Micro SmartGrid with Non Ideal Constraints

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I. Introduction

A. Practical relevance of the problem

In a typical power distribution network the complex and expensive infrastructure bears in a critical way on the cost of the energy, and moreover has a fixed configuration: the energy flow moves in an unidirectional way, so the user, in this scenario, is only a passive load for the network. This configuration has a lot of disadvantages, including the elevated losses (caused by Joule-effect) on the connecting lines from the point of common coupling,PCC, to the users, and the difficulty in using renewable energy. This kind of energy, in fact, requires a very high adaptation capability to variability-factors that electric networks generally do not have.

In the last decade, the idea of "Smart microgrid" has been developed driven by the increasing demand of energy and the need for higher quality of service, together with the introduction of distributed microgeneration of electric energy.

A smart microgrid is a portion of the low-voltage power distribution network that is managed autonomously from the rest of the network, in order to achieve better quality of service, improve efficiency and pursue specific economic interests.

Inside the microgrid there are microgenerator devices (solar panels, micro turbines, etc.) connected to the microgrid via inverters (electronic interfaces), which not only enable the injection of the produced power into the microgrid, but also can perform other tasks, denoted as "ancillary services". We will focus on one of these, the reactive power compensation.

Loads belonging to the microgrid may require a sinusoidal current not in phase with voltage. This requirement can be described as a demand of reactive power, together with active power, that are associated respectively to the out-phase and in-phase components of the current.

There is no cost to produce reactive power, but like active power flows, reactive power flows contribute to power losses on the connection lines. For this reason, it is preferable to produce reactive power as close as possible to the users that need it, in order to minimize reactive power flows.

B. State of the art

The use of the smart grids in the networked control system is one of the challenges in recent years. Because of the continuous growth of the electricity demands the use of these devices in electrical microgrid is one of the most interesting applications.

The importance of this technology is easily seen in its application in the electrical network of a city or a district, with the consequent global economic impact.

Therefore, the optimization of the amount of the power flow needed to satisfy the request is becoming a central problem of the research.

The implementation of these smart devices could organize the random appearance of users that produce private energy which is connected to the network and gives a further degree of freedom and leads to significant financial savings.

In the following, we mention some basic literature which begins addressing the micro-grid problems in terms of distributed control and energy processing, which are the key sectors where all involved technologies merge and the DIAMOND project will move a step forward.

The smart micro-grid concept – a portion of the power distribution network which is populated by a large number of small-size inverter-based micro-generators, that can be managed independently of the rest of the bigger energy grid – has been explored only quite recently in the scientific literature.

Some of the proposed analyses and solutions are interesting but they offer only preliminary results, which will be integrated and extended by the DIAMOND project. Just for reference, we mention here some publications addressing themes, which will be more deeply analyzed and widely developed by the DIAMOND project. Regarding the distributed control of micro-grids we cite:

1 T. C. Green and M. Prodanovic, "Control of inverter-based micro-grids," Electric Power Systems Research, vol. 77, no. 9, pp. 1204-1213, Jul. 2007.

2 J. A. Lopes, C. L. Moreira, and A. G. Madureira, "Defining control strategies for microgrids islanded operation," IEEE Transactions Power Systems, vol. 21, no. 2, pp. 916-924, May 2006. The control

strategies presented in these papers only cover some basic aspects of micro-grid operation, and have not yet been translated into algorithms, and therefore have not been deployed into practical, real-life installations.

Other publications have been devoted to exploit the power converters in a micro-grid to provide some ancillary services, such as reactive power compensation and voltage support. For example:

3 F. Katiraei and M. R. Iravani, ”Power management strategies for a microgrid with multiple distributed generation units,” IEEE Transactions on Power Systems, vol. 21, no. 4, pp. 1821-1831, Nov. 2006.

4 M. Prodanovic, K. De Brabandere, J. Van den Keybus, T. Green, and J. Driesen, ”Harmonic and reactive power compensation as ancillary services in inverter-based distributed generation,” IET Generation, Transmission and Distribution, vol. 1, no. 3, pp. 432-438, 2007. The main contribution of these papers has been recognizing that a smart utilization of the power inverters that belong to the micro-grid can be extremely beneficial for the overall efficiency of the micro-grid operation and for the quality of the provided service.

In the above contributions, the control law of each inverter is based on local measurements and local a-priori knowledge of the micro-grid, while there is no explicit use of any communication channel among inverters. We thus refer to this approach as local control strategy.

An alternative approach consists of defining a rigorous optimization problem, where the decision variables are the controls of each power converter connected to the micro-grid, and the cost function to be minimized includes global performance metrics as power losses, voltage drop across the distribution links, and grid stability. This approach yields large-scale, non-convex optimization problems, the solution of which requires the presence of a central controller capable of collecting data from all the devices and controlling them according to the result of its optimization process. We thus refer to this scenario as a centralized control strategy. The investigation of such approach in the literature produced notable results such as:

5 K. Turitsyn, P. Sulc, S. Backhaus, and M. . Chertkov, ”Options for control of reactive power by distributed photovoltaic generators,” Proceedings of the IEEE, vol. 99, no. 6, pp. 1063-1073, Jun. 2011.

6 J. Lavaei, A. Rantzer, and S. H. Low, ”Power flow optimization using positive quadratic programming,” in Proceedings of the 18th IFAC World Congress, 2011. These papers relate to a centralized approach which is able to guarantee the optimal performance, and can therefore be considered as a benchmark for any practical control algorithm, but is unfeasible in case of many distributed, intermittent and unreliable energy sources.

The themes related to distributed control of smart micro-grids have been recently addressed, with a slightly different perspective, also by the research community of power electronics, who devoted a special emphasis to the management of distributed energy resources to improve distribution efficiency, voltage stabilization, and load unbalance.

Among various contributions, we mention the following, which propose interesting and applicable solutions:

7 J.M. Guerrero, J.C. Vasquez, J. Matas, M. Castilla, L.G. de Vicuna: ”Control strategy for flexible microgrid based on parallel line-interactive ups systems”. IEEE Transactions on Industrial Electronics, vol. 56, n. 3, pp. 726 -736, 2009.

This paper presents a control strategy for a flexible micro-grid, equipped with several line-interactive uninterruptible power supply (UPS) systems connected in parallel, which can perform either connected to the grid or in autonomous operation. The adopted control strategy is dependent on the condition of the connection switch: power sources behave as current sources in grid connected operation, becoming voltage sources in islanded operation. In the latter case, a classic frequency-active power and voltage-reactive power droop control is implemented, to guarantee power sharing among the units without the need for any communication link. A virtual impedance is implemented to decouple the droop loops even in a mainly resistive cable typical of low voltage micro-grids.

8 R. Majumder, A. Ghosh, G. Ledwich, F. Zare: ”Power management and power flow control with back-to-back converters in a utility connected micro-grid”. IEEE Transactions on Power Systems, vol. 25, n. 2, pp. 821-834, 2010. This paper presents a method for power flow control between utility and micro-grid through a back-to-back converter used as interface between the two power systems. The solution facilitates the desired real and reactive power flow between utility and micro-grid. The back-to-back converters also provide total frequency isolation between the utility and the micro-grid at the expense of reduced conversion efficiency.

9 E. Serban, H. Serban: ”A control strategy for a distributed power generation micro-grid application with voltage- and current-controlled source converter”. IEEE Transactions on Power Electronics, vol. 25 n.12, pp. 2981-2992, 2010. This paper presents a distributed micro-grid structure capable to operate in islanded and grid-connected modes, using hybrid converters, i.e., four-quadrant PWM bidirectional converters with integrated ac transfer switches. They can perform as voltage sources, current sources, or active-rectifiers. In islanded mode, the hybrid converters operate as voltage sources, controlling the ac voltage and frequency, and implement a droop function if multiple converters operate in the same micro-grid. In grid connected operation, the converters are controlled as AC current sources, or active rectifiers if equipped with energy storage.

10 J.M. Guerrero, J.C. Vasquez, J. Matas, L.G. de Vicuna, M. Castilla: ”Hierarchical control of droop-controlled ac and dc micro-grids: a general approach toward standardization”. IEEE Transactions on Industrial Electronics, vol. 58, n. 1, pp.158-172, 2011. The paper proposes a vision on the standardization of micro-grid architecture and control. The proposal is derived from ISA-95 and other standards and regulations, to offer a smart and flexible management of a micro-grid. In particular, a hierarchical control is proposed consisting of three levels: 1) The primary control, based on the droop method, including an output-impedance virtual loop; 2) the secondary control allowing the restoration of the deviations produced by the primary control; and 3) the tertiary control managing the power flows between the micro-grid and the external electrical distribution grid. This hierarchical control concept has been implemented in simulation and experimentally validated.

11 T.L. Vandoorn, B. Meersman, L. Degroote, B. Renders, L. Vandevelde: ”A control strategy for islanded micro-grids with dc-link voltage control”. IEEE Transactions on Power Delivery, vol. 26, n. 2, pp. 703-713, 2011. This work bases the control of islanded micro-grids on the consideration that most of the distributed generators are connected to the micro-grid via a power-electronic inverter with dc link. New control methods for these inverters are developed to exploit the generation units in case of islanded operation. Good power sharing, transient behaviour and stability with no communication infrastructure is achieved.

12 A. Cagnano, E. De Tuglie, M. Liserre, R.A. Mastromauro: ”Online optimal reactive power control strategy of pv inverters”. IEEE Transactions on Industrial Electronics, vol. 58, n.10, pp.4549-4558, 2011.
This paper proposes a decentralized nonlinear auto-adaptive controller to reduce the losses in a distribution grid by optimally injecting the reactive power. In the considered scenario, the reactive power is supplied by the inverters of photovoltaic units. The designed optimization is dynamic, and thus the reactive current references are continuously updated on-line. Experimental validation is provided, showing the effectiveness of the method in reducing distribution loss by minimizing reactive power flows.

Although there are different parameters that can be optimized, in this work we consider the problem of the optimal reactive power compensation in a smart electrical microgrid, in order to minimize the active power distribution losses.
Only recent studies have started considering this aspect and our project is based on a previous work:
S. Bolognani and S. Zampieri,”A distributed control strategy for reactive power compensation in smart microgrids” [1], that led to important results in the steady state behavior of an electrical network.
First, the article traces the issue of minimization of losses to an optimization quadratic problem by the derivation of an approximate model of the power flow.
Second, it designs a distributed optimization algorithm able to converge to the global minimum power losses.
As can be seen from the previous bibliography, smart micro-grids are at the center of wide research interests. However, the current status of scientific advancements does not envelope the management of an entire micro-grid by addressing local and global optimization goals as a whole. Moreover, only few research projects address the ICT architecture to support effective distributed control strategies based on cooperative operation of the power interfaces between energy sources and distribution network, which is instead the prime objective of DIAMOND project.
In other words, compared to the scientific state-of-art it is expected that DIAMOND project will move a step forward in the direction to develop the theoretical background and to integrate various technologies to develop a truly efficient and effective management of residential micro-grids.

C. Our contribution

Starting from the results obtained by [1] Bolognani and Zampieri in their article, the problem that we focused on consists of adding upper and lower bounds to the reactive power that each compensator can exchange with the remaining network.
These box constraints complicate the initial problem, but draw the model closer to the real microgrid.
Our contribution is twofold: first we proved by a counterexample that the previous algorithm in [1] does not converge with the new bounds, second we implemented a new algorithm based on a multi-hop communication, compensator-by-compensator, with random access of these agents. We proved the convergence of the algorithm for the quadratic and box-constrained problem, under a specific condition of the compensators-graph related to the microgrid.

D. Short summary

Section II presents a summary of the model found in [1] and the most meaningful results which they demonstrated are reported. Indeed, it is important to read this section because this model is the one we used as our initial point. Section III explains the randomized algorithm that is proposed in [1] and its convergence. In section IV consequences for the addition of the constraints are explained. In particular, differences which arise for these constraints in the model and in the problem formulation are discussed. Section V presents the gossip-like algorithm and there is explained the multi-hop communication, how with this technique the compensators-graph tend to a complete graph and the proof of the convergence of the algorithm. Section VI reports the simulations and describes the results obtained through graphics and explanations. In section VII there are the conclusions of the work and some cues for the future work.

II. Model

Our project is based off of the article [1] by Saverio Bolognani and Sandro Zampieri.
The first goal they achieved was deriving a model that approximated the microgrid state.
The microgrid introduced before is modeled as a direct graph $\mathcal{G}$, where edges represent power lines and nodes represent both loads and generators connected to the microgrid; a special node is the point of connection of the microgrid with the transmission grid, called PCC, Point of Common Coupling.
In the article the study is limited to the steady state behavior of the system, which means that all voltages and currents are sinusoidal signals at the same frequency ω_0.
The system variables used for describing the model are:

- $u \in \mathbb{R}^n$, where u_v is the grid voltage at node v;
- $i \in \mathbb{Q}^n$, where i_v is the current injected by node v into the grid;
- $\xi \in \mathbb{C}^r$, where ξ_e is the current flowing on the edge e.

These variables satisfy the following constraints:

$$\begin{aligned} A^T\xi + i = 0; \\ Au + \mathbf{Z}\xi = 0; \end{aligned} \tag{1}$$

where A is the incidence matrix of $\mathcal{G}$, and $\mathbf{Z} = \text{diag}(z_e, e \in E)$ is the diagonal matrix of line impedances, z_e being the impedance of the microgrid power line corresponding to the edge e.
To characterize any node v they found a relationship between its injected current i_v with its voltage u_v, this law is called *exponential model*:

$$u_v\bar{i}_v = s_v \left|\frac{u_v}{U_0}\right|^{\eta_v}, \quad \forall v \in \mathcal{V} \setminus \{0\} \tag{2}$$

where s_v is the *nominal complex power* and η_v is a characteristic parameter of the node v. For this model the microgenerators fit with η_v = 0, that describes the behavior of constant power.
The only exception is for node PCC which is modeled as a constant voltage generator:

$$u_0 = U_0$$

The first Lemma they define is:

Lemma 1. *Let $\boldsymbol{L}$ be the complex valued Laplacian $\boldsymbol{L} := A^T\boldsymbol{Z}^{-1}A$. There exists a unique symmetric matrix $\boldsymbol{X} \in \mathbb{C}^{n\times n}$ such that*

$$\begin{cases} \boldsymbol{XL} = I - \boldsymbol{1}\boldsymbol{1}_0^T \\ \boldsymbol{X}\boldsymbol{1}_0 = 0 \end{cases} \tag{3}$$

where, we recall, $[\boldsymbol{1}_0]_v$ = 1 for $v = 0$, and 0 otherwise and I is the identity matrix.

This matrix $\mathbf{X}$ depends only on the topology of the microgrid power lines and on their impedance.
The *effective impedance*, Z^{eff}_{uv}, of the power lines for every pair of nodes (u, v) can be represented by the following:

$$Z^{eff}_{uv} = (\mathbf{1}_u - \mathbf{1}_v)^T\mathbf{X}(\mathbf{1}_u - \mathbf{1}_v) \tag{4}$$

Thanks to these results the currents i and the voltages u are therefore determined by the equations

$$\begin{cases} u = \mathbf{X}i + U_0\mathbf{1} \\ \mathbf{1}^T i = 0 \\ u_v\bar{i}_v = s_v\left|\frac{u_v}{U_0}\right|^{\eta_v}, \quad \forall v \in \mathcal{V}\setminus\{0\} \end{cases} \tag{5}$$

We can see the currents i and the voltages u as functions $i(U_0)$, $u(U_0)$ of U_0. The following proposition provides the Taylor approximation of $i(U_0)$ for large U_0.

Proposition 2. *Let $s_0 := -\sum_{v\in\mathcal{V}\setminus\{0\}} s_v$. Then for all $v \in \mathcal{V}$ we have that*

$$i_v(U_0) = (\bar{s}_v + \delta_v(U_0))\frac{1}{\bar{U}_0} \tag{6}$$

where $\delta_v(U_0)$ is infinitesimal when U_0 tends to infinity.

A. Power losses minimization problem

The metric, for the optimality of reactive power flows, is considered to be the active power losses on the power lines.

Thanks to the approximating model (5) it is possible to:

- approximate power losses as a quadratic function of the injected power;
- decouple the problem of optimal power flows into the problem of optimal active and reactive power injection.

In this microgrid it is possible to command only a subset $\mathcal{C} \subset \mathcal{V}$ of the nodes, named *compensators*.

The problem of optimal reactive power injection at the compensators can be expressed as a quadratic, linearly constrained problem, in the form

$$\min_q \quad J(q), \quad \text{where } J(q) = \frac{1}{2}q^T\text{Re}(\mathbf{X})q \tag{7}$$

$$\text{subject to } \mathbf{1}^T q = 0,$$

$$q_v = \text{Im}(s_v),\ v \in \mathcal{V}\setminus\mathcal{C},$$

$\text{Im}(s_v)$, $v \in \mathcal{V}\setminus\mathcal{C}$ being the nominal amount of reactive power injected by the nodes that cannot be commanded.

The challenging part to solving the problem (6) is that each node has only local information.

III. A randomized distributed algorithm

The algorithm proposed is based only in a local knowledge, therefore any central controller is not needed. The algorithm can be distributed across the agents of the microgrid that consist in decomposing the optimization problem into smaller issues.

A. Optimization problem decomposition

All the compensators are divided into ℓ possibly overlapping sets $\mathcal{C}_1, \dots, \mathcal{C}_\ell$, with $\bigcup_{i=1}^{\ell}\mathcal{C}_i = \mathcal{C}$ and the nodes of the same set, called *cluster*, are able to communicate to each other, and they are therefore capable of coordinating their actions and sharing their measurements.

The proposed optimization algorithm consists of the following repeated steps:

1) a set $\mathcal{C}_{i(t)}$ is chosen at a certain discrete time $t = 0, 1, 2, \dots$ where $i(t) \in \{1, \dots, \ell\}$;
2) the agents in $\mathcal{C}_{i(t)}$, by coordinating their actions and communicating, determine the new feasible state that minimizes $J(q)$, solving the optimization subproblem in which all the nodes that are not in $\mathcal{C}_{i(t)}$ keep their states constant;
3) the agents in $\mathcal{C}_{i(t)}$ actuate the system by updating their state (the injected reactive power).

Partitioning q as

$$q = \begin{bmatrix} q_{\mathcal{C}} \\ q_{\mathcal{V}\setminus\mathcal{C}} \end{bmatrix}$$

where $q_{\mathcal{C}} \in \mathbb{R}^m$ are the controllable components and $q_{\mathcal{V}\setminus\mathcal{C}} \in \mathbb{R}^{m-n}$ are not controllable. According to this partition of q, is it possible also the partition of the matrix $\text{Re}(\mathbf{X})$ as

$$\text{Re}(\mathbf{X}) = \begin{bmatrix} M & N \\ N & Q \end{bmatrix} \tag{8}$$

Introduced also the matrices m×m

$$\Omega := \frac{1}{2m}\sum_{h,k\in\mathcal{C}}(\mathbf{1}_h - \mathbf{1}_k)(\mathbf{1}_h - \mathbf{1}_k)^T = I - \frac{1}{m}\mathbf{1}\mathbf{1}^T,$$

$$\Omega_i := \frac{1}{2|\mathcal{C}_i|}\sum_{h,k\in\mathcal{C}_i}(\mathbf{1}_h - \mathbf{1}_k)(\mathbf{1}_h - \mathbf{1}_k)^T = = \text{diag}(\mathbf{1}_{\mathcal{C}_i}) - \frac{1}{|\mathcal{C}_i|}\mathbf{1}_{\mathcal{C}_i}\mathbf{1}_{\mathcal{C}_i}^T \tag{9}$$

When the cluster $\mathcal{C}_i$ is fired its nodes perform th optimization:

$$q_{\mathcal{C}}^{opt,i} := \arg\min_{q'_{\mathcal{C}}\in q_{\mathcal{C}}+\mathcal{S}_i} J(q'_{\mathcal{C}}, q_{\mathcal{V}\setminus\mathcal{C}}) = q_{\mathcal{C}} - (\Omega_i M \Omega_i)^{\#}\nabla J, \tag{10}$$

where

$$\nabla J = Mq_{\mathcal{C}} + Nq_{\mathcal{V}\setminus\mathcal{C}} = [\text{Re}(\mathbf{X})q]_{\mathcal{C}} \in \mathbb{R}^m \tag{11}$$

is the gradient of $J(q_{\mathcal{C}}, q_{\mathcal{V}\setminus\mathcal{C}})$ with respect to the decision variables $q_{\mathcal{C}}$.

This is computed via a distributed way:

if $h \notin \mathcal{C}_i$ then $\left[q_{\mathcal{C}}^{opt,i}\right]_h = q_h$, if instead $h \in \mathcal{C}_i$ then:

$$\left[q_{\mathcal{C}}^{opt,i}\right]_h = q_h - \sum_{k \in \mathcal{C}_i} \left[(\Omega_i M \Omega_i)^{\#}\right]_{hk} [\nabla J]_k \qquad (12)$$

B. Hessian estimation from local topology information

The Hessian matrix can be compute a priori thanks only to local knowledge of the mutual effective impedances between pairs compensators.
Defining $R_{hk}^{eff} = \mathrm{Re}(Z_{hk}^{eff})$ through some computation is obteined that:

$$\Omega_i M \Omega_i = -\frac{1}{2} \Omega_i R^{\text{eff}} \Omega_i. \qquad (13)$$

C. Gradient estimation via local voltage measurement

Assume that nodes in $\mathcal{C}_i$ can measure the grid voltage at their point of connection and the following about power line impedances.

Assumption 3. *All power lines in the microgrid have the same inductance/resistance ratio, i.e.*

$$\mathbf{Z} = e^{j\theta} Z$$

where Z is a diagonal real-valued matrix, whose elements are $Z_{ee} = |z_e|$. *Consequently,* $L = e^{-j\theta} A^T Z^{-1} A$, *and* $X := e^{-j\theta} \mathbf{Z}$ *is a real-valued matrix.*

Each agent $k \in \mathcal{C}_i$ compute

$$\nu_k^{(i)} := \frac{1}{|\mathcal{C}_i|} \sum_{v \in \mathcal{C}_i} |u_v||u_k| \sin(\angle u_k - \angle u_v - \theta) \qquad (14)$$

After some computations is it possible to write the estimate gradient as:

$$[\nabla J]_k = -\cos\theta (\mathrm{Im}(\mathbf{1}_k^T X \bar{s})) \qquad (15)$$

D. Description of the algorithm

The iterative algorithm proposed, based on all the above considerations, works as follows: when the cluster $\mathcal{C}_i$ is activated the state of all the system becomes $q_h(t+1) = q_h$ for all $h \notin \mathcal{C}_i$, while the node $h \in \mathcal{C}_i$ will inject the new reactive power

$$q_h(t+1) = q_h - \cos\theta \sum_{k \in \mathcal{C}_i} \left[(\Omega_i R^{\text{eff}} \Omega_i)^{\#}\right]_{hk} \nu_k^{(i)}(t), \qquad (16)$$

As we know the algorithm can be implemented by the agents of the microgrid in a distributed way. In a preliminary, offline phase, each cluster computes $(\Omega_i R^{\text{eff}} \Omega_i)^{\#}$ then, at every iteration of the algorithm:

- a cluster $\mathcal{C}_i$ is randomly chosen;
- every agent h not belonging to the cluster $\mathcal{C}_i$ holds its injected reactive power constant;
- every agent h belonging to the cluster $\mathcal{C}_i$ senses the grid voltage at its point of connection, computes $\nu_h^{(i)}$, and then updates its injected reactive power according to (15).

E. Convergence of the Algorithm

The authors proved that the algorithm converges and they analyzed the convergence rate.

1) A necessary condition for convergence: This discrete time system describes the ideal iterative optimization algorithm:

$$q(t+1) = q(t) - (\Omega_i M \Omega_i)^{\#} (M q(t) + N q_{\mathcal{V} \setminus \mathcal{C}}). \quad (17)$$

Introducing the auxiliary variable $x = q_{\mathcal{C}} - q_{\mathcal{C}}^{opt}$, where $q_{\mathcal{C}}^{opt}$ are the elements of q^{opt}, that is the solution of the optimization problem (7), corresponding to the nodes in $\mathcal{C}$, it is possible to explicitly express the previous discrete time system as:

$$x(t+1) = F_{i(t)} x(t), \quad x(0) \in \ker \mathbf{1}^T, \qquad (18)$$

where $F_i = I - (\Omega_i M \Omega_i)^{\#} M$. Therefore q(t) converges to the optimal solution qopt is if and only if x(t) converges to zero for any initial condition $x(0) \in \ker \mathbf{1}^T$. A necessary condition for this to happen is that there are no nonzero equilibria in the discrete time system .
This property can be characterized by the Proposition 5. But first Lemma 4 is needed.

Lemma 4. *x = 0 is the only point in* $\ker \mathbf{1}^T$ *such that* $F_i x = x$ *for all i if and only if*

$$span[\Omega_1 \dots \Omega_l] = \ker \mathbf{1}^T. \qquad (19)$$

The following proposition is the same condition of the previous Lemma but as a connectivity test.

Proposition 5. *x = 0 is the only point in* $\ker \mathbf{1}^T$ *such that* $F_i x = x$ *for all i if and only if the hypergraph* $\mathcal{H}$ *is connected.*

2) Convergence and rate of convergence: What they show in this section is that the connectivity of the hypergraph $\mathcal{H}$ is not only a necessary but also a sufficient condition for the convergence of the algorithm.

Assumption 6. *The sequence i(t) is a sequence of independently, identically distributed symbols in* $1, \dots, \ell$, *with non-zero probabilities* $\rho_i > 0, i = 1, \dots, \ell$

Let $v(t) := \mathbb{E}[x^T(t)Mx(t)] = J(q(t)) - J(q^{\text{opt}})$ they consider the performance metric

$$R = \sup_{x(0)\in\ker \mathbf{1}^T} \limsup\{v(t)\}^{\frac{1}{t}}$$

which describes the exponential rate of convergence to zero of $v(t)$.
But the study of this metric is usually not simple, therefore they decided to analyze its behavior indirectly through another parameter

$$\beta = \max\{|\lambda| \mid \lambda \in \lambda(F_{\text{ave}}), \lambda \neq 1\},$$

where $F_{\text{ave}} := \mathbb{E}[F_i]$.
They prove the following result:

Theorem 7. *Assume that Assumption 6 holds true and that the hypergraph $\mathcal{H}$ is connected. Then*

$$R \leq \beta < 1.$$

Corollary 8. *Assume that Assumption 6 holds true and that the hypergraph H is connected. Then the state of the iterative algorithm described in Section II-B-1 converges in mean square to the global optimal solution.*

Finally the following result shows what is the best performance (according to the bound β on the convergence rate R) that the proposed algorithm can achieve.

Theorem 9. *Consider the algorithm (15), and assume that $\mathcal{H}$ describing the clusters $\mathcal{C}_\rangle$ is an arbitrary connected hypergraph defined over the nodes $\mathcal{C}$. Let Assumption 6 hold. Then*

$$1 - \frac{\sum_{i=1}^{\ell} \rho_i |\mathcal{C}_\rangle| - 1}{m-1} \leq \beta \tag{20}$$

In case all the sets $\mathcal{C}_\rangle$ have the same cardinality c, namely —$\mathcal{C}_\rangle$— = c for all i, then

$$1 - \frac{c-1}{m-1} \leq \beta \tag{21}$$

3) Optimal communication hypergraph for a radial distribution network: They finally present an optimal case for the rate of convergence.

Assumption 10. *The distribution network is radial, i.e. the corresponding graph $\mathcal{G}$ is a tree.*

They demonstrated that the optimal clustering strategy consists in choosing clusters which resembles the physical interconnection of the electric network (edge-disjoint):

$$R = 1 - \frac{\left(\sum_{i=1}^{\ell} \rho_i |\mathcal{C}_i|\right) - 1}{m-1} \tag{22}$$

This result is interesting because states that communication between neighbors improve the performance of the rate of convergence, and this constrast with the phenomena generally observed in gossip consensus algorithms, in which long-distance communications are beneficial for the rate of convergence.

IV. Adding bounds

In order to make more realistic our model, relaxed the hypothesis that the compensators are not subject to any constraints. The new problem becomes to minimize the cost function J with two types of constraints: the sum of the power must be zero (equality constraint) and the reactive power supplied by the compensator is bounded (inequality constraints). This problem can appear trivial because, for example, the Gauss-Seidel method [2] seems a good solution, but there are some important differences between this method and our algorithm. First of all the Gauss-Seidel method update the components one by one and not in couple, secondly it isn't randomized and finally it doesn't contemplate any sort of equality constraints (in our case the sum of the current must be zero) and surely this is the worst fault. Introducing bounds, we have a new minimum of the cost function J, that generally is not necessarily equal to the non constrained minimum, especially if the bounds are tightened, and also we need to modify our algorithm in order to find a new solution for our problem which has to move closer to the real minimum.
Initially, we start from the simplest problem, where clusters are formed by just two compensators. This is the only case where we find the solution in a closed form instead for a number of compensators n greater than two instead we figure out a solution by using a numeric algorithm.
We consider a cluster where the two compensators are called $< q_1, q_2 >$, the lower and the upper bounds $[q_{\min,i}, q_{\max,i}]$ where $i = 1, 2$ is the compensators' index, and δ_i is the reactive power increment desired without bounds.
We have three possible cases:

- if the bounds are not violated, we doesn't change the value of δ and the algorithm is the same.
- if $q_1 + \delta_1 < q_{\min,1}$ or $q_2 + \delta_2 > q_{\max,2}$ we violate the minimum bound with the first compensator or the maximum bound with the second one. The best choice we can take respecting bounds is the following:

$$\delta_1 = \text{sign}(\delta_1) \min\{|\delta_1|, q_{\max,2} - q_2, q_1 - q_{\min,1}\}$$
$$\delta_2 = -\delta_1$$

In other words, in this case we choose as δ the minimum between maximum value we can subtract from q_1 and the maximum value we can add to q_2 respecting bounds.

- if $q_1+\delta_1>q_{\max,1}$ or $q_2+\delta_2<q_{\min,2}$ we violate the maximum bound with the first compensator or the minimum bound with the second one. In this case the best choice that we can take is the following:

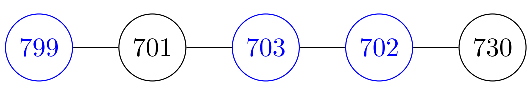
$$\delta_1=\operatorname{sign}(\delta_1)\min\{|\delta_1|,q_{\max,1}-q_1,q_2-q_{\min,2}\}$$
$$\delta_2=-\delta_1$$

Similar logic is taken in this situation.

With this algorithm we have not reached the global minimum of the convex constrained function, as this is not a trivial result, we analyze the process to get it in the following:

Lemma 11. *Adding box bounds to the original minimization algorithm, The minimization algorithm in [1] considered with reactive power limits (box bounds) does not always drive the system to the minimum of the cost function.*

Proof: Let us consider the following network with just 5 nodes and 3 compensators: in this configuration we can create just 2 clusters, formed by the previous compensators.



Where the bounds of the nodes are:

$$\begin{aligned} q_{\min} &= -10^5[1000 \quad 1.5 \quad 0.15 \quad 1.5 \quad 1.5]^T \\ q_{\max} &= 10^5[1000 \quad 1.5 \quad 0.15 \quad 1.5 \quad 1.5]^T \end{aligned} \tag{23}$$

The PCC bounds are bigger than the others because we suppose that it can compensate all the power changing. The two clusters are formed by $\{(799),(703)\}$ and $\{(702),(703)\}$.
If we suppose to start from the initial condition:

$$q=[-15000 \quad 0 \quad 15000 \quad 0 \quad 0]^T \tag{24}$$

If we choose the cluster formed by the nodes $\{702,703\}$ we obtain with the standard algorithms the following value of δ:

$$\delta=[12808 \quad -12808]^T \tag{25}$$

Obviously bounds are violated, sousing our control law:

$$\begin{aligned} &q_2+\delta_2=27808>q_{max,2} \\ &\delta=-\min(12808,0,150000)=0 \end{aligned} \tag{26}$$

Therefore with this cluster we can't decrease the cost function value. Now we can try with the second cluster, $\{799,703\}$. In this case the optimal value of δ is:

$$\delta=[-73187 \quad 73187]^T \tag{27}$$

Also in this case the bounds are violated so we can apply our algorithm.

$$\begin{aligned} &q_2+\delta_2=88187>q_{max2} \\ &\delta=-\min(73187,0,99985000)=0 \end{aligned} \tag{28}$$

So we have verified that we don't reach the minimum because we are blocked in this configuration. ■

V. A GOSSIP-LIKE ALGORITHM WITH MULTI-HOP COMMUNICATION

In this section, we present a new type of algorithm. It can be implemented by the compensators, without the supervision of any central controlled, by performing local measurements and actuation, like the previous version. The innovative strategy consists in the implementation of a multi-hop communication compensator-by-compensator. Each time a cluster is chosen, a compensator belonging to the triggered cluster can "share" its measurements with another compensator, randomly chosen among the neighbors. We will show that with this algorithm the graph of the compensators tends to a complete graph, in other words the probability of triggering any pair of compensators is strictly greater than 0. Finally we will show that this gossip-like algorithm yields the solution of the original constrained optimization problem.

A. Description of the algorithm

We assume that compensators are clustered into edge-disjoint pairs, and that the resulting clustering graph $\mathcal{H}$ is connected.

We also assume that each compensator h knows the electric distance R^{eff}_{hk} between itself and any other neighbor k in the clustering graph $\mathcal{H}$.

Each cluster is provided with a timer, governed by an independent Poisson process, which triggers the cluster after exponentially distributed waiting times.

The proposed multi-hop algorithm consists of the following, repeated steps:

1) A cluster $\mathcal{C}_{i(t)}$ is chosen at a certain discrete time $t=0,1,2,\dots$ where $i(t)\in\{1,\dots,\ell\}$;
2) Let h,k be the compensators belonging to $\mathcal{C}_{i(t)}$. One compensator between h and k is randomly chosen to be the one which will certainly actuate the system (say "$master$"); the other one (say "$slave$")

instead, as we will show in the following, starts the communication with its neighbors;

3) Let h be the *master* compensator. Then **Send** is executed.

Send

i) compensator h measures voltage u_h;

ii) compensator h computes $\delta_h^{\text{min}}, \delta_h^{\text{max}}$ as

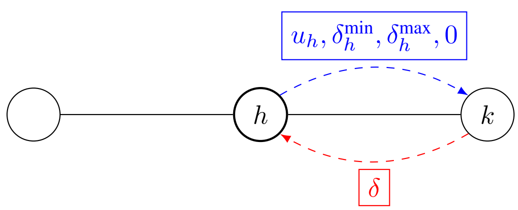
$$\delta_h^{\text{min}} = q_h - q_h^{\text{min}}, \qquad \delta_h^{\text{max}} = q_h^{\text{max}} - q_h;$$

iii) node h sends the message $\boxed{u_h, \delta_h^{\text{min}}, \delta_h^{\text{max}}, d = 0}$ to the compensator $k \in \mathcal{C}_i$;

iv) compensator h waits for a response message from k;

v) compensator h receives the message $\boxed{\delta}$ from k and actuates the system:

$$q_h^+ = q_h + \delta.$$



On the other hand, when the compensator k receives a message $\boxed{u_h, \delta_h^{\text{min}}, \delta_h^{\text{max}}, d}$ from the node h:

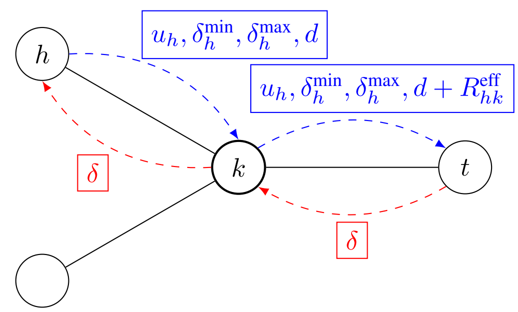
- if k has no other neighbors in the clustering graph $\mathcal{H}$ besides h, then **Respond** is executed;
- otherwise
  - with probability p, $0 < p < 1$, **Forward** is executed;
  - with probability $1-p$, **Respond** is executed.

Forward

i) Compensator k forwards the message $\boxed{u_h, \delta_h^{\text{min}}, \delta_h^{\text{max}}, d + R_{hk}^{\text{eff}}}$ to a randomly chosen neighbor (say $t \neq h$) in the clustering graph $\mathcal{H}$;

ii) compensator k waits for a response message from t;

iii) compensator k receives the message $\boxed{\delta}$ from t and forwards it back to node h.



Respond

i) Compensator k measures voltage u_k;

ii) compensator k computes the optimal step δ as

$$\delta = \left[-\frac{\cos\theta}{R_{hk}^{\text{eff}}} (\nu_k - \nu_h) \right]_{\max\{\delta_h^{\text{min}}, \delta_k^{\text{min}}\}}^{\min\{\delta_h^{\text{max}}, \delta_k^{\text{max}}\}},$$

where

$$\theta = \angle Z_{hk}^{\text{eff}} = \theta_0 \text{ known and equal for all nodes,}$$

$$\nu_k = -\frac{1}{2}|u_k|(|u_k|\sin\theta - |u_h|\sin(\angle u_k - \angle u_h - \theta)),$$

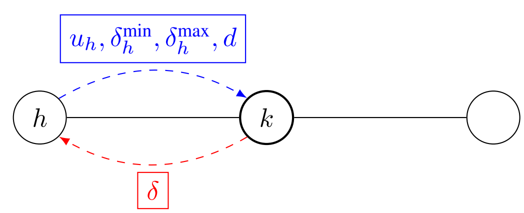
$$\nu_h = -\frac{1}{2}|u_h|(|u_h|\sin\theta - |u_k|\sin(\angle u_h - \angle u_k - \theta))$$

and where $[\cdot]_a^b = \min\{\max\{\cdot, a\}, b\}$;

iii) compensator k responds to node h with the message $\boxed{\delta}$;

iv) compensator k actuates the system:

$$q_k^+ = q_k - \delta.$$



Remark. *By assuming that the time required for the execution of the algorithm is negligible with respect to the average waiting time of the Poisson processes, we do not consider in this analysis the event of concurrent activation of different clusters. With this assumption, the* *sequence of triggering events can be described via a discrete time process.*

This probabilistic choice between **Forward** and **Respond** takes place each time a node receives a message from one of its neighbors.

Lemma 12. *Given any pair of compensators h, k, the probability of triggering the pair h, k is strictly greater than* 0.

Proof:

1) For triggering the pair h, k it is necessary to choose a cluster $\mathcal{C}_{i(t)}$ containing h or k at the step 1) of the algorithm. We consider two separate cases:
   a) if there exists a cluster $\mathcal{C}_i = \{h, k\}$ the probability of triggering the pair h, k is
   $$\mathcal{P}[i(t) = \{h, k\}] \geq \frac{1}{\ell} > 0$$
   where ℓ is the number of the clusters;
   b) Each cluster is different from $\{h, k\}$. Since each compensator belongs to at least one cluster, the probability to choose a cluster containing h or k is
   $$\mathcal{P}[i(t) = \{h, *\}] \cup \mathcal{P}[i(t) = \{k, *\}] \geq \frac{2}{\ell}$$

   Without losing generality we can suppose to choose a cluster $\mathcal{C}_i = \{h, *\}$.
2) For triggering the pair h, k, h needs to be the $master$-compensator. The probability for a compensator to be the $master$ in a two-node cluster is trivially $1/2$.
3) In order to trigger the compensator k, we need to execute **Forward** at most $d-1$ times, where d is the length of the longest path between h and k. Each time we execute **Forward**, we need to choose a neighbor belonging to the path (h, k). The probability to execute **Forward** $d-1$ times is $\underbrace{p \cdot p \cdot \ldots \cdot p}_{d-1} = p^{d-1}$, because the probability is independent for each *hop*. The probability to choose always the "right" neighbor is
   $$\mathcal{P}[s \in \textit{path}_{(h,k)}] \geq \frac{1}{\max\{\deg(j), j \in \mathcal{C}\} - 1}$$
   for every iteration of **Forward**, where $\deg(j)$ is the *degree* (number of neighbors) of node j in the communication graph $\mathcal{H}$. In the worst case, the longest path between h and k coincides with diam($\mathcal{H}$), that is the *diameter* (length of the longest path) of the graph $\mathcal{H}$.
4) To definitively trigger k (together with h) it is necessary to execute **Respond** one time. The probability to execute **Respond** is 1 if the node k has no neighbors (besides the source-node), but generally it is $1 - p$.

Since all these probabilistic events are independent one to each other, the probability of triggering the pair h, k is

$$\mathcal{P}[i(t) = \{h, k\}] \geq \frac{1-p}{\ell} \left(\frac{p}{\max\{\deg(j), j \in \mathcal{C}\} - 1} \right)^{\mathrm{diam}(\mathcal{H})-1}$$

that is a constant number $\varepsilon > 0$. ■

Theorem 13. *If the clustering graph $\mathcal{H}$ is complete, and q is an equilibrium point for each cluster, then $q = q^*$, where q^* is the optimal point for the constrained problem.*

Proof: Let us denote $m := |\mathcal{C}|$ the cardinality of the compensators set, and let be $\bar{q} = [\bar{q}_1 \ldots \bar{q}_m]^T$ any configuration of the compensators before a i-th iteration of the algorithm, $i = 1, \ldots, N$. Since $\mathbf{1}^T \Delta q = 0 \ \forall \Delta q$, we can argue that

$$\mathbf{1}^T \bar{q} = \mathbf{1}^T q^* = \mathbf{1}^T q_0$$

where q_0 is the configuration of the compensators at $t = 0$. Moreover, it is clear that

$$\begin{bmatrix} q_{\min,1} \\ \vdots \\ q_{\min,m} \end{bmatrix} =: q_{\min} \leq \bar{q}, q^* \leq q_{\max} := \begin{bmatrix} q_{\max,1} \\ \vdots \\ q_{\max,m} \end{bmatrix}$$

where $q_{\min}$ and $q_{\max}$ are two vectors containing respectively lower and upper bounds. We define the optimal step

$$\Delta q^* = q^* - \bar{q} = \begin{bmatrix} q_1^* - \bar{q}_1 \\ \vdots \\ q_m^* - \bar{q}_m \end{bmatrix} = \begin{bmatrix} \delta q_1^* \\ \vdots \\ \delta q_m^* \end{bmatrix}.$$

By actuating $\bar{q} + \Delta q^*$ the system would be in the optimal configuration in just one step. The condition $\mathbf{1}^T \Delta q^* = 0$ implies one of the two following cases:

a) $\Delta q^* = [0 \ldots 0]^T$, then $\bar{q} = q^*$ is an equilibrium point;
b) $\Delta q^* \neq [0 \ldots 0]^T$.

Let us proceed with the case b).
The condition $\mathbf{1}^T \Delta q^* = 0$ implies that $\exists \ i, j$ such that $\delta q_i^* \delta q_j^* < 0$. By defining

$$\Delta q_1 := \mathrm{sign} \ \delta q_i^* \min(|\delta q_i^*|, |\delta q_j^*|)(\mathbf{1}_i - \mathbf{1}_j)$$

(where $\mathbf{1}_i$ is 1 in position i and 0 elsewhere) we can argue that the vector $\Delta \hat{q}_1 := \Delta q^* - \Delta q_1$ has at least one component equal to 0, and necessarily $\mathbf{1}^T \Delta \hat{q}_1 = 0$.

Notice that, according to the definition, Δq_1 has just two components different from 0 and also $\mathbf{1}^T \Delta q_1 = 0$.

Since $\mathbf{1}^T \Delta \hat{q}_1 = 0$, as before we have two different cases:

a') $\Delta \hat{q}_1 = [0 \dots 0]^T$;

b') $\Delta \hat{q}_1 \neq [0 \dots 0]^T$.

Let us suppose that b') is verified. We can proceed as before, finding $\delta \hat{q}_{1,i}$ and $\delta \hat{q}_{1,j}$ discordant and defining the vectors

$$\Delta q_2 := \text{sign } \delta \hat{q}_{1,i} \min(|\delta \hat{q}_{1,i}|, |\delta \hat{q}_{1,j}|)(\mathbf{1}_i - \mathbf{1}_j),$$

$$\Delta \hat{q}_2 := \Delta \hat{q}_1 - \Delta q_2$$

where $\Delta \hat{q}_2$ is a vector with at least two components equal to 0. By repeating this process, we can argue that

1) the vector $\Delta \hat{q}_k$ has at least k components equal to 0;
2) Δq_h, $h = 1, \dots$, are all vectors with just two components different from 0, for which the condition $\mathbf{1}^T \Delta q_h = 0$ holds true. In other words they are vectors of the form $[\underline{0} \;\; \delta_h \;\; \underline{0} \;\; -\delta_h \;\; \underline{0}]^T$, where $\underline{0}$ is a vector of zeros of the appropriate dimension;
3) $q_{\min} \leq \bar{q} + \Delta \hat{q}_i \leq q_{\max}$, because
$$\begin{cases} q^* = \bar{q} + \Delta q^* \in [q_{\min} \; q_{\max}] \\ \|\Delta \hat{q}_i\| \leq \|\Delta \hat{q}_{i-1}\| \leq \dots \leq \|\Delta \hat{q}_1\| \leq \|\Delta q^*\| \end{cases}$$

At the end of the process we obtain necessarily the vector

$$\Delta \hat{q}_t = [0 \dots 0]^T, 1 \leq t \leq m.$$

which is comprehensive of the case a') (with $t = 1$). We can argue that $\Delta q^* = \sum_{i=1}^{t} \Delta q_i$, and each Δq_i is a valid choice for the algorithm described before (as said in the point 2)).

Follows directly that there cannot be any equilibrium point, except of q^*.

■

To prove the convergence of the proposed algorithm, we introduce the auxiliary variable $x = q - q^*$. In these coordinates the discrete time system

$$q(t+1) = F_{\sigma(t)}[q(t)] := \arg\min_q \quad J(q)$$

(subject to the various constraints described in previous chapters) results to be a linear time varying system of the form

$$x(t+1) = T_{\sigma(t)}[x(t)]$$

where the sequence $\sigma(t)$ is obviously a sequence of independently, uniformly, distributed symbols in $\{1, \dots, \ell\}$ Now we need to introduce the concept of *set-valued* maps.

A set-valued map $T : X \rightrightarrows X$ associates to an element of X a subset of X. T is non-empty if $T(x) \neq \emptyset$ for all $x \in X$. An evolution of the dynamical system determined by a non-empty set-valued map T is a sequence $\{x_t\}_{t \in \mathbb{Z}_{\geq 0}}$ with the property that $x_{t+1} \in T(x_t)$ for all $t \in \mathbb{Z}_{\geq 0}$. A set W is strongly positively invariant for T if $T(w) \subset W$ for all $w \in W$. The following theorem holds.

Lemma 14 (Theorem 4.5 in [3])**.** *Let (X, d) be a metric space. Given a collection of maps $T_1, \dots, T_\ell$, define the set-valued map $T : X \rightrightarrows X$ by $T(x) = \{T_1(x), \dots, T_\ell(x)\}$. Given a stochastic process $\sigma : \mathbb{Z}_{\geq 0} \to \{1, \dots, \ell\}$, consider an evolution $\{x_n\}_{n \in \mathbb{Z}_{\geq 0}}$ of T satisfying*

$$x_{n+1} = T_{\sigma(n)}(x_n).$$

Assume that

i) there exists a compact set $W \subseteq X$ that is strongly positively invariant for T;
ii) there exists a function $U : W \to \mathbb{R}$ such that $U(w') < U(w)$, for all $w \in W$ and $w' \in T(w) \backslash \{w\}$;
iii) the maps T_i, for $i \in \{1, \dots, \ell\}$, and U are continuous on W; and
iv) there exists $p \in]0, 1[$ and $h \in \mathbb{N}$ such that, for all $i \in \{1, \dots, \ell\}$ and $n \in \mathbb{Z}_{\geq 0}$
$$\mathbb{P}[\sigma(n+h) = i | \sigma(n), \dots, \sigma(1)] \geq p.$$

If $x_0 \in W$, then there exists $c \in \mathbb{R}$ such that almost surely the evolution $\{x_n\}_{n \in \mathbb{Z}_{\geq 0}}$ approaches the set

$$(N_1 \cap \dots \cap N_\ell) \cap U^{-1}(c),$$

where $N_i = \{w \in W | T_i(w) = w\}$ is the set of fixed points of T_i in W, $i \in \{1, \dots, \ell\}$.

Theorem 15. *The proposed algorithm converges to the optimal solution of the constrained problem.*

Proof: Consider the maps $T_i(x) = T_i x$ and the set-valued map $T(x) = \{T_1(x), \dots, T_\ell(x)\}$. Let W be the compact set $\{x \mid J(x) \leq J(x(0))\}$. W is strongly positive invariant for T as $J(T_i x) \leq J(x)$ for all x, i (as $T_i x$ solves the optimization subproblems initialized in x). Moreover, let U be the function $J : W \to \mathbb{R}$. It is verified that $J(w') < J(w)$, for all $w \in W$ and $w' \in T(w) \backslash \{w\}$, because of the definition of J. The continuity of the maps T_i, for $i \in \{1, \dots, \ell\}$, follows directly from the continuity of δ, that is a continuous function of u, that is a continuous function of q. The fact that U is continuous on W follows directly from the definition of J. Finally, because of Lemma 12, for all n, i we have

$$\mathbb{P}[\sigma(n+h) = i | \sigma(n), \dots, \sigma(1)] \geq \varepsilon > 0.$$

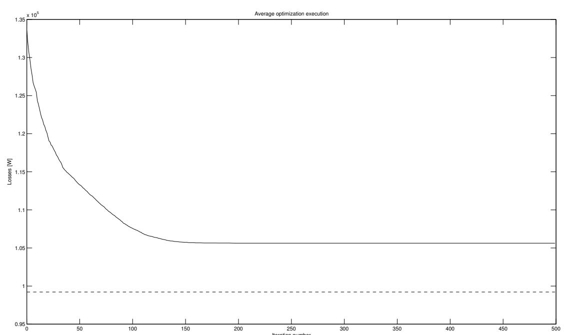
where

$$\varepsilon = \frac{1-p}{\ell}\left(\frac{p}{\max\{\deg(j), j \in \mathcal{C}\} - 1}\right)^{\operatorname{diam}(\mathcal{H})-1}$$

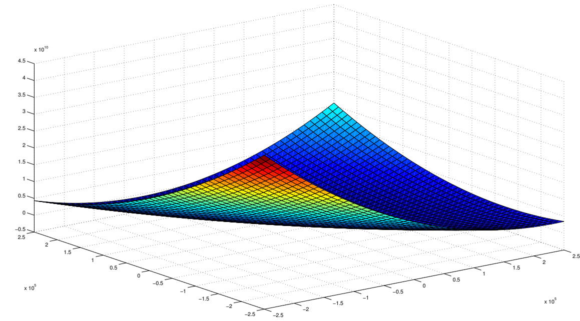
Lemma 14 then applies. Because of Theorem 13, the intersection of the fixed points of the maps T_i reduces to $x = 0$, and then $x(t) \to 0$ almost surely as $t \to \infty$. Therefore $q(t) \to q^*$ almost surely as $t \to \infty$. ■

VI. SIMULATIONS

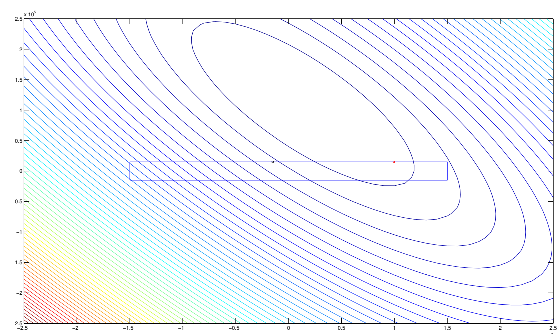
We carried on a lot of simulations and in this section we will analize the results obtained.
To perform the simulation we used the testbed implemented for testing the distributed algorithm [1], inspired from the standard testbed IEEE 37 [4]. We however assumed that load are balanced, and therefore all currents and voltages can be described in a single-phase phasorial notation.
In the first figure we can see the behaviour of the distributed algorithms with box bounds but without the multi-hop communication.



We can trivially see that we didn't reach the minimum.If we consider an easier network, for example with just 5 nodes and 3 compensators, we can plot some more interesting figures.



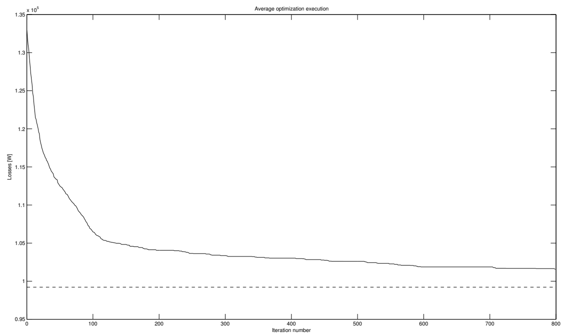
This is the cost function, it is convex and admits a unique minimum, and in the following figure we can see the projection in the two compensators with the more restrictive constraints, the red point indicates the optimal value, the black point the reached value and the blue box the bounds.



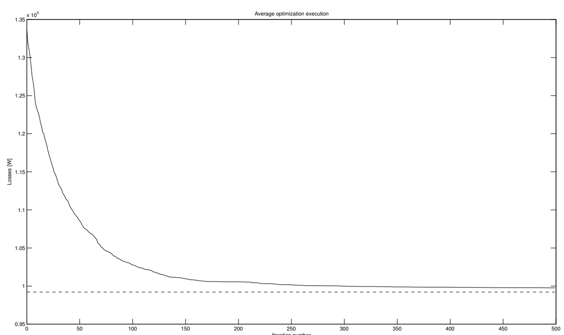
In the table below there is a summary of the performances of the algorithm.

Losses before optimization:	61253 [W]
	13.21%
cos-phi at PCC before opt.:	0.90
Optimal losses:	53499 [W]
	11.63%
Potential losses reduction:	12.66 %
Potential best cos-phi:	0.97
Losses after optimization:	61067 [W]
	13.16%
Losses reduction after opt.:	0.30%

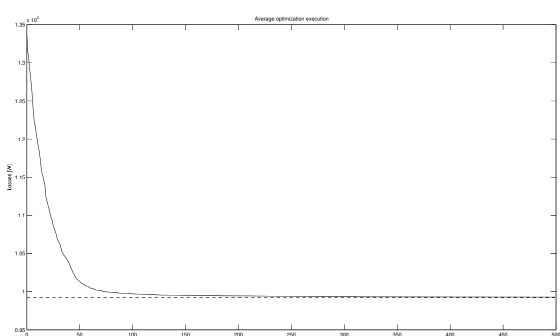
We can reach the real minimum using the multi-hop communication, and we can try different solutions changing the **Forward** probability. If we choose a small value of p, for example 0.01, we will reach all the nodes in the network and we could use all the possibile clusters formed by two nodes with a really low probability, so we expect that will be convergence with a slow rate.



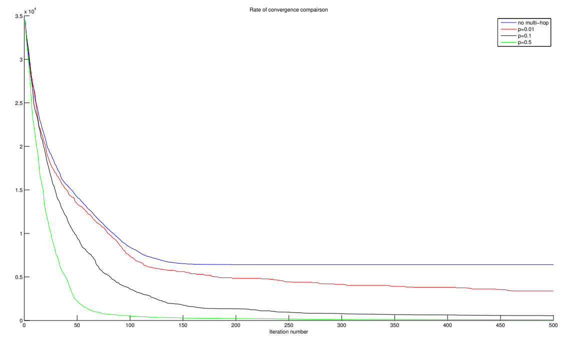
With a probability 0.1 we expect to increase the performances but the rate of convergence will be however not very fast.



In the last example we choose $p = 0.5$ and as we can see the results are incredible, because we reach the minimum quickly in perhaps 100 iterations. Obviously there is a cons because the communication cost increases.



In the last figure we can see the comparison between the rates of convergence.



VII. Conclusions

The gossip-like algorithm with multi-hop communication proposed in this work seems to be an effective way to tackle the problem of optimal reactive power flows where compensators are subject to power limits ("box-constraints"). It requires local knowledge of the problem structure and of the system state at the agent level, and it is able to drive the system to the optimal configuration whatever it is. Moreover, this work can be considered a valuable starting point for the design of a dynamic optimization algorithm, to tackle the more realistic problem in which both the reactive power demands and the agents' constraints are time-varying.

As we are able to tell which clustering choice is capable of giving the optimal performance, for a possible future development it would be interesting to implement an algorithm with which trigger clusters in a intelligent, non-randomized way. In particular it would be very important to decide in a non-probabilistic way if is more convenient to execute "Forward" or "Respond" each time. Moreover it can be interesting to develop some different multi-hop communication algorithms and compare their performances.

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