m & \lambda_l^{'T} \end{bmatrix} \begin{bmatrix} 0_m \\ Q_l^{'} \end{bmatrix} \end{aligned} \tag{22}$$

Operating as above:

$$\frac{\partial g(\lambda_l^{'})}{\partial \lambda_l^{'}} = \begin{bmatrix} 0_m \\ -\frac{|u_0^2|}{2} e_{m+1}^T Re(\mathbf{X})^{-1} \begin{bmatrix} 0_m \\ \lambda_l^{'} \end{bmatrix} - e_{m+1}^T Q_l \\ \vdots \\ -\frac{|u_0^2|}{2} e_n^T Re(\mathbf{X})^{-1} \begin{bmatrix} 0_m \\ \lambda_l^{'} \end{bmatrix} - e_n^T Q_l \end{bmatrix} = \begin{bmatrix} 0_m \\ -\frac{|u_0^2|}{2} D\lambda_l^{'} - Q_l^{'} \end{bmatrix} \tag{23}$$

where we assume that:

$$Re(\mathbf{X})^{-1} = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$$

Finally, imposing the derivative equal to zero, we find:

$$\lambda_l^{'*} = -\frac{2}{|u_0^2|} D^{-1} Q_l^{'} \tag{24}$$

which inserted in $q^*(\lambda_l)$ gives q^*. This one collects the amounts of reactive power that have to be injected into the network in order to minimize the losses. The found solution can only be implemented using a centralized algorithm which requires a global knowledge of the network topology and the existence of a central unit whose task is to collect the local information coming from compensators. The weakness of such type of solution is related to the size of the network (i.e. the number of micro-generators) that can be quite large. This leads to a very high computational cost and consequently to the need of a robust, efficient and broadband communication system. Besides, the network topology can be very changeable as a generator can become a load at every instant, for instance when it cannot satisfy the energetic demand; in this case or when the central unit is damaged, the system requires expert personnel to reset it. Furthermore, we notice that this solution calculates λ_l^* without knowing the reactive powers vector q. Since compensators supply their reactive power through an inverter which has a limitation on the maximum power delivered, and considering the fact that compensators have always to supply and not to absorb reactive power, we take into account only the positive limitation. Then we define $\lambda_c^T(q - Q_c)$ which represents the inequality constraints on the maximum reactive power generated by compensators where Q_c is the inverter limit.

$$\lambda_c = \begin{bmatrix} \lambda_c^{'} \\ 0_p \end{bmatrix} \qquad\qquad Q_c = \begin{bmatrix} Q_c^{'} \\ 0_p \end{bmatrix}$$

from which we obtain the Lagrangian function:

$$L(q, \lambda_l, \lambda_c) = \frac{q^T Re(\mathbf{X}) q}{|u_0^2|} + \lambda_l^T (q - Q_l) + \lambda_c^T (q - Q_c) \tag{25}$$

Performing calculations as above, the solution clearly becomes quite arduous. Hence, we have another reason (in addition to the disadvantages mentioned before) for abandoning this way of proceeding.

2.3 Graphic calculation of Lagrange multiplier

Exploiting the perturbation theory of duality [11], we derive a Lagrange multiplier which can be considered 'correct'. Precisely because of this theory, this parameter can be used as a valid criterion for analyzing the performance of the different multipliers estimation methods.

The calculation of a minimum point of the Lagrangian function (n+1-dimensional) corresponds to find the vector associated with the reactive powers of all the nodes which

minimizes the grid losses. Assuming the optimal powers q^* of all the nodes to be known eccept the i-th one ($q_i \neq q_i^*$), we manage to obtain a two-dimensional quadratic function (parabola) consisting in an input equal to the reactive power value relative to the i-th node and an output equal to the value of the grid losses.
In order to generate the i-th curve, we compute some grid losses values obtained by making the i-th node varying its q_i. So there are two cases: if the node is a load it is sufficient to change its absorbed reactive power; if on the other hand we deal with a compensator, this one has to be removed from the compensators set and inserted in that referring to the loads. For each node (except the PCC since the corresponding curve does not exist), we calculate 21 points in a range of ± 200 kVAR and then we identify the parabola which goes through these points. Once we have found such curve, we differentiate it obtaining the equation of a straight line. Eventually, we calculate λ_i^* which corresponds to the reactive power q_i^* i.e. the real value of the Lagrange multiplier.

Relating to the curves generation method, it is appropriate to make some considerations about PCC and compensators:
PCC: with regard to the PCC, it makes little sense to consider the losses curve as a function of its injected reactive power. As a matter of fact, the PCC can't vary this parameter at will but it must provide the reactive power required by the grid. This is particularly evident when looking at the additional equation which let us define the *Green matrix* $q\mathbf{1} = 0$. The mentioned matrix makes an element belonging to the vector q linearly dependent from the others and this element is precisely that corresponding to the PCC. In the light of these qualifications, the reason why the lambda associated with the PCC is set equal to zero has to be found in the fact that we cannot create the respective perturbation curve;
Compensators: focusing on compensators, we notice that if these devices have no power supply limitations (or they do not reach them) they tend to approach the operating point corresponding to losses minimization. As they already work at their minimum point, they are characterized by $\lambda = 0$. In the event that the inverter limitations come into play, we can imagine it to be substituted by a load which supplies an amount of reactive power equal to the limitation of the inverter. Therefore, we realize that $\lambda \leq 0$ and that the minimum of the curve is located on the right. Precisely because of these reasons, it has also been decided not to create the curves associated with the compensators as we assume them to be operating within the respective limits. If we did not make this hypothesis, the camparison between the estimated lambda and the graphical ones would become overly complicated.

At any rate, if we want all the compensators to respect the limits allowed, it is necessary to follow the next steps:

1. compute the vector q^* omitting the respective limitations;
2. check if there is a compensator which exceeds the limits allowed;
3. insert the compensators which exceed the allowed limits into the loads set and set their supplied reactive power equal to the limitation value;
4. recalculate the vector q^* in the new configuration;
5. if there are some other compensators which exceed the allowed limitations, the whole sequence of actions (starting from the second point) has to be repeated until it is sure that all the devices respect the limits allowed

In order to generate the curves, estimate the graphical lambda and save all these data, it has been implemented the function 'LambdaGraphicsCreator' which is to be found in the appendix. This subroutine allows to calculate the losses via distributed or direct ('offline') algorithm.
The following two graphs represent real lambda in the case of a load and a compensator:

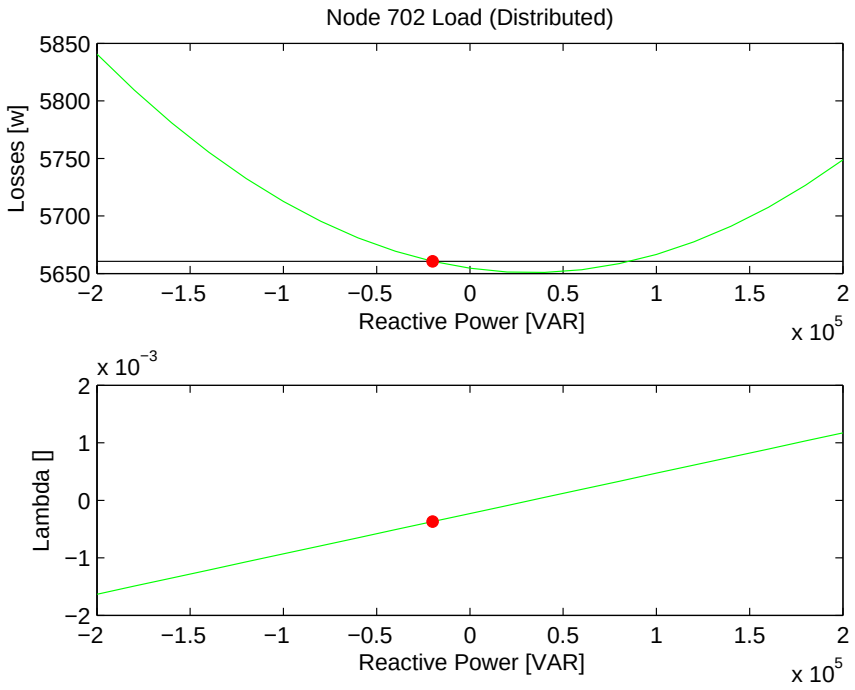

Figure 3 *Real lambda relating to a load*

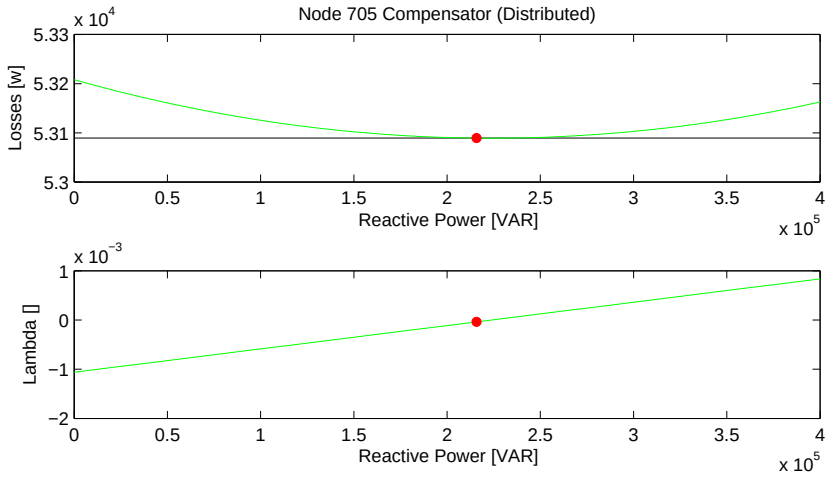

Figure 4 *Real lambda relating to a compensator*

3 A pricing strategy via gradient estimation

Since lambda calculation is complex and difficult to implement, we go back to the theory of duality in order to obtain a function which is able to compute the parameter λ having as input the reactive powers vector q. In this way, we manage to obtain the parameter in a transitory configuration with q not being at its optimum. Beginning from the Lagrangian with constraints on both loads and compensators (25) and deriving with respect to vector q, we obtain:

$$q(\lambda) = \frac{|u_0^2|}{2} Re(\mathbf{X})^{-1} \lambda \tag{26}$$

where we assume $\lambda = \begin{bmatrix} \lambda_c' \\ \lambda_l' \end{bmatrix}$ and q is supposed to be different from q^* in order to stress its generality. Solving for λ we have:

$$\lambda(q) = -\frac{2Re(\mathbf{X})}{|u_0^2|} q \tag{27}$$

where $Re(\mathbf{X})q$ is the gradient [1].

3.1 Gradient estimation using PCC voltage

In order to calculate λ without using matrix $\mathbf{X}$ and vector q, which require the whole network knowledge, we use $Re(\mathbf{X})q$ estimation in place of its real value.
Starting from the voltages vector equation:

$$u = e^{j\theta} Re(\mathbf{X}) i + u_0 \mathbf{1} \tag{28}$$

and from the current vector estimation:

$$i \cong \frac{\overline{s}}{\overline{u_0}} \tag{29}$$

we find:

$$Re(\mathbf{X})\overline{s} \cong e^{-j\theta}\overline{u_0}(u - u_0\mathbf{1}) \tag{30}$$

At first we assumed θ (phase shift of transmission lines) to be constant and precisely equal to θ_{ave} (i.e. equal to the average of the different phase shifts). Seeing as the results obtained adopting this hypothesis were not correct, we then decided to consider θ as the vector of phase shifts between each node and the PCC.
Noting that

$$cos\theta Xq = Re(\mathbf{X})q = -Im[Re(\mathbf{X})\overline{s}] \cong -Im[cos\theta\, e^{-j\theta}\overline{u_0}(u - u_0\mathbf{1})] \tag{31}$$

and substituting this expression into (27)

$$\lambda(u) = \frac{2}{|u_0^2|} Im\left[cos\theta\, e^{-j\theta}\overline{u_0}(u - u_0\mathbf{1})\right] \tag{32}$$

Knowing also that $|u_0^2| = u_0\overline{u_0}$ and $cos\theta$ are real, we obtain

$$\lambda(u) = 2cos\theta\, Im\left[\frac{e^{-j\theta}(u - u_0\mathbf{1})}{u_0}\right] \tag{33}$$

As the resulting λ parameters are uncorrectly overestimated, we need to devise another method to estimate the gradient. We presume that the wrong results come from an incorrect interpretation of parameter θ which shouldn't depend only on the impedance between two nodes but also on the phase shifts of the other branches.
To show this, we have solved the function (33) for θ. Using the graphical lambda and the voltages obtained by losses minimization, we manage to find the experimental values of θ which, besides not corresponding to those we started with, they also don't seem to be a linear combination of theabove-mentioned θ.

3.2 Distributed gradient estimation

For these reasons, we try to find out a different way to calculate the gradient estimation in order to have a phase shift θ less subject to such errors. Going back to the equation:

$$\nu_k^{(i)} := Im\left[\left(e^{-j\theta}\frac{1}{|C_i|}\sum_{v\in C_i}\overline{u_v}\right)u_k\right] \tag{34}$$

which represents the gradient estimation in a cluster, instead of considering a cluster, we focus on a pair of grid nodes (parent-child) and we rewrite the formula as:

$$\nu_c = Im\left[\frac{e^{-j\theta}}{2}(\overline{u_p}+\overline{u_c})u_c\right] \tag{35}$$

$$\nu_p = Im\left[\frac{e^{-j\theta}}{2}(\overline{u_p}+\overline{u_c})u_p\right] \tag{36}$$

where p stands for 'parent' and c stands for 'child'. Defining:

$$\Delta\nu_{p,c} := \nu_p - \nu_c = Im\left[\frac{e^{-j\theta}}{2}(\overline{u_p}+\overline{u_c})(u_p-u_c)\right] \tag{37}$$

from which:

$$\nu_c = \nu_p - \Delta\nu_{p,c} \tag{38}$$

Given that

$$\lambda_c = -\frac{2cos\theta\nu_c}{|u_0^2|}$$

and inserting (38)

$$\begin{aligned}\lambda_c &= -\frac{2cos\theta\nu_p}{|u_0^2|} + \frac{2cos\theta}{|u_0^2|}Im\left[\frac{cos\theta\, e^{-j\theta}}{2}(\overline{u_p}+\overline{u_c})(u_p-u_c)\right] \\ &= \lambda_p + \frac{cos\theta^2}{|u_0^2|}Im\left[e^{-j\theta}(\overline{u_p}+\overline{u_c})(u_p-u_c)\right]\end{aligned} \tag{39}$$

where θ is the phase shift between two nodes of the transmission lines. The results we get by using such method are substantially different from those previously found and this can be seen looking at the following graph.

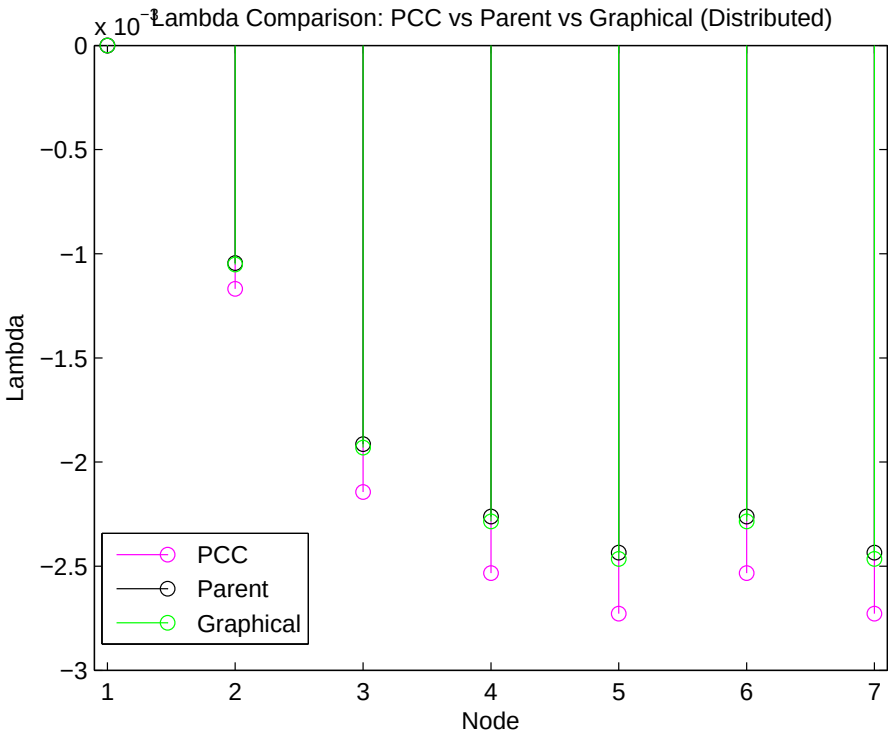

Figure 5 *Comparison of lambda obtained with different methods*

Performances and limitations will be discussed later.

4 Communication strategy

The analysis of the λ estimation formula (39) shows that, in order to obtain a correct update of the values, each child node belonging to the tree needs to receive the correct updated data from its respective parent. In this way, the algorithm converges to the optimal solution imposing an ordered communication starting from the PCC and going through each next level of the tree. The overall process takes a number of steps equal to the maximum number of tree levels minus one. On the other hand, using a random communication between the various nodes, the correct result would be obtained in a number of steps not less than (equal to) that we would get exploiting an ordered communication. In the case of random communication, it can easily happen that the temporal lambda calculated by the node turns out to be wrong. As a matter of fact, if a node updates its lambda value making use of that not correctly updated by the parent node in combination with the voltages corresponding to the new configuration, then it finds a value which does not reflect neither the previous state nor the up-to-date one. Translated into economic terms, employing an incorrect lambda value may lead to price instability therefore sug-

gesting it is preferable to use lambda values correctly updated in the minimum number of steps. In particular, since electric powers are paid hourly rates on average cost basis, there is no point in tracking the real value instant-by-instant incurring the risk of finding transition values not so close to the real ones. Anyhow, it is worth recalling that the absorption variations of the grid are very slow (almost stationary) compared to the time complexity of the algorithms for the minimization of the losses. Hence, we can consider the lambda estimation as belonging to an upper management level and so not needing a reconstruction speed equal to that of the inferior levels. The explanation for this is found in the fact that lambda represents a parameter which does not directly affect the physical stability of the grid, but influences the prices characterized by larger ranges of variation in temporal indices. Eventually, as regards the data to be exchanged, these are two: the voltage value and the lambda associated with the parent node in relation to its children. In view of such a small amount of data, it is convenient to opt for a centralized communication strategy.

Much more delicate is the correct management of the grid information flow choosing an ordered communication or a centralized one. In the case of an ordered communication (i.e. through each level), the best solution consists in making each parent node to pass its data packets to all its children at each instant. This kind of communication is by far the most convenient in terms of execution times, however it appears to be more difficult to implement. Indeed, it requires that the grid allows to transfer various types of information along different paths at the same time and besides, that the nodes are able to read and write simultaneously. Thus, it turns out to be much easier to impose a single communication between two nodes at each instant, activating the various nodes according to an ordered sequence. The weak point of this strategy is its time complexity as it takes a number of steps equal to the number of nodes minus one to converge to the optimal solution. Opting for a centralized communication, on the other hand, it is possible to make the central node ask for the necessary data (voltages) to each other node following a certain order or let each single node send all its data to the central node independently.

5 Simulations and validation

In order to test the Lagrange multipliers calculation methods, we have created some simple grids. These give us a better understanding of the way the Lagrangian parameters vary and an idea of the factors by which they are influenced.

In particular, we created three different grid configurations: a 6-node and a 12-node grid both presenting aligned nodes and a 7-node grid with symmetric bifurcation (i.e. a 'Y' shape).

First of all we analyse the behaviour of real lambda (i.e. the graphical ones) trying to find a connection between such behaviour and the grid context which takes into consideration the load and the compensator configurations.

5.1 Real lambda analysis

To determine graphical λ, we implemented the function 'LambdaTester()' with argument 'CalDist' which is to be found in the appendix. Let us assume that all the edges length and the respective phase angles are equal. In the first test configuration we consider the 6-node linear grid in which the first node coincides with the PCC while all the others represent the loads. We set all the loads to zero ($S = 0 + j0$) except the third one which is enabled to absorb active and reactive power. As a consequence, the power flows only in the grid portion within the PCC and the third node. Besides, since the subsequent nodes are all off and the respective edges are interested by no current we should expect the corresponding lambda to be null.

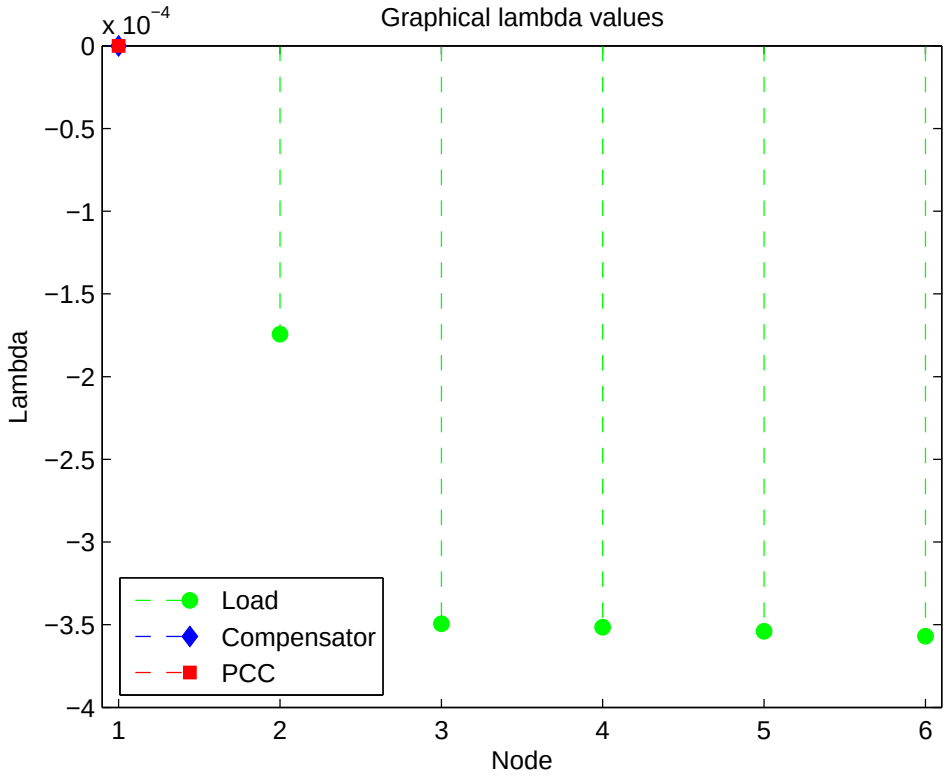

Figure 6 *6-node linear grid whith the 3^{rd} node active and all the others off*

However, looking at the graph in figure 6, which shows the graphical lambda obtained, we notice indeed that the lambda of the third node is very similar to those characterizing the subsequent ones. Investigating the matter, we realize that these nodes can be excluded from the grid but, in the case they inject an infinitesimal amount of reactive power, their contribution is similar to that we would obtain injecting such power quantity at the third node. This fact appears to be true because we have no power flow interesting the edges. Consequently, the losses of the grid are equal to zero and a minimal increment cannot alter them. As regards the second node, we find out that its value is approximately intermediate between the PCC one and that relating to the third node.
We exploit the same grid to do another simulation. This time we enable only the last node to absorb current while all the others are off. Actually, we want to analyze the way the lambda associated with the inactive nodes varies knowing that in this new situation we have a power flowing through the edges.

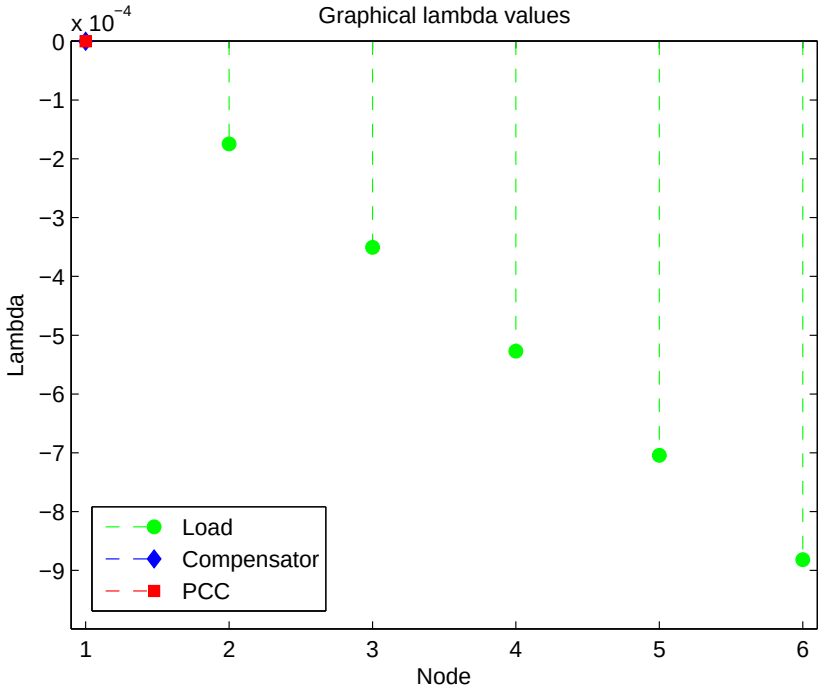

Figure 7 *6-node linear grid whith the last node active and all the others off*

Examining the graph in figure 7, it is evident that lambda parameters grow linearly from 0 associated with the PCC to the maximum value relating to the sixth node which is enabled to absorb current from the grid lines. This can be easily explained bearing in mind the assumption made on the edges all characterized by the same length and phase angle and therefore interested by identical losses. It follows that the injection of reactive power would make the overall grid losses linearly decrease depending on the proximity

of the node which is absorbing. In support of this consideration, we notice that it would be the maximum benefit if the sixth node started injecting reactive power as this would lead to lower its demand and besides to a considerable decrease of the reactive power circulating through the lines. Conversely the second node, being the furthest one, would not be of great support. In fact, it would only reduce the power circulating in the first edge which is exclusively related to the PCC.

From these two analysis, we deduce that in a grid having identical transmission lines, the effect of an absorbing node, expressed by λ multipliers, is exactly the same for all the downstream nodes while as regards the upstream ones its influence follows a linear law of increase.

Now, we focus on the 12-node grid (presenting aligned nodes) where the fifth and tenth nodes are active while all the others are off. On the basis of this configuration, we analyze how λ vary and how they affect each other.

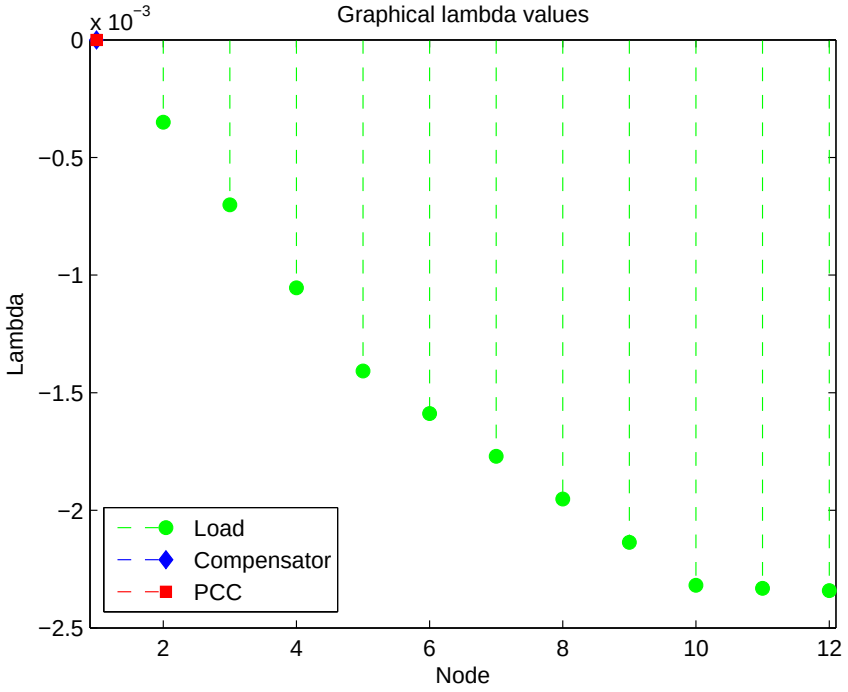

Figure 8 *12-node grid with the 5^{th} and 10^{th} nodes and all the others off*

Looking at figure 8, we notice a superposition of the various effects due to the absorptions at the different nodes. Particularly, the function behaviour changes identifying three sections each one characterized by a different constant slope:

- in the first tract, the slope is greater since the first edges are interested by the losses due to the power transfer of both nodes;

- in the second tract, the slope decreases owing to the power absorbed by the tenth node;

- finally, in the third tract, the slope is almost horizontal because there is no power flow through the transmission lines.

We continue doing a test where the 12-node grid impedances are all different.

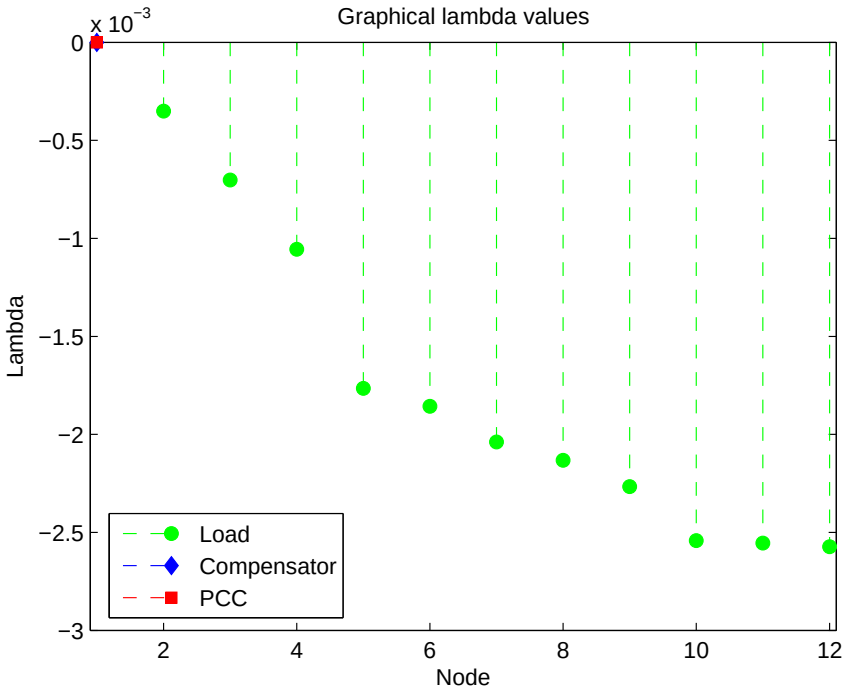

Figure 9 *12-node grid whose impedances are all different*

By observing the graph presented in figure 9, we see how λ vary according to broken lines as in this configuration the different grid portions are no more identical. Thus, we can conclude that in a real network, where all the nodes absorb or generate power and where all the transmission lines are different, it becomes difficult to have a priori estimation of λ behaviour.

Eventually, we test the 12-node grid in which all the nodes are active and with the insertion of two compensators occupying positions 6 and 11. We should expect λ corresponding to the positions occupied by compensators to be close to zero and besides to have a symmetric behaviour similar to a parabola for the loads placed between the two compensators. This is due to the fact that all the grid nodes are identical as it can be noticed from figure 10.

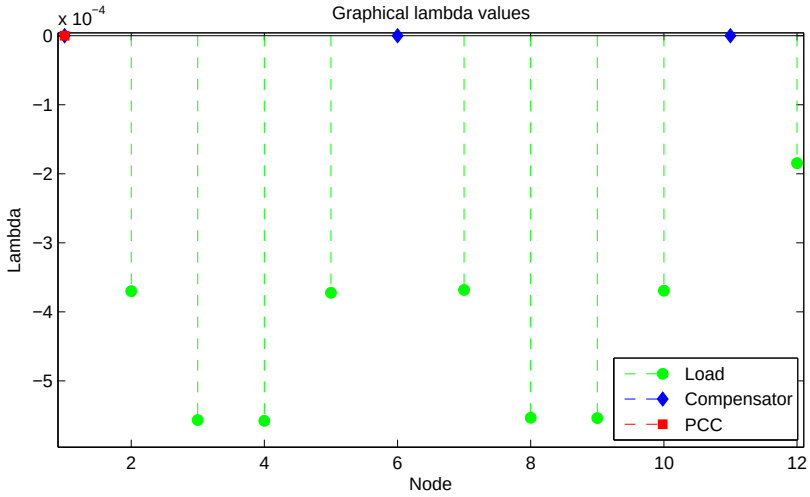

Figure 10 *12-node grid with all nodes active and the insertion of 2 compensators*

5.2 Distributed λ analysis

We are now going to test the performance of the λ estimation we found comparing the obtained results with the graphical lambda. In order to do that, we use the function 'LambdaTester()' with argument 'Com' (Compare) which is to be found in the appendix. At first we validate the results on the 7-node 'Y' grid so we can verify their correctness in the presence of branches. Then we adopt two linear grids formed by 12 and 25 nodes respectively to check the correctness of the obtained results when the tree-width increases.

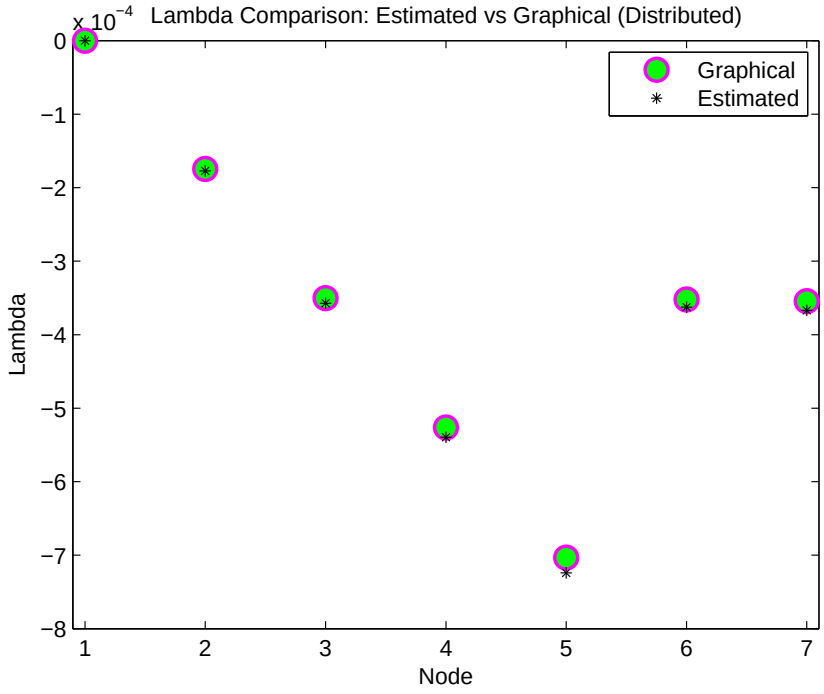

Figure 11 *Test 1: 'Y' grid with only an active leaf*

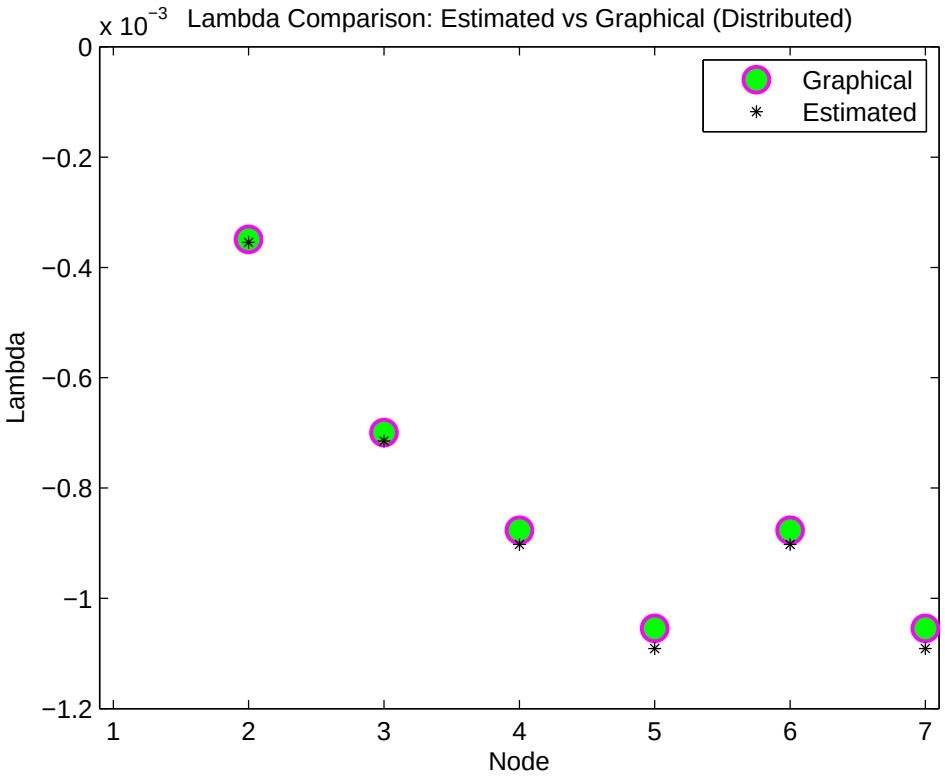

Figure 12 *Test 2: 'Y' grid with node 2 and all the leaves active*

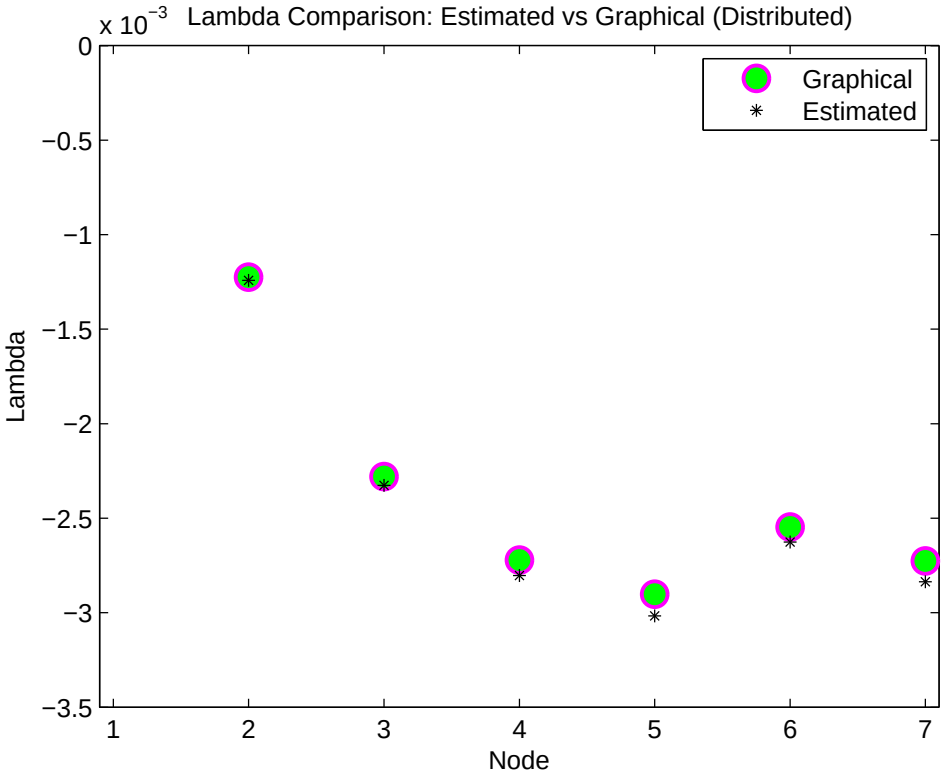

Figure 13 *Test 3: 'Y' grid with all nodes active and different powers*

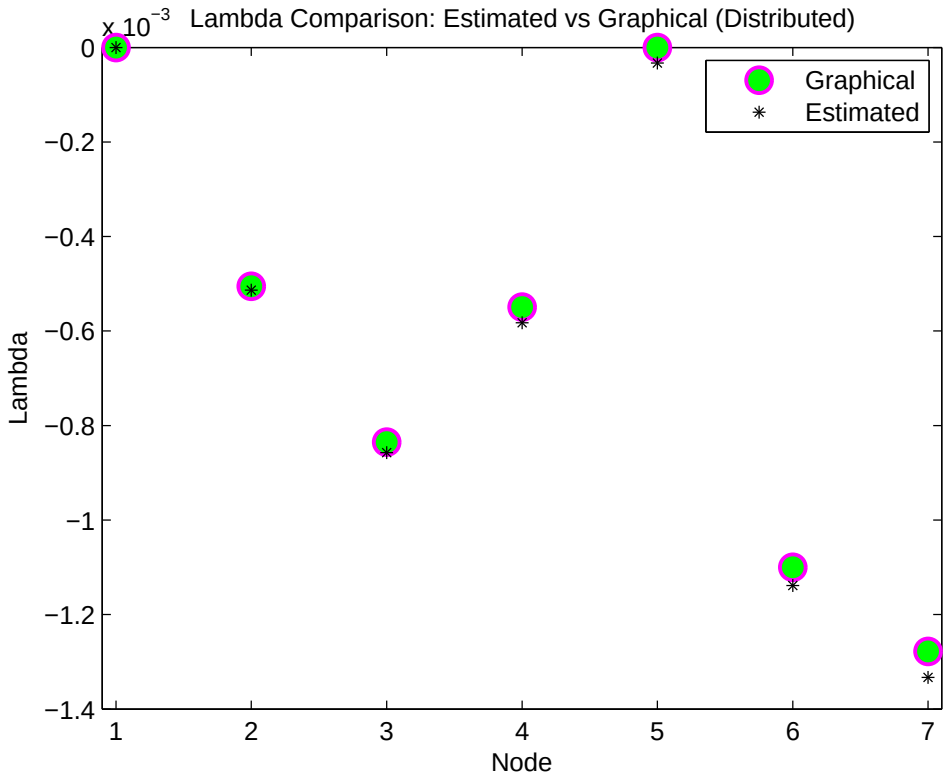

Figure 14 *Test 4: 'Y' grid with all nodes active and the insertion of a compensator among the terminal nodes*

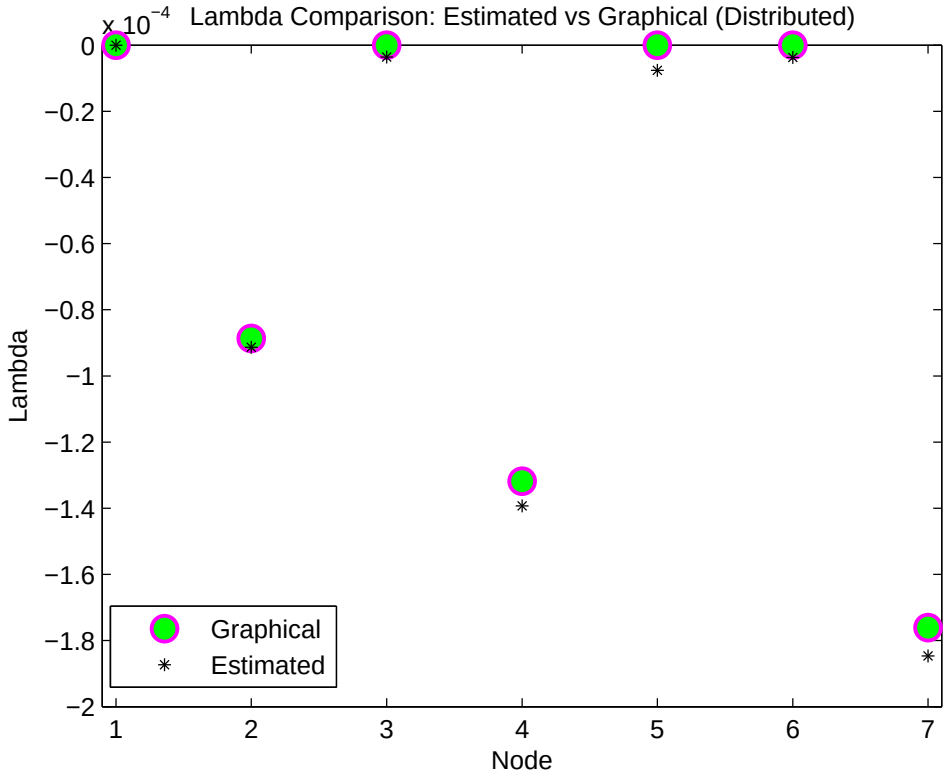

Figure 15 *Test 5: 'Y' grid with more compensators*

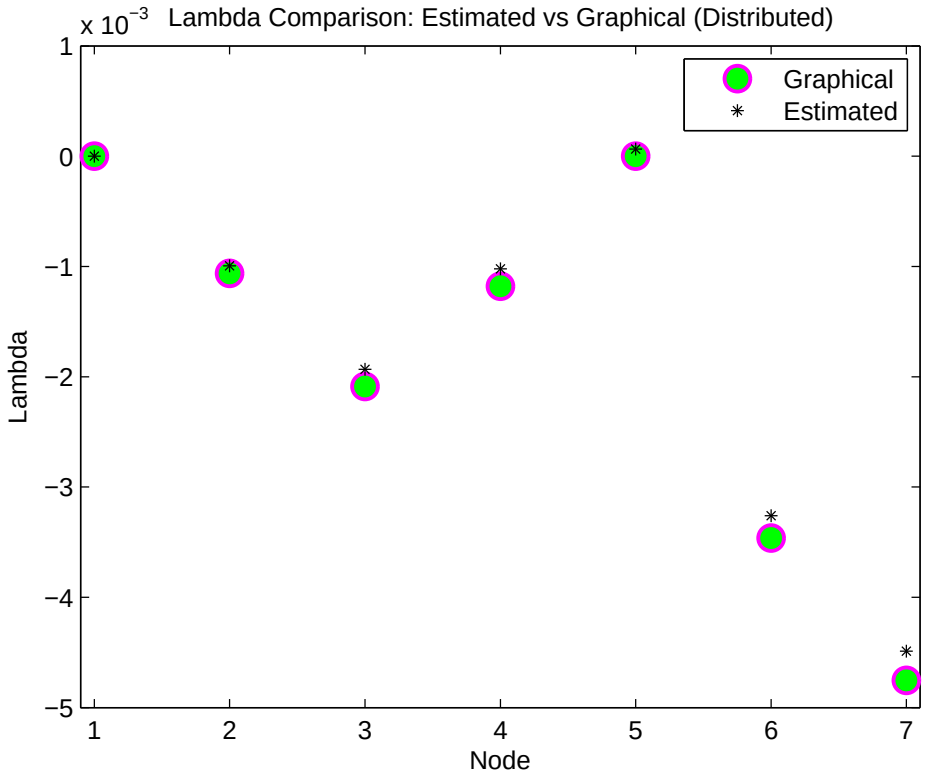

Figure 16 *Test 6: 'Y' grid with more compensators and different values for each branch impedance*

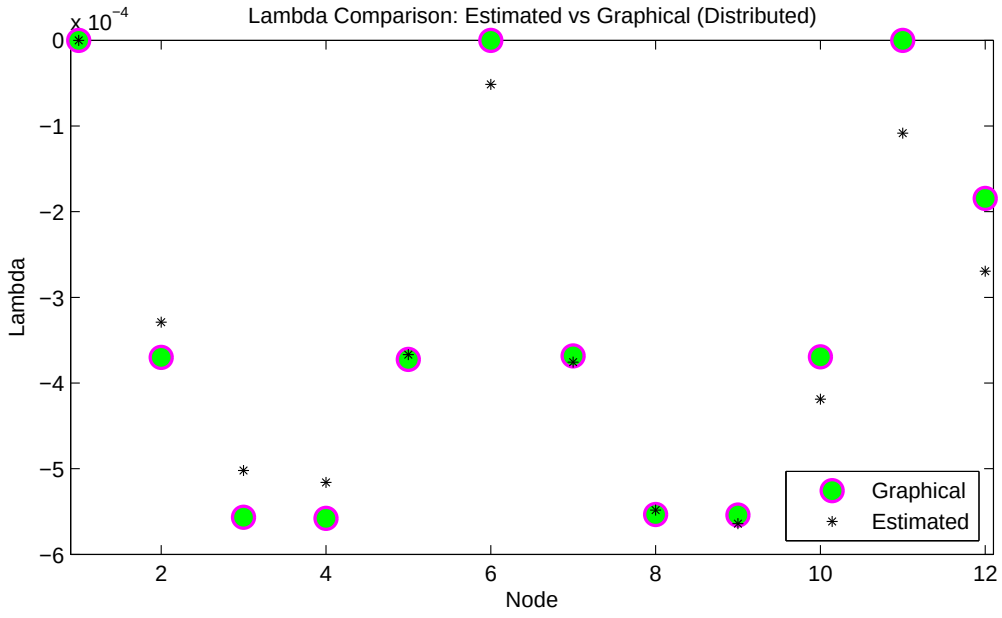

Figure 17 *Test 7: 12-node grid with 2 compensators and all the nodes active*

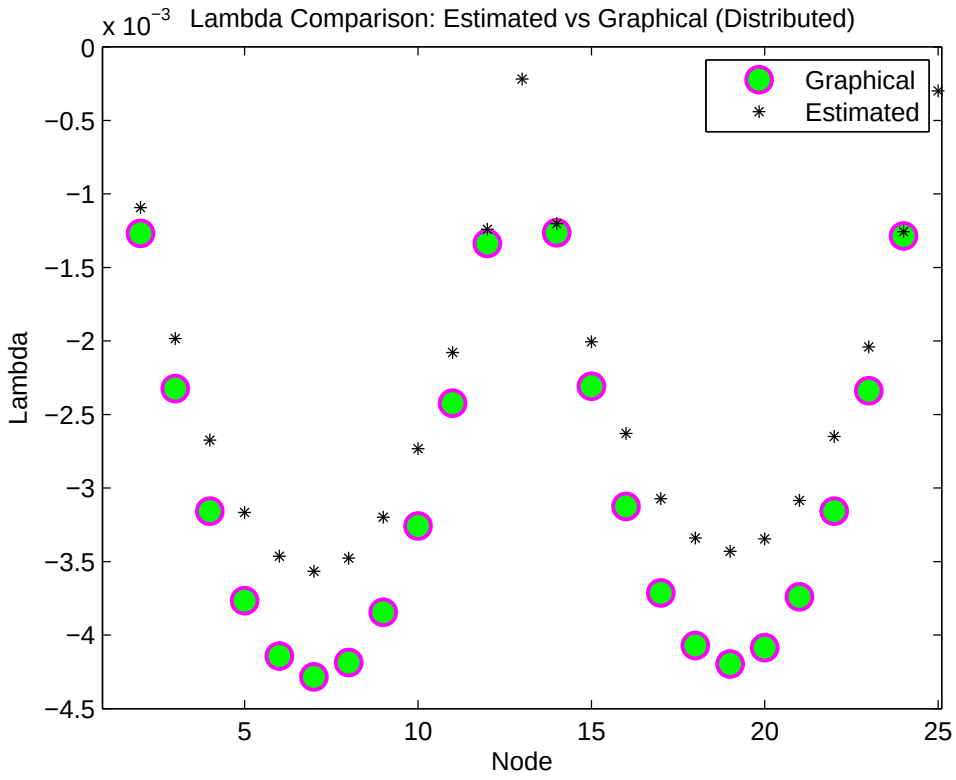

Figure 18 *Test 8: 25-node grid with 2 compensators and all the loads active*

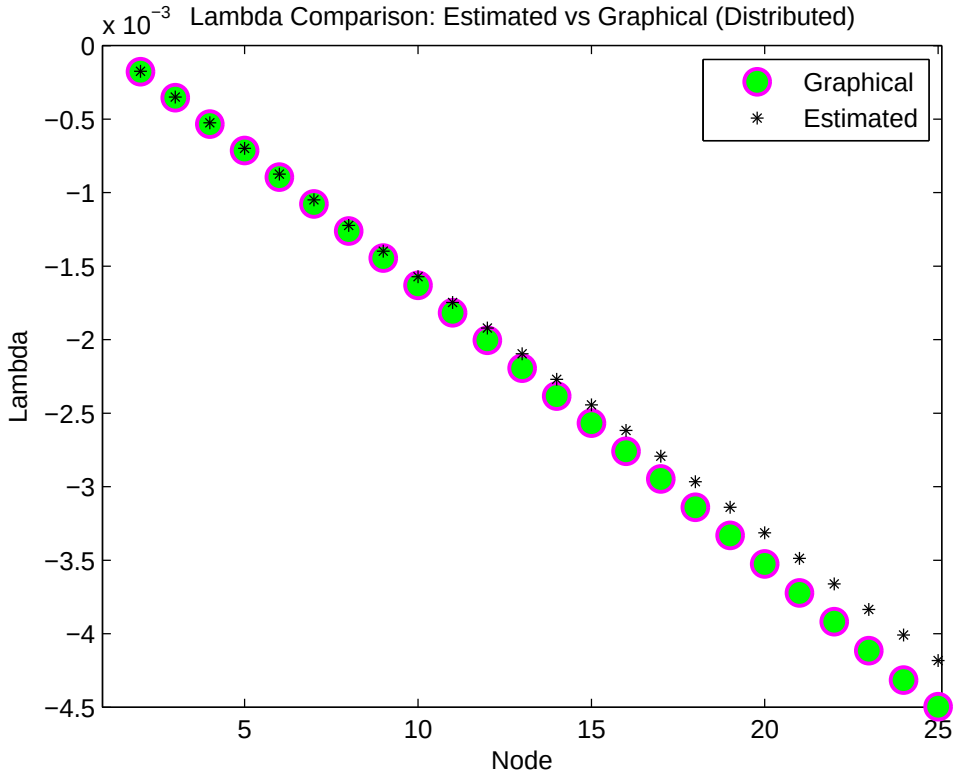

Figure 19 *Test 9: 25-node grid with only one active node*

By reference to tests 1 and 5, we notice how the estimated λ have good performances in the case of a branched grid.
On the other hand in test number 6, where there are grid portions having different impedance phase angles, it is evident that the algorithm performance worsens considerably precisely because there is no grid homogeneity. In fact, analysing the mathematical procedure by means of which we defined the estimations, we realize that we imposed the assumption of all the phase angles being the same. Hence, in this case, we can define the *Green matrix* by means of modulus and phase $\mathbf{X} = e^{j\theta}X$. If we omit such hypothesis, we are not able to extract the term $e^{j\theta}$ anymore. Therefore, it is not possible to make the previous reductions, compelling us to operate with the complex matrix $\mathbf{X}$ which is difficult to handle.
From tests 7 and 8, we see that if the tree-width increases also the error undergoes an increase. However, the estimated λ give a clear information about the behaviour of the real ones, showing that, even if the error is not neglibible, it does not make the parameter unusable. Furthermore, considering the residential use of the microgrid, where it is reasonable to assume that there are approximately 100-150 nodes, it is unlikely to have a grid width which exceeds the 25 levels.
Eventually, focusing on test 9, it is immediately evident that the error increases with depth and it can be interpreted as a sum of estimation errors. In particular, from equation (39) it is seen that the second term causes an error for each calculation but this must be added to the estimation error of the parent contained in the first term.
We notice that, in order to decrease the errors due to the tree-depth, we should use the information about compensators. As a matter of fact, a compensator, which does not work in the saturation area (or shows a slight saturation), could set its λ to zero, thus blocking the error propagation of the error accumulated by the upstream nodes.

6 Pricing and Storage strategy

6.1 Shadow prices interpretation

In scientific literature [11] [12], it is possible to find an interesting economic interpretation of Lagrange multipliers. Let us consider a simple convex problem with no equality constraints (14).

Being $f_0(x)$ the cost function to minimize, let $-f_0(x)$ be the profit of a productive process which operates according to the variable x; if each constraint $f_i(x) \leq u_i$ represents

a limit on some resource u_i, then the negative perturbed optimal cost function $-p^*$ shows how much profit a process could make with some resource available to it. Furthermore, if p^* is differentiable near $u = 0$ then, as previously said, we find:

$$\lambda_i^* = -\frac{\partial p^*(0,0)}{\partial v_i}$$

In other words, λ_i tells us approximately how much more profit the process could make for a small increase in availability of resource u_i. For these reasons, economists usually give the Lagrange multiplier the name of *shadow price*. It is nevertheless true that reactive power can be used to maintain the voltage within the limits of working and to avoid the risk of congestion. Thus, only if these important issues were resolved without using reactive power directly, then the parameter λ could assume the economic meaning depicted above. Actually, avoiding such problems with alternative methods is expensive and difficult to effect and consequently reactive power turns out to be necessary. In this case, the parameter λ we have obtained cannot be used because the Lagrangian does not take into account neither the voltage limits nor the network capacity. These two constraints have been neglected for the following reasons:

- voltage constraints: currently, the constraint violation involves a penalty charged to the administrator of the network; this tax is not proportional to the caused damaged and there is not a network rate of perfomance based on the closeness to the nominal voltage value. This leads to the impossibility of obtaining a cost function implementable in the Lagrangian.

- congestion constraints: in this case, building the cost function is made difficult by the need of introducing a new parameter, a sort of index which can evaluate the network risk of congestion. Moreover, the smart grid model previously used does not consider the limitations on transmission lines.

6.2 Storage strategy

Postulated the necessary preconditions, we are ready to define a Storage strategy in order to evaluate what is truly convenient to do in the case a compensator does not manage to supply the whole demanded active and reactive power due to the inverter operating limits.

The main objective is to determine a law based on the shadow price λ which let us decide what is the best choice for a compensator working in the limitation area. In particular, we want to obtain a concrete indication of whether or not is to opt for active power

supplying. As a matter of fact, in some situations it may be preferable to store active power in batteries providing more reactive power.
At first we assume that the compensator privileges active power sales at the expense of the reactive one, checking if in case it could be more convenient to store a certain amount of active power in order to produce more reactive power.
To study the problem, we are going to introduce and analyze a pair of functions which will provide cost and gain of the compensator. However, we must before define the quantities relative to the power supplying limitations.
We indicate with: $\overline{P}$ the amount of active power which should be provided by the compensator; with $\overline{Q}$ the amount of reactive power required in order to minimize the overall grid losses; and with $\overline{A}$ the apparent power; these expressions are linked by the Boucherot theorem:

$$\overline{A}^2 = \overline{P}^2 + \overline{Q}^2 \tag{40}$$

Now, suppose that the inverter, by means of which the node can inject power into the grid, has maximum apparent power capacity equal to M. As a consequence, for each compensator we can individuate two different operating modes:

- $M \geq \overline{A}$: in this case, the compensator does not operate in the limitation area and so we have no storage problem;
- $M < \overline{A}$: in this case, the node works in the limitation area. As a consequence, it must decide how to operate.

In this situation, it is assumed that the node sells all the possible active power obtaining at time t_0 the following expression:

$$M(t_0)^2 = P(t_0)^2 + Q(t_0)^2 = \overline{P}^2 + Q(t_0)^2 \tag{41}$$

Obviously, referring to this configuration, there is no storage and therefore active power is completely sold to the electric grid. Keeping the same $\overline{P}$ and $\overline{Q}$, let us think that at the next instant it occurs an infinitesimal increment in the reactive power supply dQ obtaining:

$$Q(t_1) = Q(t_0) + dQ$$

$$Q(t_1)^2 = Q(t_0)^2 + dQ^2 + 2dQ\,Q(t_0) \tag{42}$$

Recalling that M rapresents the inverter limitation, we express the power supplied at instant t_1 as:

$$M(t_1)^2 = P(t_1)^2 + Q(t_1)^2 = P(t_1)^2 + Q(t_0)^2 + dQ^2 + 2dQ\,Q(t_0)$$

Defining at this point the amount of active power which is not injected into the grid $dS(t_1)$ as the difference between the power generated at instant t_1 and that one erogated at t_0, we have that such quantity will be stored using storage appliances.

$$dS(t_1)^2 = \overline{P}^2 + Q(t_1)^2 - M(t_0)^2 = \overline{P}^2 + Q(t_1)^2 - \overline{P}^2 - Q(t_0)^2 = Q(t_1)^2 - Q(t_0)^2$$

$$dS(t_1)^2 = Q(t_0)^2 + dQ^2 + 2dQ\,Q(t_0) - Q(t_0)^2 = dQ^2 + 2dQ\,Q(t_0)$$

$$dS(t_1) = dQ\sqrt{\left(1 + 2\frac{Q(t_0)}{dQ}\right)} \tag{43}$$

6.3 Profits analysis with Lagrange multiplier

Once defined a rule which indicates how much power we have to store for small variations, we are now ready to define cost and gain laws. If active power storage in appropriate devices did not cause any losses, it would always be convenient for a node to produce the reactive power required and later sell active power. However, the storage process involves losses due to the device used to store the energy, to the storage method and to the amount of energy stored. Hence, it is reasonable to suppose that losses are linear and proportional to the energy stored and considering the storage inefficiency η, we define a cost function based on storage as:

$$dC = \eta dS\,\$_P = \eta dQ\sqrt{\left(1 + 2\frac{Q(t_0)}{dQ}\right)}\,\$_P \tag{44}$$

where $\$_P$ indicates the unit price of active power. Exploiting this equation, we introduce a new cost which is to be interpreted as a loss of profits due to the reactive power losses caused by storage inefficiency.

It is time to define a gain function which uses the shadow price λ directly. In particular, we presume that a minimal variation of the reactive power injected leads to a saving in terms of active power purchased from the PCC equal to:

$$dP = \lambda dQ \tag{45}$$

from which we derive the following gain equation:

$$dG = \lambda dQ \, \$_P \tag{46}$$

Obviously, the goal is to have profits higher than costs in case accepting them to be equal. The objective storage inequality is then:

$$dG \geq dC \tag{47}$$

$$\lambda dQ \, \$_P \geq \eta dS \, \$_P$$

$$\lambda dQ \geq \eta dS = \eta dQ \sqrt{\left(1 + 2\frac{Q(t_0)}{dQ}\right)}$$

$$\lambda \geq \eta \sqrt{\left(1 + 2\frac{Q(t_0)}{dQ}\right)}$$

From which:

$$1 \leq \frac{\lambda}{\eta \sqrt{\left(1 + 2\frac{Q(t_0)}{dQ}\right)}}$$

being $Q(t_0) \cong 0$, we have that $\frac{Q(t_0)}{dQ} \cong 0$ and thus we obtain the following inequality:

$$1 \leq \frac{\lambda}{\eta} \tag{48}$$

which represents the condition for which the compensator should opt for energy storage. Analyzing the equations parameters, we notice that in the present state inefficiencies average about 10-7% i.e. $\eta = 0.1 - 0.07$, while shadow price values are all around 0.01 (even under conditions of strong absorption it does not exceed 0.05).

These observations make it possible to realize that in the case we use the shadow price previously found directly, energy storage is never convenient and it is more suitable to inject active power instead of decreasing grid losses. On the other hand, this result is not surprising given that the Lagrangian function does not take into account voltage and network congestion limitations.

6.4 Profits analysis with a given reactive power price

Let us change the gain equation as that relative to cost is very realistic. Suppose we know the price that would be paid for each unit of reactive power produced equal to $\$_Q$, we define the new gain function as:

$$dG = dQ\,\$_Q \tag{49}$$

We notice that reactive power price is time variant and a function of λ but it also gives information regarding the network state in terms of performance. We are now going to investigate this last issue.

Imposing the inequality concerning storage convenience, we obtain:

$$dG \geq dC \tag{50}$$

$$dQ\,\$_Q \geq \eta dS\,\$_P = \eta dQ\sqrt{\left(1+2\frac{Q(t_0)}{dQ}\right)}\,\$_P$$

$$\$_Q \geq \eta\sqrt{\left(1+2\frac{Q(t_0)}{dQ}\right)}\,\$_P$$

From which:

$$1 \leq \frac{\$_Q}{\eta\sqrt{\left(1+2\frac{Q(t_0)}{dQ}\right)}\,\$_P}$$

Recalling that $Q(t_0) \cong 0$, it follows

$$1 \leq \frac{\$_Q}{\eta\,\$_P}$$

$$\$_Q \geq \eta\,\$_P \cong 0.1\,\$_P \tag{51}$$

From the last inequality analysis, we notice that a storage policy is convenient when reactive power price is at least equal to the storage inefficiency percentage relating to active power.

In particular, it is plausible that in some cases reactive power importance is very high (even higher than active power one) and hence its price allows us to opt for storage. This fact is based upon the foregoing considerations about reactive power relevance within the limits imposed by voltage bounds and network congestion.

7 Conclusions

The goal of this project was trying to develop a reactive power pricing method. Particularly, the initial idea was to make use of the Lagrange multipliers in their sense of economic shadow prices.
λ parameters have been calculated using two different methods:

- directly applying the theory of duality (centralized system).
  The results so obtained have an error of overestimation compared to graphical lambda, nevertheless they reflect the trend

- using the gradient estimation (distributed system).
  In this case, results are better than those we obtain with the first method, however we notice that the deepest levels of the tree representing the microgrid are more affected by the estimation error

We realize that the obtained λ do not give sufficient information to be interpretated as shadow prices. As a matter of fact, the objective function in the Lagrangian only contains the information related to the reactive power compensation instead of considering the one associated with voltage limits and congestion constraints. These issues have not been deepened due to the difficulty of implementing them in the Lagrangian in the form of a cost function.
So we choose to analyse the possible economic management of a compensator first using the information given by the λ parameter and then considering reactive power price which is known. On the basis of cost and gain considerations, we have attained favorable economic conditions that make the compensator able to decide if it is worth storing active power or whether it is preferable to inject less reactive power than that required.
Future developments, regarding the possibility to use the λ multiplier as a shadow price, may see the redefinition of the cost function in order to take into account all the effects connected with reactive power. Alternatively, it might be possible to define different type of Lagrangian according to the various objective functions and use the respective λ.
If these ways were not permissible, we ought to completely abandon the Lagrange theory and the multipliers in their sense of economic parameters, choosing to redefine the whole pricing problem.

A Geometric interpretation of Lagrange multipliers

We assume that for all u and v we have

$$p^*(u,v) \geq p^*(0,0) - \lambda^{*T}u - 1nu^{*T}v \tag{52}$$

To establish this inequality, suppose that x is any feasible point for the perturbed problem, then

$$\begin{aligned} p^*(0,0) = g(\lambda^*,\nu^*) &\leq f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \nu_i^* h_i(x) \\ &\leq f_0(x) + \lambda^{*T}u + \nu^{*T}v \end{aligned} \tag{53}$$

where the first inequality follows from the definition of $g(\lambda,\nu)$ and the second is due to $\lambda \geq 0$.

We conclude that

$$f_0(x) \geq p^*(0,0) - \lambda^{*T}u - \nu^{*T}v$$

from which we obtain (52).

To show (18), suppose that $u = te_i$ is the perturbation and that $v = 0$ where e_i is the i-th unit vector.

Then we have

$$\lim_{t\to 0} \frac{p^*(te_i,0) - p^*}{t} = \frac{\partial p^*(0,0)}{\partial u_i}$$

The inequality (52) states that for $t > 0$

$$\frac{p^*(te_i,0) - p^*}{t} \geq -\lambda_i^*$$

while for $t < 0$ we have the opposite inequality.

Taking the limit $t \to 0$ with $t > 0$, yields

$$\frac{\partial p^*(0,0)}{\partial u_i} \geq -\lambda_i^*$$

while taking the limit with $t < 0$, yields the opposite inequality; so we can conclude that

$$\frac{\partial p^*(0,0)}{\partial u_i} = -\lambda_i^*$$

and the same method can be used to prove

$$\frac{\partial p^*(0,0)}{\partial v_i} = -\nu_i^*$$

B Matlab functions

B.1 LossesOffline

This function is defined as:
$[\mathbf{losses\_alg}, \mathbf{sc\_alg}, \mathbf{u\_alg}, \mathbf{i\_alg}] = \mathbf{LossesOffline}(\mathbf{G}, \mathbf{C}, \mathbf{D})$
The function calculates the optimal solution in one step exploiting the knowledge of the whole grid. As regards the calculation, refer to the article "A distributed control strategy for reactive power compensation in smart microgrids" by Saverio Bolognani and Sandro Zampieri [1]
Function inputs:

- **G** describes the grid (see README or look at the ./examples/)
- **C** describes the compensators (see README or look at the ./examples/)
- **D** describes the values of the power absorbed at each node

Function outputs:

- **losses_alg** minimal losses of the grid after the algorith execution
- **sc_alg** power erogated by compensators in the final configuration
- **u_alg** node voltage in the final configuration
- **i_alg** node current in the final configuration

B.2 FinalLosses

This function is defined as:
$[\mathbf{FinalLosses}, \mathbf{sc_{alg}}, \mathbf{u_{alg}}, \mathbf{i_{alg}}] = \mathbf{LossesDistributed}(\mathbf{G}, \mathbf{C}, \mathbf{D}, \mathbf{NOITER}, \mathbf{RUNS})$
The function calculates the grid losses using the distributed algorithm. This one exploits the local exchange of information among compensators described in the article "A distributed control strategy for reactive power compensation in smart microgrids" by Saverio Bolognani and Sandro Zampieri [1].
Function inputs:

- **G** describes the grid (see README or look at the ./examples/)
- **C** describes the compensators (see README or look at the ./examples/)

- **D** describes the values of power absorbed by each node
- **NOITER** number of iterations of the distributed algorithm
- **RUNS** number of executions of the distributed algorithm

Function outputs:

- **FinalLosses** minimal losses of the grid after the algorithm execution
- **sc_alg** power erogated by compensators in the final configuration
- **u_alg** node voltage in the final configuration
- **i_alg** current voltage in the final configuration

B.3 LossesDistributed

This function is defined as:
$[] = \mathbf{LambdaGraphicsCreator}(\mathbf{G}, \mathbf{C}, \mathbf{D}, \mathbf{method}, \mathbf{namefile})$
The function allows to calculate the lambda multipliers graphically exploiting the sensitivity analysis. This one is applied to the function of minimum relating to the duality theory. It creates the sensitivity curves associated with each load through the identification of a parabola. The curve is obtained by a finite number of losses values and using different values of the reactive power absorbed by the load. In particular, you can decide to calculate these losses values via 'method' argument using the distributed algorithm or the offline one. It generates the graphs of these curves for each load indicating with a red point the working point and with a black horizontal line the losses value corresponding to the analyzed configuration. Then it calculates the lambda multipliers differentiating the found parabola and computes the value assumed by the curve at the working point. Eventually, it creates a graph where the lambda relative to PCC, compensators or loads are identified by different colours and shapes. It is worth recalling that for the theory of duality, both compensators and PCC work at their minimum. This means that their lambda is equal to zero if they are not operating in saturation. The function saves the calculated data relating to the lambda values and to the identified curves in a file the name of which can be specified.
Function inputs:

- **G** describes the grid (see README or look at the ./examples/)
- **C** describes the compensators (see README or look at the ./examples/)

- **D** describes the values of power absorbed by each node
- **method** indicates the method used for the losses calculation:
    1. if not specified it uses the distributed algorithm implemented by the function 'LossesDistributed';
    2. if specified (for any value) it uses the offline algorithm implemented by the function 'LossesOffline'
    3. **namefile** if not specified it saves the results in the default file 'GraphicLambdaData.mat'. Otherwise, it saves the data using the name of the indicated file. namefile argument must have the following structure between quotes 'name.mat' where name refers to the name of the desired file.

Since data are saved in a file, there are no output arguments.

B.4 EstimateLambda

This function is defined as:
lambda = **EstimateLambda**(**G**, **C**, **u_alg**)
The function takes as input the node-voltage vector and returns an estimate of lambda multipliers via local gradient estimation. If there is not the argument 'namefile', it creates the graph of the estimated lambda whose colour changes according to the type of node (loads, compensators or PCC). If instead there is the argument 'namefile', it opens the file (namefile) returned by the function 'LambdaGraphicsCreator' and compares the estimated lambda with the graphic real ones.
Function inputs:

- **G** describes the grid (see README or look at the ./examples/)
- **C** describes the compensators (see README or look at the ./examples/)
- **u_alg** node-voltage vector
- **namefile** name of the file returned by the funtion 'LambdaGraphicsCreator'

Function outputs:

- **lambda** vector of the lambda estimated at each node

B.5 LambdaTester

This function is defined as:

[] = **LambdaTester**(**action**)

The function calculates (via distributed or offline method), estimates or compares the lambda values of the grid. It has been created in order to allow rapid lambda-checks on a generic grid. Notice that to compare the results first it is necessary to calculate the graphical lambda using the appropriate function and then we need to update the file in which the graphical lambda have been saved.

Function inputs:

- **action** string which, according to the values assumed, commands the following actions:

    1. **'Est'**: lambda estimation via distributed algorithm based on the voltages computed by means of distributed minimum.
    2. **'Com'**: compares the estimated lambda with those calculated graphically. It must be preliminarily executed the graphical calculation of lambda values 'CalDist' or 'CalOff'.
    3. **'CalDist'**: calculates and saves the graphical lambda obtained via distributed algorithm
    4. **'CalOff'**: calculates and saves the graphical lambda obtained via offline algorithm

Since data are saved in a file, there are no output arguments.

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