Stabilization of Linear Systems over a non-ideal Channel

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Abstract:
This paper is concerned with the regulator design of a Networked control system (NCSs). A new model of the NCS channel is provided under consideration of the main non-idealities that affects the data transmission, such as packet drop, power constraint at the channel input and quantization noise. In terms of the given model, a regulator design method is derived based on LQG approach. An example is given to show the effectiveness of this method. Moreover, a revision work on the related literature has been carried out, in which the SNR-constrained problem has been translated into a LQG one, by the choice of suitable weight matrices.

1 Introduction

Today, an increasing number of applications demands remote control of plants over unreliable networks. The recent evolution of sensor web technology enables the development of wireless sensor networks that can be immediately used for estimation and control. In these systems issues of communication delay, data loss, and time-synchronization play critical roles. This work explores the theoretical foundations for estimation and control system design problems while explicitly accounting for realities of the underlying wireless communication network.
In this paper we consider the problem of stabilizing a possibly unstable system across a communication channel (e.g. , Networked Controlled System), where the plant is modeled as a discrete time LTI dynamical systems subject to additive measurement and process noise.
The main issue is to give a model representation that captures the main non-idealities of the digital channel, such as quantization noise, packet loss and power constraints. The next step is to design an optimal control and estimation scheme in order to achieve desiderable features in terms of some performance index.

In this scenario both communication and control fields become tightly coupled. Classical control theories provide a wealth of analytical results but they critically rely on the assumption that the underlying communication technology is ideal. It means that signals coming from sensors and actuators over a communication network are "perfect", more precisely they can't be corrupted by delay or information loss. Instead communication protocols assume the plant and the source process to be stationary and stable and they don't require any feedback control loop. It's of paramount importance to understand how these two approaches can be combined.

2 Control over a communication network

Networked Control Systems (NCSs) are systems where the control loop, consisting of a discrete-time plant, is closed over a communication channel. A schematic representation of a NCS is depicted in Figure 1.

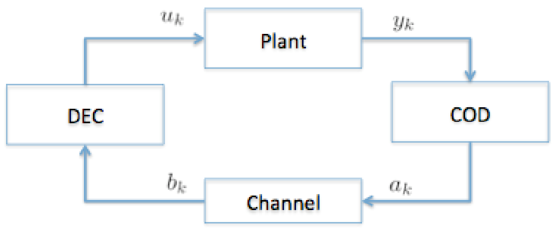


Figura 1: Scheme of control system over a communication channel

The plant output y_k is measured and preprocessed by a causal Coder/Estimator (COD) which sends data a_k across a communication channel. On the other side, a causal Decoder/Controller (DEC) processes the received data b_k and computes the control input u_k necessary to achieve some desirable feature, first of all the closed-loop stability.

It's a standard practice to decouple the Coder and Decoder design into two parts: one related to the source(plant), and the other related to the channel (see Figure 2).

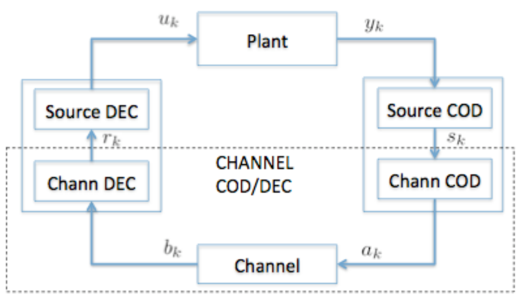


Figura 2: Decoupling of the COD/DEC blocks

The goal of the Source Coder/Decoder design is to achieve some objectives, e.g. stabilization and performance optimization. On the other hand the Channel Coder/Decoder blocks implement a suitable digital code of the input analog signal, according to the chosen protocol. More precisely, the Channel Coder computes the binary code a_k which is sent across the channel. Then the received code word b_k will be reconverted into the analog signal b_k by the Channel Decoder, in order to guarantee the minimum transmission error, i.e. $r_k \approx s_k$.

For a given communication protocol, the objective is the Source Coder/Decoder design, using the typical control and estimation techniques. This approach allows us to focus our attention on the design of the optimal control strategy mantaining the existing channel models. In fact the overall Coder/Decoder blocks design would require the change of the most commonly used protocols, with a large impact from the economic and implementative point of view.

2.1 Previous work

These problems have stimulated a strong research interest within the control community and the literature on Networked Control Systems with random delay measurement and control or packet loss is wide and diverse. We focus our attention on four main fields: NCS with lossy channels, NCS with SNR-limited channels, NCS with rate-limited channels and NCS with limited informations.

2.1.1 NCS with lossy channels

We consider the literature known as NCS with lossy channels or NCS subject to packet loss: the authors in [6] discuss control and estimation problems where the observation and control packets may be lost or delayed. The plant is modelled as a discrete-time linear stochastic system with intermittent observation and control packets (see Figure 3):

$$\begin{cases} x_{k+1} = Ax_k + Bu_k^a + w_k \\ u_k^a = \nu_k u_k^c \\ y_k = \gamma_k C x_k + v_k \end{cases}$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}$, $u^a \in \mathbb{R}$ is the control input from the actuator, $u^c \in \mathbb{R}$ is the control input from the controller, w, v, x are Gaussian, uncorrelated and white. The unreliability of the communication network is handled in a stochastic framework by assigning probabilities to the successful transmission of packets : γ_k and ν_k are i.i.d Bernoulli random variables that models the correct information delivery between sensor/controller and controller/actuator respectively. We notice that neither packet delay nor quantization noise have been taken into consideration.

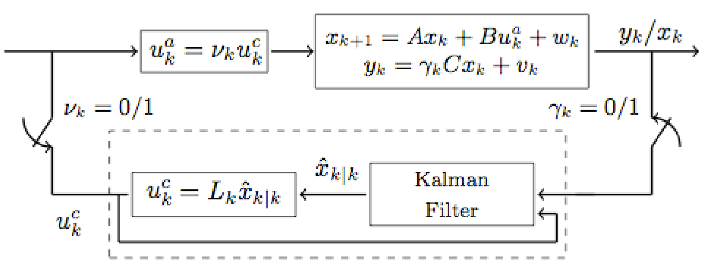


Figura 3: Feedback control over a lossy network

In relation to this scheme the problem of estimation and control over a unreliable network is handled. It's shown that for network protocols where successful transmissions of packets is acknowledged at the receiver side (TCP-like protocols) the classic separation principle holds, and consequently the controller and estimator can be designed independently. Moreover, the optimal LQG controller is a linear function of the estimated state.

However, unlike standard LQG control, the gain of the optimal observer does not converge to a steady-state value and is a function of the packet arrival process. Furthermore, in analyzing the infinite-horizon problem, it's shown that exists a critical threshold for the arrival probabilities $\bar{\gamma} := P(\gamma_k = 1)$ and $\bar{\nu} := P(\nu_k = 1)$, under which the LQG optimal controller fails to stabilize the closed-loop system. It means that the underlying communication channel isn't enough reliable to ensure stability. Those critical probabilities are related to the solvability of a Modified Algebraic Riccati Equation (MARE). For example for $\bar{\nu}$ the MARE is $S = \Pi(S, A, B, W, U, \nu)$ where:

$$\Pi(S, A, B, W, U, \nu) := A'SA + W - \nu A'SB(B'SB + U)^{-1}B'SA$$

It is known that if B is rank one the critical probability, $\bar{\nu}$, is related to $max_i |\lambda_i^u(A)|^2$ and that in the opposite case if B is square and invertible then $\bar{\nu}$ is related to $\prod_i |\lambda_i^u(A)|^2$. Finding a close form for $\bar{\nu}$ in the general case is instead an open research problem. Similar formulae are valid also for the MARE related to $\bar{\gamma}$ which is $P = \Pi(P, A', C', Q, R, \gamma)$.

2.1.2 NCS with SNR-limited channels

In this context usually an AWGN channel is considered with the further constraint that the input power must be limited. The following are the major works in this area:

- The paper [1] has considered feedback stabilization problem where power constraint on the discrete-time input channel is imposed. It's assumed that either the states or the output of the plant are available at the controller and estimator side, and the minimal SNR compatible with stabilization using linear feedback is derived (see Figure 4).

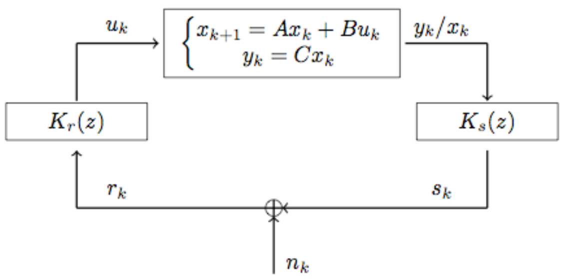


Figura 4: Feedback control over a communication link

The plant is modeled as a possibly unstable noise-free discrete-time SISO dynamical system with state equations:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_k \end{cases}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R}$, it's also assumed that the triple (A, B, C) is minimal. The sent message s_k

is delivered over an AWGN channel with input-output relation:

$$r_k = s_k + n_k$$

where r_k is the received signal and n_k is zero-mean white Gaussian noise with variance σ_n^2, which represents the quantization noise as an additive disturbance. The channel input s_k is assumed to be a stationary stochastic process with power $||s||_{Pow} := \mathbb{E}\left[||s_k||^2\right]$ and it is required to satisfy the power constraint:

$$||s||_{Pow} < \mathcal{P}_d \tag{1}$$

for some predetermined power level $\mathcal{P}_d$. As such, we notice that the maximum admissible SNR that fits the power constraint is $\mathcal{P}_d/\sigma_n^2$. If $T(z)$ is the closed-loop transfer function from channel noise n_k to channel input s_k, then the power in the channel input is given by $||s||_{Pow} = ||T(z)||^2_{\mathcal{H}_2}\sigma_n^2$ and the condition (1) can be written as:

$$||T(z)||^2_{\mathcal{H}_2} < \frac{\mathcal{P}_d}{\sigma_n^2} \tag{2}$$

The main goal of the work is to find the minimum SNR that allows the feedback loop stability. From the former discussion it turns out that this problem can be translate in computing the minimum value of $||T(z)||^2_{\mathcal{H}_2}$ that satisfies (2). The main results can be exposed as follows.

For state feedback stabilization a lower bound for $||T(z)||^2_{\mathcal{H}_2}$ is found, and it depends on the unstable eigenvalues $\left\{\lambda_i^u, i = 1, \ldots, m\right\}$ of A . More precisely:

$$\inf_{K_r, K_z} ||T(z)||^2_{\mathcal{H}_2} = \left(\prod_{i=1}^{m} \lambda_i^u\right) - 1 \tag{3}$$

So the smallest value of SNR that ensures stability is given by:

$$\frac{\mathcal{P}_d}{\sigma_n^2} > \left(\prod_{i=1}^{m} \lambda_i^u\right) - 1 := SNR_{min} \tag{4}$$

For output feedback stabilization a similar bound is found and it depends also on the non-minimum phase zeros and the relative degree of the transfer function $T(z)$, through the coefficients η and δ as follows:

$$\frac{\mathcal{P}_d}{\sigma_n^2} > \left(\prod_{i=1}^{m} \lambda_i^u\right) - 1 + \eta + \delta := SNR_{min} \tag{5}$$

- In the work [2] the authors consider the problem of minimizing the variance of a plant output in response to a stochastic disturbance (see Figure 5) .

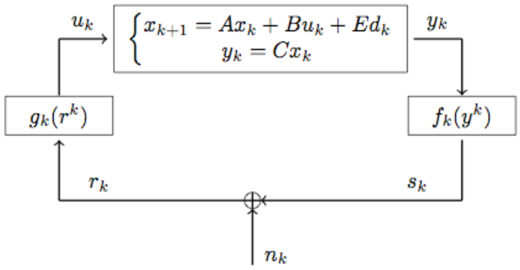


Figura 5: Feedback control over a communication link

The plant is modeled as a possibly unstable discrete-time SISO dynamical system with state equations:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + Ed_k \\ y_k = Cx_k \end{cases}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R}$, $d \in \mathbb{R}$. The process disturbance d_k is a zero mean Gaussian i.i.d noise. It's also assumed that the triple (A, B, E, C) is minimal and the transfer function has relative degree one and is minimum phase; noiseless measurements of the plant are avaliable.

The sent message s_k is delivered over an AWGN channel with input-output relation:

$$r_k = s_k + n_k$$

where n_k is a zero mean Gaussian white noise sequence with variance σ_n. A precompensation (the encoder f_k, see Figure 5) can be introduced before the channel (for example, s_k can be a filtered version of the ouput).

We notice that there's no delay in the feedback path, and the input channel is required to satisfy the istantaneous power constraint:

$$\mathbb{E}\left[||s_k||^2\right] \leq P \tag{6}$$

The goal of this paper is to select the optimal control strategy with respect to the power limit (6), for which the channel input depends causally on the plant output, and the control input depends causally on the sequence of the channel ouput (e.g. , $s_k = f_k(y^k)$ and $u_k = g_k(r^k)$).

In other words the authors aims to find the optimal coding and decoding sequences $f_k(y^k)$ and $g_k(r^k)$ that minimize the mean square value of the system output at the fixed time $k = N + 1$, subject to (6):

$$J_{N+1}^{opt} := \inf_{f_k, g_k, k=0,\ldots,N} \mathbb{E}\left[||y_{N+1}||^2\right]$$

Without the precompensation the solution to this problem, in case of noise-free measurements, requires to solve a "cheap control" linear quadratic Gaussian problem (LQG). On the contrary in this context the separation principle, present within the LQG optimal control theory, may no longer be valid.
The optimal control at the last-time step is derived, and it's found to be a linear function of the estimated-state at the same time. Moreover, a lower bound on the achievable performance is derived. Only suboptimal strategies are designed for the infinite-horizon problem.

2.1.3 NCS with rate-limited channels

In the context of NCS with rate-limited channels the key problem is to characterize the minimal average data-rate (i.e. bit per sample) that allows to achieve a given control objective when the channel is assumed to transmit data without errors or delays. Given that, rate-distortion theory seems to be the natural framework to deal with the problem. However standard results in this field rely upon coding arbitrarily long sequences and do not take stability nor causality into account. It thus becomes clear that some work must be done in order to adapt classic information theory to control problems. A landmark result in this direction was published in [5] where it was shown that for a noisy plant model it is possible to find causal coders, decoders and controllers such that the resulting closed loop system is mean square stable iff the average data-rate, say $\mathcal{R}$, satisfies :

$$\mathcal{R} > \sum_{i=1}^{n_p} log_2|p_i| \tag{7}$$

where p_i denoth the i^{th} unstable plant pole. We have previously seen that bounds similar to (7) arise as solutions to different problems with different assumption on the channel, this is a sign that the quantity on the right hand side of (7) is a fundamental measure of the difficulty of stabilizing a system.

Another relevant paper in this area is [7]. The main contribution of this work is that it shows , for a specific class of source-coding schemes, that average data-rate constraints can be enforced by imposing signal to noise ratio (SNR) constraints in a related analog additive noise channel. In other words this paper establish a relationship between SNR constraints, discussed in section 2.1.2, and average data-rates constraints in noiseless digital channel. More in particular, given a linear discrete-time SISO plant (see Figure 6) such that:

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \qquad \begin{bmatrix} e(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} d(t) \\ u(t) \end{bmatrix}$$

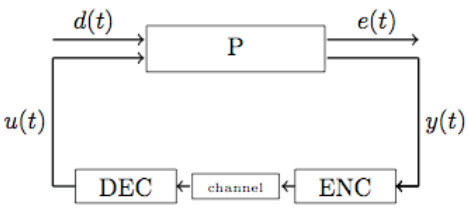


Figura 6: NCS with rate-limited channel

where transparent feedback is assumed, the goal is to characterize the minimal average data-rate that allows one to achieve a prescribed degree of fidelity, first of all the closed-loop mean square stability. The situation described through this scheme arises naturally if, for example, P corresponds to the interconnection on an LTI plant and LTI controller that has been designed without taking into account data rate limits in the feedback path. The channel in this context is noiseless and transmits data without errors or delays; only rate-distortion is assumed.

Focusing on a particular class of souce-coding schemes, a lower bound for $\mathcal{R}$ is derived, which is at most 1.254 bits per sample away from the absolute minimum rate established by (7). This rate penality is compensated by the simplicity of the class chosen. Furthermore this average data-rate limit can be enforced by imposing signal-to-noise ratio (SNR) constraints in a related analog additive noise communication channel.

2.1.4 NCS with limited informations

The main goal of this class is to develop a theory of stabilization of LTI systems using only a finite number of fixed control values and of measurement levels. The quantization of controls and measurements induces a quantization, or partition, in the system state space.

In the paper [3], for example, the control strategy adopted is called Control Lyapunov Function (CLF) for systems with control inputs. The plant is here assumed unstable, single input and stabilizable, and governed by the following equations:

$$x_{k+1} = Ax_k + Bu_k \tag{8}$$

From these hypothesis it follows that exists a quadratic Lyapunov function $V(x)$ for the closed-loop system (CLF) such that:

$$V(x) = x^\top Px \quad P > 0 \quad P = P^\top$$

and given that, it is always possible to find a a linear static state-feedback control that stabilize the system. More precisely, for a given $V(x)$, exist a set of control values $\mathcal{U} = \{u_i \in \mathbb{R} : i \in \mathbb{Z}\}$ and a function $f : \mathcal{X} \to \mathcal{U}$ such that $f(x) = -f(-x)$, and such that $\forall x \neq 0$:

$$\Delta V(x) = V(Ax + Bf(x)) - V(x) < 0$$

With respect to this notation f is called quantizer and it's a symmetric, bijective function that can take a countable number of levels. The measure of the coarseness of the quantizer is expressed by its density. Let $\mathcal{Q}(V)$ denote the set of all quantizer for a given CLF $V(x)$ for the system (8), for $g \in \mathcal{Q}(V)$ and $0 < \varepsilon \leq 1$, let let $\#g[\varepsilon]$ denote the number of levels that g takes in $[\varepsilon, 1/\varepsilon]$. Than we define the quantization density η_g of g as follows:

$$\eta_g := \lim_{\varepsilon \to 0} \sup \frac{\#g[\varepsilon]}{-ln\varepsilon}$$

Moreover, a quantizer f is said to be coarsest for $V(x)$ if it has the smallest quantization density, i.e. :

$$f = \arg \inf_{g \in \mathcal{Q}(V)} \eta_g$$

It's worth remarking that this definition of density allows us to measure quantizers for which the number of quantization values grows logarithmically, with the length of the interval that includes them (e.g. any uniform quantizer has infinite density, viceversa a finite quantizer has density equal

to zero).
In this paper it's shown that the coarsest quantizer that quadratically stabilizes the plant follows a logarithmic law. Moreover, the best quadratic Lyapunov function that allows the coarsest logarithmic quantizer it's the same that arise in the solution of the expensive control LQR problem. A closed form for the smallest logarithmic base compatible with the closed-loop stability is then derived and it depends exclusively on the unstable eigenvalues of the system.

All these works suffer from some limitations: they cosider the channel non-idealities one at time, even if the others are not negligible in real application. The aim of this paper is to bring packet loss, delay, SNR constrains and quantization errors into the same picture in a simple and understandable way, even at the price of loosing some optimality in the design. Only little efforts (see [4])have been made in this direction. Our goal is to get deeper into this subject.

2.2 Our Contribution

The paper is organized as follows. In section 3 we describe the setup, in particular we choose to use an LQG approach to the problem. In section 4 a revision of the work [1] is exposed using the LQG approach. Two different schemes are proposed: the former, Scheme 1, send the control law through the channel while the latter, Scheme 2, send the measured output. The channel is assumed to be AWGN. The aim of this section is to demonstrate, with a simulative approach, that both the schemes requires the same minimum value of the SNR derived in [1]. A proof of this fact in the scalar case is given in the Appendix. In section 5 we propose a new channel model that captures the essence of todays' communication systems, i.e. quantization error, delays, packet drop and power constraints, yet being amenable to analysis. Furthermore we propose two different algorithms to compute the variance of the quantization error, for both scalar and multidimensional channel, when the quantizer is assumed to be adaptive. In section 6 we want to apply a simpler version of this channel model to a scheme similar to Scheme 2 except for the presence of quantization and measurement noise. We derive the equation for the estimator and the controller in order to minimize the output variance. Since the estimator gain depends on the output variance it is not clear if in this context the separation principle can be applied. If the cost functional is the output variance the simulations show that the controller gain is that of the standard LQG approach. In this context a lower bound on the SNR for stabilizability is derived. On the contrary, changing the cost functional, the controller gain obtained by simulations is different from that of the LQG approach: this is a clear sign that the separation principle is no longer valid. Finally in section 7 we propose several different schemes to stabilize the plant when the channel model of section 5, without packet delay, is used. In particular we want to test the algorithms described in section 5 and compare the output variance of the schemes for different values of SNR and different probabilities of packet loss. Section 8 draws conclusions.

3 Problem Formulation

We consider the problem of stabilizing a possibly unstable system across a communication channel. In our work we will model the plant as a SISO discrete time linear time invariant dynamical systems subject to additive measurement and process noise. More specifically:

$$\begin{cases} x_{t+1} = Ax_t + Bu_t + w_t \\ y_t = Cx_t + v_t \end{cases}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, $v_t \sim \mathcal{N}(0, R)$, $w_t \sim \mathcal{N}(0, Q)$, $x_0 \sim \mathcal{N}(0, P_0)$, and $w_t \perp v_t \perp x_0$. We also assume that the pairs (A, B) and (A, Q) are controllable, the pairs (A, C) and (A, W) are observable, and $R > 0$.
A simple and appropriate model channel is developed in section 5. It's important that this model renders correctly the relevant beformentioned non-idealities.

For such a system (with partial state/output measurements) the linear quadratic Gaussian (LQG) methodology has proved to be a useful technique for designing output feedback controllers. We define the performance index as follows:

$$J = \frac{1}{T} \sum_{t=0}^{T} \mathbb{E}[x_t^T W x_t + u_t^T U u_t]$$

And in the context of infinite horizon LQG control we define:

$$J = \lim_{T \to +\infty} \frac{1}{T} \mathbb{E}[\sum_{t=0}^{T} x_t^T W x_t + u_t^T U u_t] = \lim_{t \to +\infty} \mathbb{E}[x_t^T W x_t + u_t^T U u_t]$$

and the last equivalence holds if the processes are stationary and ergodic. The choice for the matrices W, U depends on the avaliable measurement information set, and it will be discussed in the next section. In our context the plant control input will always be available at the Kalman estimator site.

4 Revision on literature using LQG approach

In this section a revision of the work [1] will be exposed using the LQG approach, i.e instead of characterizing the closed-loop system stability under the constraint (1) in terms of transfer function properties, we apply the LQG approach to solve the problem.

We consider the system described in the first part of Section 2.1.2 : let $K(z) := K_s(z)K_r(z)$, $T(z)$ the transfer function from n to s and $G(z)$ the transfer function from u to y/x. The following relation holds:

$$T(z) = \frac{K_r(z)K_s(z)G(z)}{1 + K_r(z)K_s(z)G(z)} = \frac{K(z)G(z)}{1 + K(z)G(z)}$$

We are dealing with a SISO linear time-invariant for which the two schemes in Figure 7 and 8 share the transfer function $T(z)$:

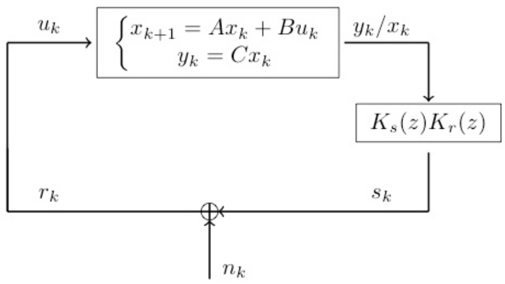


Figura 7: Scheme 1

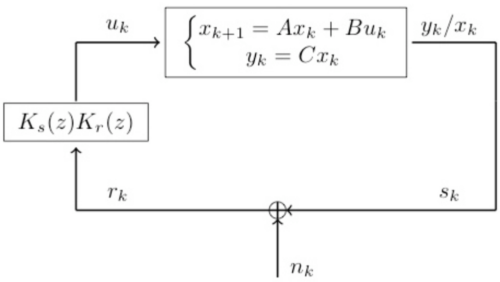


Figura 8: Scheme 2

4.1 Scheme 1: SNR-constrained feedback stabilization

The stabilization problem under the constraint (1) for the scheme presented in Figure 7 will be translate in a LQG problem applied at the same scheme. The equivalent system model at the controller side has state variable description:

$$\begin{cases} x_{k+1} = Ax_k + Bs_k + Bn_k \\ y_k = Cx_k \end{cases}$$

where s_k is the control signal and Bn_k is the noise process. Since the separation principle holds, controller and estimation design can be decoupled (see Figure 9). In the next section a solution to the feedback stabilization problem is derived, using as measurements set the output or the state of the plant model.

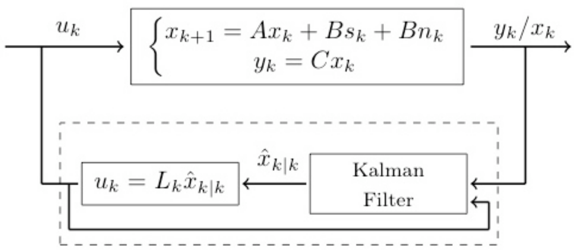


Figura 9: Scheme 1 feedback stabilization

4.1.1 SNR-constrained output feedback stabilization

For this scheme the deterministic component is given by s_k while Bn_k represents the process noise of convariance $Q = \sigma_n^2 BB^\top$, there's no output noise. We also assume that the pairs (A, B) and (A, Q) are controllable, the pairs (A, C) and (A, W) are observable. These conditions ensure the convergence properties and consistency of the Kalman Filter independently from the initial conditions. The Kalman Filter estimate state $\hat{x}_{k|k}$ fits the following recursive equation:

$$\hat{x}_{k|k} = (I - KC)(A\hat{x}_{k-1|k-1} + Bu_{k-1}) + Ky_k \tag{9}$$

Where $K = PC^\top(CPC^\top)^{-1}$ is the static Kalman filter gain, and the error variance matrix $P = \mathbb{E}\left[(x_k - \hat{x}_{k|k})^2\right]$ satisfy the Algebraic Riccati Equation (ARE):

$$P = A\left[P - PC^\top(CPC^\top)^{-1}CP\right]A^\top + \sigma_n^2 BB^\top$$

The feedback-loop performance can be characterized in terms of the index cost:

$$J_T\left(x_0, K(z)\right) = \frac{1}{T}\sum_{k=0}^{T-1} \mathbb{E}_{n_k}\left[x_k^\top W x_k + s_k^\top U s_k\right]$$

where $W \geq 0$ and $U \geq 0$ are suitable matrices. If s_k and x_k are strictly stationary processes, the ergodic theorem holds, for $T \to +\infty$:

$$\begin{aligned} J\left(x_0, K(z)\right) &= \lim_{T\to+\infty} J_T\left(x_0, K(z)\right) = \\ &= \lim_{k\to+\infty} \mathbb{E}_{n_k}\left[x_k^\top W x_k + s_k^\top U s_k\right] \end{aligned}$$

taking $W = CC^\top$ and $U = \rho$, with $\rho \in [0, +\infty)$ the asintotic index cost can be written as:

$$\begin{aligned} J\left(x_0, K(z)\right) &= \lim_{k\to+\infty} \mathbb{E}_{n_k}\left[||y_k||^2 + \rho||s_k||^2\right] \\ &= \lim_{k\to+\infty} \rho\mathbb{E}_{n_k}\left[\frac{1}{\rho}||y_k||^2 + ||s_k||^2\right] \end{aligned} \tag{10}$$

The goal is to find, over all the control strategies that guarantee the feedback loop stability, the one that get the minimum value of the index cost (10):

$$\arg\min_{K(z)} \rho\mathbb{E}\left[\frac{1}{\rho}||y_k||^2 + ||s_k||^2\right] = \arg\min_{K(z)} \mathbb{E}\left[\frac{1}{\rho}||y_k||^2 + ||s_k||^2\right]$$

let $\lambda = \frac{1}{\rho} \in (0, +\infty)$, and taking the limit $\lambda \to 0$ in the infinite-horizon, we get the expected power constraint:

$$\lim_{\lambda\to 0}\min_{K(z)}\lim_{k\to+\infty} \mathbb{E}\left[\lambda||y_k||^2 + ||s_k||^2\right] = \min_{K(z)}\lim_{k\to+\infty} \mathbb{E}\left[||s_k||^2\right] := \mathcal{P}_{min}$$

And it can be demonstrated that this limit exists and it can be reached by the same time-invariant controller L^* that solves the former LQG problem, with $W = CC^T$ and $\rho \to +\infty$. The optimal control law $u_k = L^*\hat{x}_{k|k}$ is derived, solving the Algebraic Riccati Equation (ARE):

$$S = A^\top SA - A^\top SB(B^\top SB + \rho)^{-1}B^\top SA + W \tag{11}$$

and from this:

$$L^* = -(B^\top SB + \rho)^{-1}B^\top SA \tag{12}$$

Closed-loop transfer function from the feedback output

In this section the closed-loop Transfer function $T(z)$ for the feedback output system is derived and it can be compared to the one presented in [1]. From the equation (9) and the optimal control law $u_k = L^* \hat{x}_{k|k}$ we get:

$$\hat{x}_{k|k} = (I - KC)(A + BL^*)\hat{x}_{k-1|k-1} + Ky_k$$

$$u_k = L^*(I - KC)(A + BL^*)\hat{x}_{k-1|k-1} + L^* Ky_k$$

from this expression, let $F = (I - KC)(A + BL^*)$, the transfer function $K(z)$ from y_k to s_k is derived:

$$K(z) = L^* F(zI - F)^{-1} K + L^* K$$

And then $T(z)$ is:

$$T(z) = \frac{G(z)K(z)}{1 - G(z)K(z)}$$

Comparisons

Let's consider the theoretical bound $\mathcal{P}_{min}$ derived in [1] for the minimum feasible power at the channel input, with respect to the output feedback problem:

$$\frac{\mathcal{P}_{min}}{\sigma_n^2} = \left(\prod_{i=1}^{m} \phi_i\right) - 1 + \eta + \delta \tag{13}$$

where $\{\phi_i, i = 1, \ldots, m\}$ are the unstable eigenvalues of the system. Let r be the relative degree of the open-loop transfer function $G(z)$ and $\{\alpha_i, i = 1, \ldots, q\}$ is the set of the distinct non minimum-phase zeroes of $G(z)$. The coefficients that appears in (13) are obtained as follows:

$$\eta = \sum_{l=1}^{q} \sum_{i=1}^{q} \frac{\gamma_i \gamma_l}{\alpha_i \bar{\alpha}_l - 1} \qquad \delta = \begin{cases} 0, & \text{se } r = 1 \\ \sum_{i=1}^{r} |\beta_k|^2, & \text{se } r > 1 \end{cases}$$

$$\gamma_l = (1 - |\alpha_l|^2) \left(\prod_{i=1}^{m} \frac{1 - \alpha_l \bar{\phi}_i}{\alpha_l - \phi_i} - \sum_{k=0}^{r-1} \beta_k \alpha_l^{-k} \right) \prod_{k=1_{k \neq l}}^{q} \frac{1 - \alpha_l \bar{\alpha}_k}{\alpha_l - \alpha_k}$$

$$B_\phi(z) = \prod_{i=1}^{m} \frac{z - \phi_i}{1 - z\bar{\phi}_i} \qquad \beta_k = \frac{1}{k!} \frac{d^k}{dz^k} \left[\prod_{i=1}^{m} \frac{z - \phi_i}{1 - z\bar{\phi}_i} \right]_{z=0}$$

In the case of in two-dimensional space system with only one zero (α) the former formulae become:

$$\frac{\mathcal{P}_{min}}{\sigma_n^2} = \left(\prod_{i=1}^{m} \phi_i\right) - 1 + \eta$$

con

$$\eta = \frac{|\gamma|^2}{|\alpha|^2 - 1} \qquad e \qquad \gamma = (1 - |\alpha|^2)(B_\phi^{-1}(\alpha) - B_\phi(0))$$

if the zero is unstable ($|\alpha| \geqslant 1$). On the contrary if $|\alpha| < 1$ then $\eta = 0$, we notice that the same bound is achieved in the case of state feedback stabilization (see equation (15)).

4.1.2 SNR-constrained state feedback stabilization

If we assume that state measures are avaliable to design the feedback loop, Kalman Filter is no longer needed and the problem is reduced in computing the optimal control law $u_k = L^* x_k$, according to (11), (12) in the past sections.

Closed-loop transfer function from the feedback state

In this section the closed-loop Transfer function $T(z)$ for the feedback state system is derived and it can be compared to the one presented in [1]. Let us consider the equivalent system from u_k to s_k, the state system equation are:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k \\ s_k = L^* x_k \end{cases}$$

The transfer function from u_k to s_k is $H(z) = G(z)K(z) = L^*(zI - A)^{-1}B$ and from $u_k = s_k + n_k$ we obtain the transfer function $T(z)$ from n_k to s_k:

$$T(z) = \frac{H(z)}{1 - H(z)} \tag{14}$$

Comparisons

Let's consider the theoretical bound $\mathcal{P}_{min}$ derived in [1] for the minimum feasible power at the channel input, with respect to the state feedback problem:

$$\frac{\mathcal{P}_{min}}{\sigma_n^2} = \left(\prod_{i=1}^{m} \phi_i\right) - 1 \tag{15}$$

where $\{\phi_i, i = 1, \ldots, m\}$ are the system unstable eigenvalues.

4.1.3 Matlab Simulation

These results have been supported by system simulation using Matlab. Indeed, adopting LQG methodology we find the same results presented in [1]). In particular we find the following equivalence:

- The expected asintotic cost $\mathbb{E}_{n_k}\left[\lambda||y_k||^2 + ||s_k||^2\right]$, over $N = 50$ simulations with random initial conditions, when $\lambda \to 0$.
- The theoretical bound $\mathcal{P}_{min}$ of the minimal feasible power at the channel input, computed in [1].
- The norm $||T(z)||^2_{\mathcal{H}_2}$ of the transfer function $T(z)$ from n to s, computed as indicated in the previous Sections 4.1.1 and 4.1.2, multiplied by σ_n^2.

Examples

In the following we report the results found for a 2-dimensional system characterized by the matrices:

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix} \qquad C^T = B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where $W = C^T C$, $U = \rho$ and random initial condition, $x_0 \in \mathcal{N}(0, 4I_2)$. In the following grid we compute $\mathcal{P}_{min}$ with three different methods:

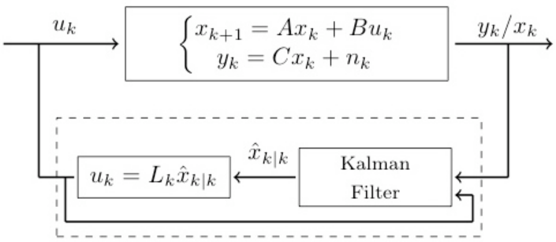


Figura 10: Scheme 2 feedback stabilization

- Theoretical bound: computed from the article [1];
- Transfer function: the norm $||T(z)||^2_{\mathcal{H}_2}$ of the Transfer Function from n to r computed with $\rho = 100000$ multiplied by σ_n^2;
- Matlab Simulation : expected cost $\mathbb{E}_{n_k}\left[\lambda||y_k||^2 + ||s_k||^2\right]$ over $N = 50$ realizations with random initial condition and $\rho = 1/\lambda = 100000$.

	Theoretical	Transfer Function	Simulation
state	0.071225	0.071225	0.071218
output	0.089289	0.089298	0.089552

Tabella 1: $\lambda_1, \lambda_2 = \{1.5, 1.9\}, z = 1.2, \sigma_n = 0.1$

	Theoretical	Transfer Function	Simulation
state	0.0429	0.0429	0.042859
output	0.069739	0.069739	0.069867

Tabella 2: $\lambda_1, \lambda_2 = \{0.9, 2.3\}, z = 1.1, \sigma_n = 0.1$

4.2 Scheme 2: SNR-constrained output feedback stabilization

We now apply the same methods for the Scheme 2 (see Figure 8) to compute the optimal control strategy. With respect to this configuration only the output feedback stabilization can be considered as a point-to-point connection channel is employed so we can't send a multi-dimensional signal (i.e. , the vector state). The equivalent system at the controller side is:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k \\ y_k^{eq} = Cx_k + n_k \end{cases} \tag{16}$$

where u_k is the input signal and n_k is the output noise, with covariance $R = \sigma_n^2 > 0$, and there's no process noise.

Since the separation principle holds, controller and estimation design can be decoupled (see Figure 10).

We also assume that the pair (A, C) is observable, however $(A, Q^{\frac{1}{2}}) = (A, 0)$ isn't stabilizable, so we can't ensure the uniqueness of the asintotic solution of the error covariance $P = \mathbb{E}\left[(x_k - \hat{x}_{k|k})^2\right]$. Indeed the associated Algebraic Riccati Equation is homogeneous, and the convergence properties depends on the particular choice of the initial conditions:

$$P = A\left[P - PC^\top(CPC^\top + \sigma_n^2)^{-1}CP\right]A^\top$$

A first approach is to study the solution convergence for $Q \to 0$. We now introduce the index cost:

$$J_T(x_0, K(z)) = \frac{1}{T}\sum_{k=0}^{T-1}\mathbb{E}_{n_k}\left[x_k^\top W x_k + u_k^\top U u_k\right]$$

If u_k and x_k are strictly stationary processes, the ergodic theorem holds, for $T \to +\infty$:

$$\lim_{T\to+\infty} J_T(x_0, K(z)) = \lim_{k\to+\infty}\mathbb{E}_{n_k}\left[x_k^\top W x_k + u_k^\top U u_k\right]$$

Taken $W = CC^\top$ and $U = \rho$, with $\rho \in [0, +\infty)$, from the output relation (16) we can derive the following representation of the index cost:

$$J_T(x_0, K(z)) = \frac{1}{T}\sum_{k=0}^{T-1}\mathbb{E}_{n_k}\left[||y_k||^2 + \rho||u_k||^2\right]$$

where y_k is the plant output (not the regulator output y_k^{eq}).

For such a scheme, the input channel subject to the power constraint (1) is $s_k = y_k$.

The goal is to find, over all the control strategies that guarantee the feedback loop stability, the one that get the minimum value of the index cost. More precisely:

$$\arg\min_{K(z)}\mathbb{E}\left[||y_k||^2 + \rho||u_k||^2\right]$$

And taking the limit $\rho \to 0$ in the infinite-horizon problem, we get the expected power limit:

$$\lim_{\rho\to 0}\min_{K(z)}\lim_{k\to+\infty}\mathbb{E}\left[||y_k||^2 + \rho||u_k||^2\right] =$$

$$= \min_{K(z)}\lim_{k\to+\infty}\mathbb{E}\left[||y_k||^2\right] := \mathcal{P}_{min}$$

And it can be demonstrated that this limit exists and it can be reached by the same time-invariant controller L^* that solves the former LQG problem, with $W = CC^T$ and $\rho \to 0$. The optimal control law $u_k = L^*\hat{x}_{k|k}$ is derived, solving the Algebraic Riccati Equation (ARE):

$$S = A^\top SA - A^\top SB(B^\top SB + \rho)^{-1}B^\top SA + W$$

and we get:

$$L^* = -(B^\top SB + \rho)^{-1}B^\top SA$$

Closed-loop transfer function from the feedback output

In this section the closed-loop Transfer function $T(z)$ for the feedback output system is derived and it can be compared to the one presented in [1]. From the equation (9) and the optimal control law $u_k = L^*\hat{x}_{k|k}$ we get:

$$\hat{x}_{k|k} = (I - KC)(A + BL^*)\hat{x}_{k-1|k-1} + Ky_k$$

$$u_k = L^*(I - KC)(A + BL^*)\hat{x}_{k-1|k-1} + L^*Ky_k$$

Let: $F = (I - KC)(A + BL^*)$ thus we get the transfer function $K(z)$ from y_k^{eq} to u_k:

$$K(z) = L^*F(zI - F)^{-1}K + L^*K$$

And the Transfer Function $T(z)$ is:

$$T(z) = \frac{K(z)G(z)}{1 - K(z)G(z)}$$

4.2.1 Matlab Simulation

These results have been supported by system simulation using Matlab. Indeed, adopting LQG methodology we find the same results presented in [1]). In particular we find the following equivalence:

- The theoretical bound $\mathcal{P}_{min}$ of the minimal feasible power at the channel input, computed in [1].
- Transfer function: the norm $||T(z)||^2_{\mathcal{H}_2}$ of the Transfer Function from n to r computed as exposed in Section 4.2, multiplied by σ_n^2.
- The expected cost $\mathbb{E}_{n_k}\left[||y_k||^2 + \rho||u_k||^2\right]$, when $\rho \to 0$, over $N = 50$ realizations with arbitrary initial conditions .

Examples

In the following we report the results found for a 2-dimensional system characterized by the matrices:

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix} \qquad C^T = B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where $W = C^TC$, $U = \rho$ and random initial condition, $x_0 \in \mathcal{N}(0, 4I_2)$. In the following grid we compute $\mathcal{P}_{min}$ with three different methods:

- Theoretical bound: computed from the article [1];
- Transfer function: the norm $||T(z)||^2_{\mathcal{H}_2}$ of the Transfer Function from n to r computed with $\rho = \frac{1}{100000}$ multiplied by σ_n^2;
- Matlab Simulation : the expected cost $\mathbb{E}_{n_k}\left[||y_k||^2 + \rho||u_k||^2\right]$ over $N = 50$ realizations with arbitrary initial conditions and $\rho = \frac{1}{100000}$.

	Theoretical	Transfer Function	Simulation
output	0.089289	0.089324	0.089318

Tabella 3: $\lambda_1, \lambda_2 = \{1.5, 1.9\}, z = 1.2, \sigma_n = 0.1$

	Theoretical	Transfer Function	Simulation
output	0.069739	0.069861	0.070098

Tabella 4: $\lambda_1, \lambda_2 = \{0.9, 2.3\}, z = 1.1, \sigma_n = 0.1$

4.3 Transfer Function

In the previous sections we compared the $\mathcal{H}_2$-norm of the transfer function, obtained for the two different schemes, and we found that they were equal. Here we want to compare the functions themselves instead of their $\mathcal{H}_2$-norm. Even if the transfer function is $K(z)G(z)$ in both cases, $K(z)$ could be different because the schemes at the Kalman filter site are different and so the AREs involved in the calculation of the gain are different. Surprisingly the simulations confirm that, on the contrary, the two transfer function aren't only equal in $\mathcal{H}_2$-norm but they are exactly the same function. This is due to the fact that also the ARE for the controller gain is different in the two schemes and this two diversity compensate one for each other. In the following we give some examples:

- $\lambda_1, \lambda_2 = \{1.5, 1.9\}, z = 1.2, \sigma_n = 0.1$

  Scheme 1 : Transfer Function

  $$T(z) = \frac{-1.951z^3 + 3.479z^2 - 1.366z + 7.584^{-17}}{z^4 - 3.983z^3 + 4.831z^2 - 1.661z}$$

  Scheme 2 : Transfer Function

  $$T(z) = \frac{-1.96z^3 + 3.466z^2 - 1.337z + 2.227^{-16}}{z^4 - 3.966z^3 + 4.774z^2 - 1.613z + 0.0001629}$$

- $\lambda_1, \lambda_2 = \{0.9, 2.3\}, z = 1.1, \sigma_n = 0.1$

  Scheme 1 : Transfer Function

  $$T(z) = \frac{-2.163z^3 + 4.326z^2 - 2.141z - 8.259^{-17}}{z^4 - 4.406z^3 + 5.93z^2 - 2.497z}$$

  Scheme 2 : Transfer Function

  $$T(z) = \frac{-2.202z^3 + 4.405z^2 - 2.18z - 2.421^{-16}}{z^4 - 4.411z^3 + 5.945z^2 - 2.507z + 0.0001529}$$

If a SISO scalar system is considered, than the Transfer Function equivalence can be analytically proved (see Appendix). A possible approach to demonstrate the same result in the general case could be carried out through the method of Lagrange multipliers. However this subject is not further developed in this paper.

5 Channel Model

In the various works cited in section 2.1 several channel models have been proposed, however all of them focus only on one non-ideality at time. The aim of this section is to find a channel model that captures the essence of todays' communication systems in all of his relevant aspect yet being amenable to analysis. It is a standard practice, in the communication field, to characterize a channel through its capacity C, in this context it is worth to recall the *Shannon's channel capacity theorem*:

Theorem: Let $\bar{P}$ be the average signal power at the input of the channel and suppose the noise is white thermal noise of power N in the band W. By sufficiently complicated encoding systems it is possible to transmit binary digits at a rate

$$C = Wlog_2\left(1 + \frac{\bar{P}}{N}\right) [bit/s]$$

with as small a frequency of errors as desired. It is not possible by any encoding method to send at higher rate and have an arbitrarily low frequency of errors.

Throughout this theorem it is immediate to show that, in the context of modeling an AWGN channel, the capacity constraint can be translated into a signal-to-noise ratio (SNR) limit. In fact suppose that we have the following constraint on the transmitted signal:

$$\lim_{T\to+\infty}\frac{1}{T}\sum_{t=0}^{T}\mathbb{E}[||s_t||^2] = P_s \le \bar{P} = N(2^{\frac{C}{W}}-1) \tag{17}$$

so that

$$SNR = \frac{P_s}{N} \le (2^{\frac{C}{W}}-1) = SNR^*$$

this means that given a channel with capacity C, and so of $SNR^* = (2^{\frac{C}{W}}-1)$, it is possible to stabilize the systems only if the required SNR is lower than SNR^*, otherwise it is not possible to transmit the signal through the channel. Nonetheless, this is a necessary but not sufficient condition for controlling the plant, since the reliable transmission with rate C, by the Shannon theorem, is obtained at the cost of infinitely long decoding delays, which are not suitable for control purposes. However, if we bound the maximum decoding delay to τ_{max}, we need to admit a certain erasure probability ϵ and a certain distortion of the signal due to quantization errors.

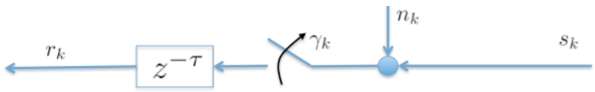


Figura 11: Channel model

Given that it is hereby proposed a model of the channel, see figure 11, where the loss of accuracy due to the quantization error is modeled as an additive gaussian noise $n_t \sim \mathcal{N}(0,\sigma_n^2)$. Furthermore the decoding delay $\tau \in \mathbb{N}$ is taking into account and the probability to not being able to correctly decode the message is represented by the binary variable $\gamma_t \in \{0,1\}$. The successful trasmission probability is $\mathbb{P}[\gamma_t = 1] = \bar{\gamma} = 1-\epsilon$. The parameters that characterize this model are:

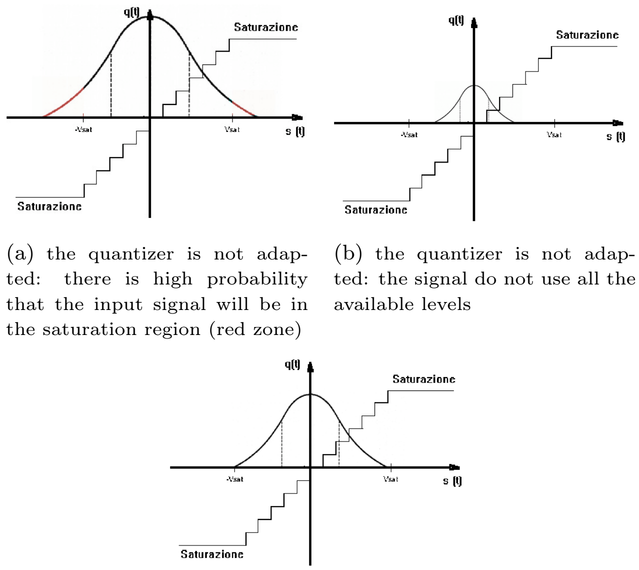
$$SNR^*,\quad \sigma_n^2 = \mathbb{E}[||n_t||^2],\quad \bar{\gamma} = \mathbb{P}[\gamma_t = 1],\quad \tau = \text{cod/dec delay}$$

They can be designed via appropriate choice of the Channel Cod/Dec, however they are all coupled since augmenting the successful probability $\bar{\gamma}$ it might require increasing the delay τ or the equivalent noise variance σ_n^2. Therefore some trade-offs are likely to appear in the context of feedback control systems, since all the terms negatively impact the performance of the closed loop system.

Assuming that the channel error $n(t)$ depends only on the presence of the quantizer, which is supposed to be uniform, and given the probability density function of the input signal $s(t)$, it is possible to derive a relation between SNR^* and σ_n^2. In particular we say that the channel is *adaptive* if at each instant time it satisfy the equality

$$SNR = \frac{P_s(t)}{\sigma_n^2(t)} = \frac{P_s}{\sigma_n^2} = SNR^*$$

where we suppose the signal to be stationary. This condition means that the dynamic range $(-v_{sat}, v_{sat})$ of the uniform quantizer is always adapted to the input signal, i.e. the dynamic range is chosen in order to reach the minimum feasible value for σ_n^2 subject to the constraint $SNR < SNR^*$, figure 12.



(a) the quantizer is not adapted: there is high probability that the input signal will be in the saturation region (red zone)

(b) the quantizer is not adapted: the signal do not use all the available levels

(c) this quantizer is adapted

Figura 12: Example of different input signal with the same quantizer

For a uniform quantizer with a fixed number of bits b, and consequently $L = 2^b$ levels, the SNR in given by:

$$SNR = 3k_f^2 L^2$$

where k_f is the *shaping factor* of the channel:

$$k_f = \frac{\sqrt{P_s}}{v_{sat}}$$

Lets suppose that $s(t)$ is a gaussian signal then it must be $v_{sat} \geqslant k\sqrt{P_s}$, where usually $k=3$, in order to guarantee that the probability of saturation is negligible. Given that:

$$SNR = 3\frac{P_s}{v_{sat}^2}L^2 \leqslant 3\frac{P_s}{k^2 P_s}L^2 = \frac{3}{k^2}L^2 = SNR^*$$

We will say that the channel is adapted if $v_{sat} = k\sqrt{P_s}$ and so $SNR = SNR^*$, which is the best signal to noise ratio feasible with the given channel. It follows that:

$$\frac{P_s}{\sigma_n^2} = SNR^* \Rightarrow \sigma_n^2 = \frac{P_s}{SNR^*}$$

If the variance P_s is constant and independent from the input noise $n(t)$ the optimal choice will be $\sigma_n^2 = \frac{P_s}{SNR^*}$. Unfortunately in our schemes this never happens since $P_s = f(\sigma_n^2)$ is a function of the input noise. For this reason deriving the optimal value of σ_n^2 is a much more complex and non linear problem since we have to solve the equation:

$$\sigma_n^2 = \frac{P_s}{SNR^*} = \frac{f(\sigma_n^2)}{SNR^*}$$

In order to compute σ_n^2 two different tecniques are here proposed:

1. Fixed point method
In the scalar case it is possible to find the fixed point of the map $\sigma_n^2 = \frac{P_s}{SNR^*} = \frac{f(\sigma_n^2)}{SNR^*}$ through graphical analysis. The value of σ_n^2 is the intersection point between the two graph $y_1 = SNR^* \sigma_n^2$ and $y_2 = f(\sigma_n^2)$. The first one is a straight line that assign to each value of σ_n^2 the power P_s of the input signal needed to guarantee $SNR = SNR^*$. The latter must be obtained from simulations: for each value of σ_n^2 the related system must be implemented and the power P_s is then obtained as sampling variance of the input signal.

2. Iterative method
Another possible way is to initialize $\sigma_n^2(1) = 0$ and then repeat the following steps until we arrive at convergence:

   (a) implement the system with $\sigma_n^2(i)$ and find the power $P_s(i)$ as sampling variance of the input signal;

   (b) set the quantization error for the next step as $\sigma_n^2(i+1) = P_s(i)/SNR^*$;

If the input signal is multidimensional, for example a vector of length 2, the same procedure can be used allocating b_1 bits to the first component and b_2 bits to the second one, with the constraint $b_1 + b_2 = b$. The problem is so reduced to an equivalent one where there are two scalar channel with b_1 and b_2 bits each. For the sake of simplicity let $b_1 = b_2 = b/2$ then the maximum signal to noise ratio for each channel is :

$$SNR_c^* = \frac{3}{k^2} L_c^2 = \frac{3}{k^2} * 2^{2b/2} = \frac{3}{k^2} * 2^b = \frac{SNR^*}{2^b}$$

from which can be derived:

$$\sigma^2_{(n,i)} = \frac{P_{(s,i)}}{SNR_c^*}$$

where $\sigma^2_{(n,i)}$ is the variance of the i-th component of the quantization noise and $P_{(s,i)}$ is the variance of the i-th component of the input signal. In vector notation:

$$Q_n = Var\left[n(t)\right] = \left[\begin{array}{c|c} \frac{P_{(s,1)}}{SNR_c^*} & 0 \\ \hline 0 & \frac{P_{(s,2)}}{SNR_c^*} \end{array}\right]$$

in this case only the iterative method can be used to estimate the value of Q_n.

6 A first channel model application

We now want to apply the channel model derived in the previous section to the scheme of figure 13.

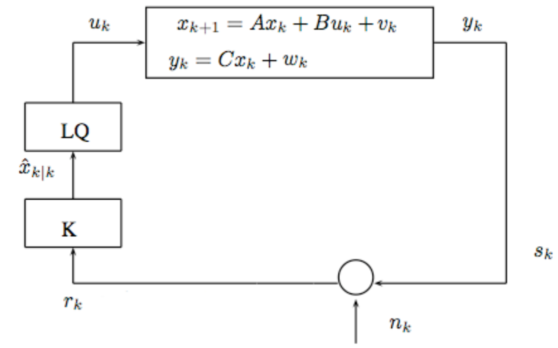


Figura 13: Scheme 2 with measurement and process noise

This is the same scheme of figure 8 in which process and measurement noise, $v(t)$ and $w(t)$, are added. The main difference relies in the channel model: with respect to Braslavsky context, in which the quantization noise was fixed, here instead we want to use an adaptive quantizer so that $SNR = SNR^*$. In other words the scheme is the same used in section 4.2 but the model of the channel is that described before, without considering the packet loss or the delay. This non idealities will be considered in a second moment. The equation of the system are:

- controller
$$u(t) = L\hat{x}(t|t)$$

- system
$$x(t+1) = Ax(t) + Bu(t) + v(t)$$
$$y(t) = Cx(t) + w(t)$$
where $Var(v(t)) = Q$ e $Var(w(t)) = R$. Let the variance of the output $y(t)$ be P_y.

- Equivalent system at the Kalman estimator site
$$x(t+1) = Ax(t) + Bu(t) + v(t)$$
$$y'(t) = Cx(t) + w(t) + n(t)$$
where in adaptive condition $Var(n(t)) = N = \alpha P_y$ with $\alpha = \frac{1}{SNR^*} < 1$.

- Kalman predictor
$$\hat{x}(t+1|t) = A\hat{x}(t|t-1) + Bu(t) + G\left[y'(t) - C\hat{x}(t|t-1)\right]$$
so:
$$e(t+1|t) = x(t+1) - \hat{x}(t+1|t) = Ax(t) + Bu(t) +$$
$$+v(t) - A\hat{x}(t|t-1) - Bu(t) - G\left[y'(t) - C\hat{x}(t|t-1)\right] =$$
$$= Ae(t|t-1) + v(t) - G\left[Ce(t|t-1) + w(t) + n(t)\right] =$$
$$= (A - GC)e(t|t-1) + v(t) - G(w(t) + n(t))$$

- Kalman estimator
The equation of the Kalman estimator are needed in order to compute the control:
$$\hat{x}(t|t) = \hat{x}(t|t-1) + K\left[y'(t) - C\hat{x}(t|t-1)\right]$$

Given that:

$$u(t) = L\hat{x}(t|t) = L\hat{x}(t|t-1) + LK\left[y'(t) - C\hat{x}(t|t-1)\right] =$$

$$= L\left[I - KC\right]\hat{x}(t|t-1) + LKy'(t)$$

Substituting the input given by the controller in the Kalman predictor equation:

$$\hat{x}(t+1|t) = A\hat{x}(t|t-1) + Bu(t) + G\left[y'(t) - C\hat{x}(t|t-1)\right] =$$

$$= A\hat{x}(t|t-1) + BL\left[I - KC\right]\hat{x}(t|t-1) + BLKy'(t) +$$
$$+ G\left[y'(t) - C\hat{x}(t|t-1)\right]$$
$$= \left[A + BL\right]\hat{x}(t|t-1) + \left[BLK + G\right]\left[y'(t) - C\hat{x}(t|t-1)\right] =$$
$$= \left[A + BL\right]\hat{x}(t|t-1) + \left[BL + A\right]K\left[Ce(t|t-1) + w(t) + n(t)\right]$$

The real output of the system is:

$$y(t) = Cx(t) + w(t) = C\left[e(t|t-1) + \hat{x}(t|t-1)\right] + w(t)$$

it follows that the equation of the feedback loop system are:

$$\begin{bmatrix} \hat{x}(t+1) \\ e(t+1) \end{bmatrix} = \begin{bmatrix} (A+BL) & (A+BL)KC \\ 0 & A(I-KC) \end{bmatrix}\begin{bmatrix} \hat{x}(t) \\ e(t) \end{bmatrix} +$$
$$+ \begin{bmatrix} 0 \\ I \end{bmatrix} v(t) + \begin{bmatrix} (A+BL)K \\ -AK \end{bmatrix}\begin{bmatrix} w(t) + n(t) \end{bmatrix}$$
$$y(t) = \begin{bmatrix} C & C \end{bmatrix}\begin{bmatrix} \hat{x}(t) \\ e(t) \end{bmatrix} + w(t)$$

where we use $G = AK$, $\hat{x}(t) = \hat{x}(t|t-1)$ and $e(t) = e(t|t-1)$. Let P be the matrix variance of the state and:

$$\bar{A} = \begin{bmatrix} (A+BL) & (A+BL)KC \\ 0 & A(I-KC) \end{bmatrix}$$

since $\begin{bmatrix} \hat{x}(t) \\ e(t) \end{bmatrix}$, $v(t)$ e $[w(t) + n(t)]$ are uncorrelated, it follows:

$$P = \bar{A}P\bar{A}' + \begin{bmatrix} 0 \\ I \end{bmatrix} Q \begin{bmatrix} 0 & I \end{bmatrix} +$$
$$+ \begin{bmatrix} (A+BL)K \\ -AK \end{bmatrix}\begin{bmatrix} R + N \end{bmatrix}\begin{bmatrix} K'(A+BL)' & -(AK)' \end{bmatrix}$$
$$N = \alpha P_y \qquad P_y = \begin{bmatrix} C & C \end{bmatrix} P \begin{bmatrix} C \\ C \end{bmatrix} + R$$

In the scalar case with $b = c = 1$ the equations become:

$$\begin{bmatrix} \hat{x}(t+1) \\ e(t+1) \end{bmatrix} = \begin{bmatrix} (a+l) & (a+l)k \\ 0 & a(1-k) \end{bmatrix}\begin{bmatrix} \hat{x}(t) \\ e(t) \end{bmatrix} +$$
$$+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t) + \begin{bmatrix} (a+l)k \\ -ak \end{bmatrix}\begin{bmatrix} w(t) + n(t) \end{bmatrix}$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} \hat{x}(t) \\ e(t) \end{bmatrix} + w(t)$$

With variance:

$$P = \bar{A}P\bar{A}' + q\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} (a+l)k \\ -ak \end{bmatrix}(r + \alpha P_y)\begin{bmatrix} (a+l)k & -ak \end{bmatrix}$$
$$P_y = \begin{bmatrix} 1 & 1 \end{bmatrix} P \begin{bmatrix} 1 \\ 1 \end{bmatrix} + r$$

Substituting the second equation in the first one we get:

$$P = \bar{A}P\bar{A}' + q\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + r\begin{bmatrix} (a+l)^2k^2 & -ak^2(a+l) \\ -ak^2(a+l) & (ak)^2 \end{bmatrix} +$$
$$+ \begin{bmatrix} (a+l)k \\ -ak \end{bmatrix}\left\{\alpha\begin{bmatrix} 1 & 1 \end{bmatrix} P \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha r\right\}\begin{bmatrix} (a+l)k & -ak \end{bmatrix} =$$
$$= \bar{A}P\bar{A}' + q\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + r(1+\alpha)\begin{bmatrix} (a+l)^2k^2 & -ak^2(a+l) \\ -ak^2(a+l) & (ak)^2 \end{bmatrix} +$$
$$+ \alpha\begin{bmatrix} (a+l)k \\ -ak \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} P \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} (a+l)k & -ak \end{bmatrix}$$
$$= \bar{A}P\bar{A}' + \alpha\bar{B}P\bar{B}' + \bar{Q} \tag{18}$$

Where:

$$\bar{B} = \begin{bmatrix} (a+l)k & (a+l)k \\ -ak & -ak \end{bmatrix}$$

$$\bar{Q} = q\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + r(1+\alpha)\begin{bmatrix} (a+l)^2k^2 & -ak^2(a+l) \\ -ak^2(a+l) & (ak)^2 \end{bmatrix}$$

this is solvable using the vectorized form:

$$vecP = \left[\bar{A} \otimes \bar{A} + \alpha\bar{B} \otimes \bar{B}\right] vecP + vec\bar{Q}$$

$$vecP = \left[I - (\bar{A} \otimes \bar{A} + \alpha\bar{B} \otimes \bar{B})\right]^{-1} vec\bar{Q} \tag{19}$$

and $J = Var(y(t)) = P_y = \sum_i \left[vecP\right]_i + r$.

Notice that this equation is solvable if and only if the difference equation:

$$vecP(k+1) = \left[\bar{A} \otimes \bar{A} + \alpha\bar{B} \otimes \bar{B}\right] vecP(k) + vec\bar{Q}$$

converges and this happens iff the matrix $\left[\bar{A} \otimes \bar{A} + \alpha\bar{B} \otimes \bar{B}\right]$ is stable. If $\alpha = 0$, i.e without quantization error, this condition is equivalent to the stability of $\bar{A}$. In this case $\bar{A}$ is a triangular matrix and so it is immediate to find out the two constraints:

$$|a + l| < 1 \Rightarrow -a - 1 < l < -a + 1$$

$$|g - a| < 1 \Rightarrow a - 1 < g < a + 1$$

however these are necessary but not sufficient conditions when $\alpha \neq 0$.

6.1 Minimization of the cost $J = E\left[y^2\right]$

Let

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{11} \end{bmatrix}$$

then the cost functional is $J = p_{11} + 2p_{12} + p_{22} + r$. Notice that p_{12} must alway been null since $\hat{x}(t|t-1)$ and $e(t|t-1)$ are uncorreleted, while p_{11} and p_{22} must always be positive since P is positive-semidefinite. Given that the cost functional becomes $J = p_{11} + p_{22} + r$, which is a monotonically increasing function of p_{11} and p_{22}. From equation 18 we derive, see Appendix:

$$p_{11} = (a+l)^2\left[(p_{11} + 2kp_{12} + k^2p_{22}) + \alpha k^2(p_{11} + 2p_{12} + p_{22}) + r(1+\alpha)k^2\right]$$

$$p_{12} = (a+l)\left[a(1-k)(p_{12} + kp_{22}) - \alpha ak^2(p_{11} + 2p_{12} + p_{22}) - r(1+\alpha)ak^2\right]$$

$$p_{22} = a^2(1-k)^2p_{22} + q + \left[(1+\alpha)r + \alpha(p_{11} + 2p_{12} + p_{22})\right](ak)^2$$

In order to find the couple (l^*, k^*) that minimize J notice that p_{22} is independent on l and monotonically increasing on p_{11}. Moreover for $l = -a$ we have $p_{11} = p_{12} = 0$. Given that, since p_{11} is always positive, both p_{11} and p_{22} are minimum for $l^* = -a$, which thus is the optimal value. The value of k^* can be obtained imposing:

$$k^* = arg\min_k p_{22} = \frac{p_{22}}{(1+\alpha)(p_{22}+r)}$$

Substituting this value in p_{22} we get:

$$p_{22} = a^2 p_{22} + q - \frac{1}{1+\alpha}\frac{a^2 p_{22}^2}{p_{22}+r}$$

This is a MARE with $\bar{\gamma} = \frac{1}{1+\alpha}$. Given that it should be easy to include in this model also the packet loss, the reader is invited to compare this result with that obtained in section 7.2.1.

Notice that since the system is scalar the critical probability for the solvability of the MARE is known:

$$\gamma_c = 1 - \frac{1}{a^2}$$

Given that the problem has solution, i.e. p_{22} converges, iff:

$$\bar{\gamma} = \frac{1}{1+\alpha} > \gamma_c = 1 - \frac{1}{a^2}$$

which implies:

$$\alpha < \frac{1}{a^2-1} \quad \Rightarrow \quad SNR^* > a^2 - 1 \tag{20}$$

This constraint is the same obtained in section 4, however this is a more general results since it is derived for a plant with measurement and process noise. In the LQG context the main results of [1] is thus obtained as a corollary. In the following section we derive the same result with a different technique which doesn't require the use of a MARE.

Analysis with the LQG controller gain $l^* = -a$

The simulations confirm that the minimum value of $J = E[y^2]$ is always obtained when $l = -a$. This value is the controller gain that can be obtained with the standard LQG approach when $\rho = 0$, see Appendix. With this constraint the previous equations becomes $p_{11} = 0$, $p_{12} = 0$ and:

$$p_{22} = a^2(1-k)^2 p_{22} + q + [(1+\alpha)r + \alpha p_{22})]\,(ak)^2$$

and so:

$$p_{22} = \left[a^2(1-k)^2 + \alpha(ak)^2\right] p_{22} + q + (1+\alpha)r(ak)^2$$

Existence of the solution

The algebraic equation is solvable iff the difference equation

$$p_{22}(t+1) = \left[a^2(1-k)^2 + \alpha(ak)^2\right] p_{22}(t) + q + (1+\alpha)r(ak)^2$$

converges and this happens iff there is at least a value of k such that $|a^2(1-k)^2 + \alpha(ak)^2| = a^2(1-k)^2 + \alpha(ak)^2 = f_a(k) < 1$. Clearly values of k such that this constraint is satisfy exist iff:

$$f_a(k^*) < 1 \quad \text{where} \quad k^* = arg\min_k f_a(k)$$

In order to find k^* we impose the first derivative equal to zero

$$\frac{df_a(k)}{dk} = 2a^2\,[k - 1 + \alpha k] = 0$$

and so $k^* = 1/(1+\alpha)$, since the second derivative is always positive this must be a minimum point. The corresponding value of $f_a(k^*)$ is:

$$f_a(k^*) = a^2 \frac{\alpha}{1+\alpha}$$

Given that, the constrained problem has a solution (in particular we can choose $k = k^*$) iff

$$a^2 \frac{\alpha}{1+\alpha} < 1 \Rightarrow \alpha < \frac{1}{a^2-1}$$

In oder words the minimum value of SNR^* needed to guarantee the stabilization of the plant is $SNR^* > a^2 - 1$. This is the same bound derived in (20).

Minimization of the cost functional

If the problem is solvable, i.e. $\alpha < \frac{1}{(a^2-1)}$, there is an interval of values such that the feedback loop system is stable. We now want to find the one that minimizes the cost functional $J = E\left[y^2\right] = p_{22} + r$. There are three possible way of proceding:

1. Exhaustive search
   Given a grid of values k, for each point the cost function is calculated solving equation (19) where $l = -a$. The desired value of k is that one corresponding to the minimum value of J;

2. First iterative method
   In order to minimize $J = p_{22} + r$ we have to minimize p_{22}. Consider the recursive equation:

   $$p_{22}(t+1) = \left[a^2(1-k(t))^2 + \alpha(ak(t))^2\right] p_{22}(t) + q + (1+\alpha)r(ak(t))^2$$

   Let $p_{22}(1) = 0$ and iterate until convergence:

   (a) $k(t) = arg\min_k p_{22}(t+1) = \frac{p_{22}(t)}{(1+\alpha)(p_{22}(t)+r)}$

   (b) $p_{22}(t+1) = \left[a^2(1-k(t))^2 + \alpha(ak(t))^2\right] p_{22}(t) + q + (1+\alpha)r(ak(t))^2$

   Then $k(\infty)$ is the desired value, see appendix for the proof.

3. Second iterative method
   Let $\sigma_n^2 = 0$ and repeat until convergence:

   (a) solve the ARE to find out the gain of the Kalman filter with input noise $\sigma_n^2(k)$;

   (b) compute the output variance $P_y(k)$. This can be done in two different ways. One approach is to implement the system and compute the sampling variance of $y(t)$, otherwise it can be calculated with a procedure similar to that previously described. If the variance of the input noise is given and it is independent from the input signal we have:

   $$P = \bar{A}P\bar{A}' + q\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} + (r + \sigma_n^2(k))\begin{bmatrix}0 & 0\\0 & (ak)^2\end{bmatrix}$$

   in vectorized form:

   $$vecP = \left[\bar{A}\otimes\bar{A}\right] vecP + vec\bar{Q}(k)$$

   (c) let $\sigma_n^2(k+1) = \alpha P_y(k)$

The desired gain is that of the Kalman filter when the noise variance is fixed at $\sigma_n^2(\infty)$. Notice that this is the iterative scheme proposed in section 5.

The simulations confirm that these three methods are equivalent, i.e. they all give the same value of g.

6.2 Analysis with other cost functionals

All the previous work refer to the cost functional $J = E\left[y^2\right]$, in this case the simulations show that the optimal value for the controller gain is that of the standard LQG approach and the value of the Kalman filter gain is that of a system where the quantization error is calculated with the iterative procedure described in section 5.

These results don't seem to be correct if we use a functional $E\left[y^2 + \rho u^2\right]$ where the cost depends not only on the output but also on the control. The cost functional in this case becomes, see Appendix:

$$J = p_{11}(1 + \rho l^2(\alpha k^2 + 1)) + 2p_{12}(1 + \rho l^2 k(\alpha k + 1)) + \\ + p_{22}(1 + \rho l^2 k^2(\alpha + 1)) + R(1 + (1 + \alpha)\rho l^2 k^2)$$

Notice that the coefficient of p_{22} in the cost functional depends on l so that in general the choice $l = -a$ is not the optimal one. In confirmation the simulations show that the optimal value for the controller gain is different from that of the standard LQG. This is an evidence that in this context the separation principle is not valid. Notice that in this case it is not guaranteed that the choice of a scheme where the controller and the estimator are independent is optimal, whatever the values of their gains are. A future direction of research could be to investigate more this case to demonstrate analytically that the separation principle is violated and to study the properties of the cost functional. In this setting even the unicity of the minimum is not guaranteed so that also the proposed iterative algorithms may fail.

To summarize we have shown that the simple channel model developed in section 5, even without packet loss or delay, can lead to very interesting and difficult situations where it is not clear which is the optimal configuration and even if this is unique.

Using the predictor instead of the estimator

Instead of using the control $u(t) = L\hat{x}(t|t)$ it is possible to implement the control $u(t) = L\hat{x}(t|t-1)$, see appendix C. It is worth to mention that with this control law the same results are obtained, i.e $l^* = -a$ with $J = E\left[y^2\right]$ and $l^* \neq -a$ otherwise, but the stability condition is more conservative since it must be:

$$\alpha < \frac{1}{(a^2 - 1)a^2}$$

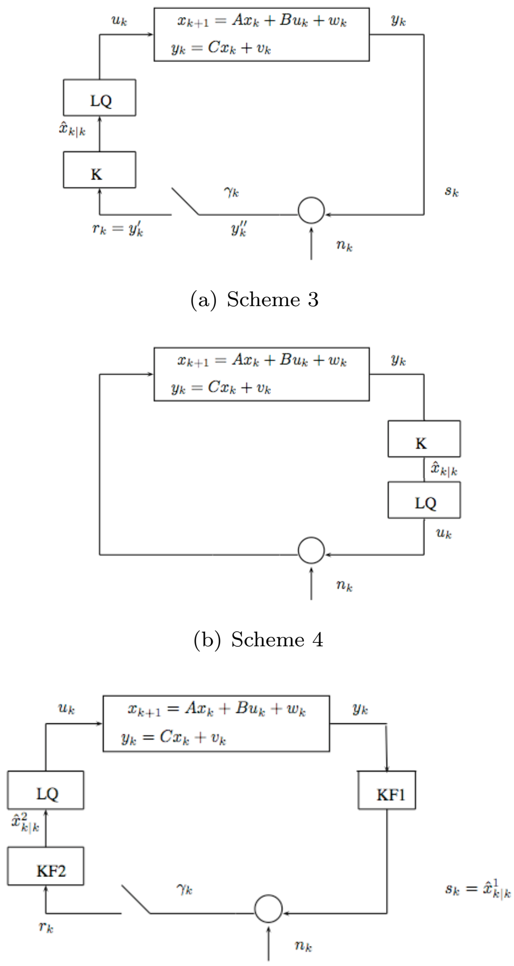
This is due to the fact that using the predictor, instead of the estimator, a delay in added in the loop.

7 Adaptive quantization and packet loss

In the following we propose several schemes to stabilize the plant when the cost functional is the output variance. In this section we will use the channel model described in section 5 without packet delays. The reader should refer to [4] for a possible approach, in the LQG context, that takes into account the delays.
First of all we want to verify the convergence of the algorithms, for the calculation of σ_n^2, proposed in section 5. In particular we will show the equivalence between the iterative method and the fixed point method for the scalar case. Given that, we will compare the cost functional for the schemes under adaptive conditions, i.e. the variance of the input noise will be assumed equal to that one calculated with the algorithms described above.

7.1 Proposed Schemes

We take into account three different schemes, figure 14:



(a) Scheme 3

(b) Scheme 4

(c) Scheme 5

Figura 14: Proposed schemes.

- Schemes 3 and 4 are the equivalents of schemes 1 and 2 of section 4 when process and measurement noise are added and when the channel model is that one described in section 5, without packet delay. Notice that in Scheme 4 there is no packet loss, this is because if the input of the communication channel is the control law, it is not clear what happens if this is lost. In particular the system could become open-loop and so unstable.;

- Scheme 5 is totally different from the previous configurations since the channel input is the estimate state and so is multidimensional. For the sake of simplicity we will assume that the same number of bits will be used for each component and so the noise variance matrix Q_n can be computed with the iterative method described in section 5, a graphical solution of this problem is not possible. Finally notice that the input signal is the estimate state and not the estimate error due to the presence of packet loss.

7.2 Theoretical analysis of the schemes with fixed quantization error

We have shown in the previous section that if the quantizer is adaptive it is not clear weather the separation principle is valid or not. In this context, since the schemes are much more complicated than that one depicted in section 6, we will assume that the principle is valid and so we will design separately the estimator and the controller. If this assumption will prove to be false then the schemes hereby described will be only sub-optimal. For the derivation of the equations let's suppose that the noise variance N is fixed, then the separation principle is valid and the optimal approach for each scheme is the separate design of the kalman filter and the controller.

7.2.1 Scheme 3

System equations

$$x_{k+1} = Ax_k + Bu_k + w_k$$
$$s_k = y_k = Cx_k + v_k$$
$$r_k = y'_k = \gamma_k(y_k + n_k) = \gamma_k Cx_k + \gamma_k(v_k + n_k)$$

Where w_k, v_k, n_k are zero mean gaussian independent noise of variance Q, R, N, while γ_k is a Bernoulli process with $Pr(\gamma_k = 1) = \bar{\gamma}$.

Kalman Filter
The system from the Kalman filter site is:

$$x_{k+1} = Ax_k + Bu_k + w_k$$
$$y'_k = \gamma_k y''_k = \gamma_k(Cx_k + v''_k)$$

where $v''_k = v_k + n_k$ is a white noise of variance $R^{eq} = R + N$. We choose a sub-optimal Kalman filter with costant gain K:

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K(y'_{k+1} - C_k\hat{x}_{k+1|k})$$

Substituting the expression for C_k and y'_k:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + \gamma_{k+1}K(y''_{k+1} - C\hat{x}_{k+1|k})$$

so

$$e_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k} =$$
$$= Ax_k + Bu_k + w_k - A(\hat{x}_{k|k-1} + \gamma_k K(y''_k - C\hat{x}_{k|k-1})) - Bu_k =$$
$$= Ae_{k|k-1} + w_k - A\gamma_k K(Cx_k + v''_k - C\hat{x}_{k|k-1})) =$$
$$= Ae_{k|k-1} + w_k - A\gamma_k K(Ce_{k|k-1} + v''_k)) =$$
$$= A(I - \gamma_k KC)e_{k|k-1} + w_k - \gamma_k AKv''_k$$

Notice that the prediction error is independent from the input, this implies that the separation principle is valid.

The variance $P_{k+1|k} = E[e_{k+1|k}e'_{k+1|k}|y''_{0:k}, \gamma_{0:k}]$ is:

$$P_{k+1|k} = A(I - \gamma_k KC)P_{k|k-1}(I - \gamma_k KC)'A' + Q + \gamma_k^2 AKR^{eq}K'A'$$

Taking the expectation over all the realizations of γ_k, i.e. $\bar{P}_{k+1|k} = E[P_{k+1|k}]$, we have:

$$\bar{P}_{k+1|k} = E_{\gamma_{0:k}}[P_{k+1|k}] = E_{\gamma_k}[E_{\gamma_{0:k-1}}[P_{k+1|k}|\gamma_k]] =$$
$$= E_{\gamma_k}[E_{\gamma_{0:k-1}}[A(I - \gamma_k KC)P_{k|k-1}(I - \gamma_k KC)'A' + Q + \gamma_k^2 AKR^{eq}K'A'|\gamma_k]] =$$
$$= E_{\gamma_k}[A(I - \gamma_k KC)\bar{P}_{k|k-1}(I - \gamma_k KC)'A' + Q + \gamma_k^2 AKR^{eq}K'A'] =$$
$$= E_{\gamma_k}[A(I - \gamma_k KC)\bar{P}_{k|k-1}(I - \gamma_k KC)'A'] + Q + \bar{\gamma}AKR^{eq}K'A' =$$
$$= \bar{\gamma}A(I - KC)\bar{P}_{k|k-1}(I - KC)'A' + (1 - \bar{\gamma})A\bar{P}_{k|k-1}A' + Q + \bar{\gamma}AKR^{eq}K'A'$$

In order to minimize $\bar{P}$ with respect to K it is convenient to define the following operators:

$$\mathcal{L}_\lambda(K, P) = \lambda A(I - KC)P(I - KC)'A' + (1 - \lambda)APA' + Q + \lambda AKRK'A'$$
$$\Phi_\lambda(P) = APA' + Q - \lambda APC'(CPC' + R)^{-1}CPA'$$

In the following we will use the definition of stability for an estimator:

Definition: Let $\tilde{x}_{k|k} = f(\tilde{y}_k, \gamma_k)$ be a generic estimator, where f is a measurable function, and $\tilde{e}_{k|k} = x_k - \tilde{x}_{k|k}$ and $\tilde{P}_{k|k}$ its error and error covariance, respectively. We say that the estimator is mean square stable iff $lim_{t\to\infty} E\left[\tilde{P}_{k|k}\right] \leqslant M$ for some $M > 0$ and for all $k \geqslant 1$.

Proposition Let (A, C) be observable, $(A, Q^{1/2})$ controllable and $R^{eq} > 0$ then:
(a) if A is unstable and $\bar{\gamma} < \gamma_c$ then there is no kalman filter with costant gain that is also stable;
(b) if instead $\bar{\gamma} > \gamma_c$ the optimal costant gain Kalman filter exist and the corrispondent gain is $\bar{K} = \bar{P}C'(C\bar{P}C' + R^{eq})^{-1}$ where $\bar{P}$ is the solution of the following MARE $\bar{P} = \Phi_{\bar{\gamma}}(\bar{P})$.

Proof In the following proof we will use theorem 2 at pg 10 of [4].
(a) First we prove by contradiction that, if A is unstable and $\bar{\gamma} < \gamma_c$, there is no stable estimator with constant gain. In fact, suppose that such a filter exists and has constant gain $\tilde{K}$. Then also a sequence $\bar{P}_{k|k}$ bounded for all k exists. From $e_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k} = Ae_{k|k} + w_k$ we have that $\bar{P}_{k+1|k} = A\bar{P}_{k|k}A' + Q$ is bounded for all k. Then the sequence $\bar{P}_{k+1|k} = \mathcal{L}_{\bar{\gamma}}(\tilde{K}, \bar{P}_{k|k})$ is a bounded sequence and so for 2(g) $S^* = \mathcal{L}_{\bar{\gamma}}(\tilde{K}, S^*)$ has a solution. For 2(h)also $P^* = \Phi_{\bar{\gamma}}(P^*)$ has a solution, but this is an absurd because for 2(i) $P^* = \Phi_{\bar{\gamma}}(P^*)$ cannot have a solution if $\bar{\gamma} < \gamma_c$.
(b) For 2(i) $P^* = \Phi_{\bar{\gamma}}(P^*)$ has a solution, this implies that $P^* = \mathcal{L}_{\bar{\gamma}}(K_{P^*}, P^*)$ has a solution too because $\Phi_{\bar{\gamma}}(P^*) = \mathcal{L}_{\bar{\gamma}}(K_{P^*}, P^*)$. For 2(g) the Kalman filter with constant gain K_{P^*} has error variance P^* for every initial condition. For any other stable kalman filter with constant gain T for 2(g) must be $S^* = \mathcal{L}_{\bar{\gamma}}(T, S^*)$ where S^* is the error

variance. Finally for 2(h) $P^* \leqslant S^*$ and so the filter with gain K_{P^*} is optimal.

Given that the desired Kalman filter is:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + \gamma_{k+1} K_{P^*}(y'_{k+1} - C\hat{x}_{k+1|k}) =$$
$$= A\hat{x}_{k|k} + Bu_k + \gamma_{k+1} K_{P^*}(y'_{k+1} - CA\hat{x}_{k|k} - CBu_k) =$$
$$= (I - \gamma_{k+1} K_{P^*} C)A\hat{x}_{k|k} + \gamma_{k+1} K_{P^*} y'_{k+1} + (I - \gamma_{k+1} K_{P^*} C)Bu_k$$

where we use $\gamma_k y''_k = \gamma_k y'_k$.

LQ controller
Since the separation principle is valid we can use the classical LQG approach:

$$u_k = L\hat{x}_{k|k}$$

where the gain L is obtained by $L = -(B'SB + U)^{-1}B'SA$, with S solution of the algebraic riccati equation (ARE):

$$S = A'SA + W - A'SB(B'SB + U)^{-1}B'SA$$

7.2.2 Scheme 4

This is a standard LQG problem with the only difference that the control u^a_k at the actuator site is the corrupted version of the output of the controller u^c_k: $u^a_k = (u^c_k + n_k)$.

System equations

$$x_{k+1} = Ax_k + Bu^a_k + w_k$$
$$y_k = Cx_k + v_k$$

Kalman Filter and LQ controller
The system from the regulator filter site is:

$$x_{k+1} = Ax_k + Bu^c_k + Bn_k + w_k$$
$$y_k = Cx_k + v_k$$

This is a standard LQG problem where the process noise is $w'_k = Bn_k + w_k$ so that $Q' = BNB' + Q$.

7.2.3 Scheme 5

System equations

$$x_{k+1} = Ax_k + Bu_k + w_k$$
$$y_k = Cx_k + v_k$$

$$s_k = \hat{x}^1_{k|k} \qquad r_k = \gamma_k(\hat{x}^1_{k|k} + n_k) \qquad u_k = L\hat{x}^2_{k|k}$$

First Kalman filter
This is the standard Kalman filter with constant gain $K_{P_1} = P_1 C'(CP_1C' + R)^{-1}$ and P_1 solution of the ARE. So that:

$$\hat{x}^1_{k+1|k+1} = (I - K_{P_1} C)A\hat{x}^1_{k|k} + K_{P_1} y_{k+1} + (I - K_{P_1} C)Bu_k$$

Notice that the estimate error variance is

$$P^1_{k|k} = P_1 - P_1 C'(CP_1C' + R)^{-1}CP_1$$

Second Kalman filter
We want to calculate the minimum variance estimate, $\hat{x}^2_{k|k}$, of the state x_k with the observation $r_k = \gamma_k(s_k + n_k)$ of the signal received after the transmission. The scheme from the second Kalman filter site is:

$$x_{k+1} = Ax_k + Bu_k + w_k$$
$$r_k = \gamma_k(\hat{x}^1_{k|k} + n_k) = \gamma_k(x_k - e^1_{k|k} + n_k)$$

We are in the same situation of Scheme 3 with the substitution $C = I$ and $v''_k = n_k - e^1_{k|k} \in \mathcal{N}(0, N + P^1_{k|k})$, where we use the independence between n_k and $e^1_{k|k}$. Using the previous formulae we get that the optimal gain is $K_{P_2} = P_2(P_2 + N + P^1_{k|k})^{-1}$ where P_2 is the solution of the MARE $P = APA' + Q - \bar{\gamma}AP(P + N + P^1_{k|k})^{-1}PA'$. Given that, the equation for the second filter is:

$$\hat{x}^2_{k+1|k+1} = (I - \gamma_{k+1} K_{P_2})A\hat{x}^2_{k|k} + \gamma_{k+1} K_{P_2} r_{k+1} + (I - \gamma_{k+1} K_{P_2})Bu_k$$

Notice that if $\gamma_{k+1} = 0$ we have $\hat{x}^2_{k+1|k+1} = A\hat{x}^2_{k|k} + Bu_k$, i.e the system is open loop; on the other hand if $\gamma_{k+1} = 1$ then:

$$\hat{x}^2_{k+1|k+1} = (I - K_{P_2})A\hat{x}^2_{k|k} + K_{P_2} r_{k+1} + (I - K_{P_2})Bu_k$$

Finally notice that the MARE just defined and that one of Scheme 3 do not have the same γ_c since this is a function of the unstable eigenvalues of A, which are the same, but also of the C rank which is different.

LQ Controller
This is the same filter of Scheme 3:

$$u_k = L\hat{x}_{k|k}$$

where the gain L is obtained by $L = -(B'SB + U)^{-1}B'SA$, with S solution of the algebraic riccati equation (ARE):

$$S = A'SA + W - A'SB(B'SB + U)^{-1}B'SA$$

7.3 Matlab analysis with adaptive quantizer

Given the schemes previously described we now want to investigate what happens if the quantizer is adaptive. In order to do that we need to find the equivalent noise variance in adaptive conditions with the algorithms described in section 5. The cost functional relative to this noise variance can be computed using the following algorithm for each scheme:

1. compute the SNR^* corresponding to the available number of *bits*:

$$L = 2^{bit}, \quad SNR^* = 3 * L^2 \quad \text{in the scalar case}$$
$$L_c = 2^{bit/2}, \quad SNR^* = 3 * L_c^2 \quad \text{in the multi-dim case}$$

2. compute the error variance in adaptive condition, this can be done with the iterative method which is suitable for all the schemes. Notice that it is best to mediate this value over a sufficiently high number of realizations;

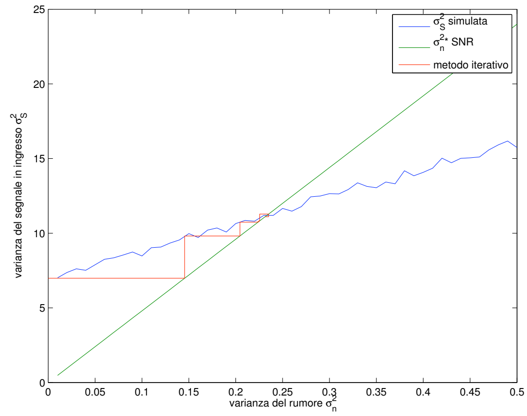
3. compute the cost functional, relative to the quantization error just derived, over a sufficiently high number of different simulations with random initial conditions.

7.3.1 Matlab analysis without packet loss

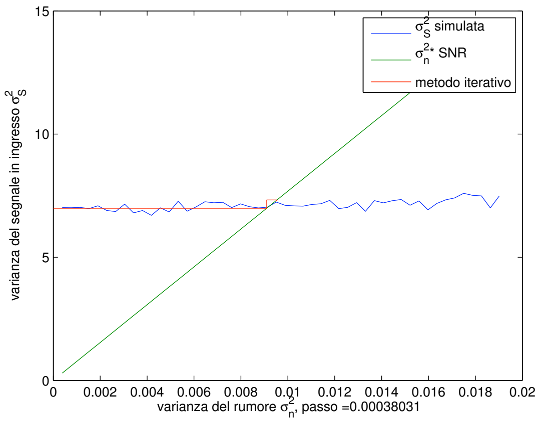
Firstly we want to show that in the scalar case, i.e. schemes 3 and 4, the iterative method and the fixed point method are equivalent. Figures 15 and 16 show the results obtained for various system with the following structure:

$$A = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix} \qquad C^T = B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

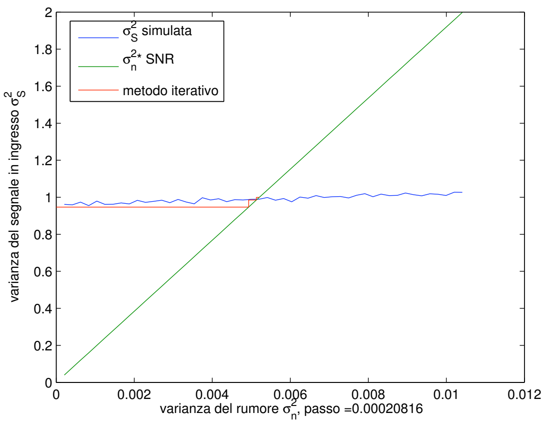
$$W = C^T C \qquad U = \rho = 1/1000 \qquad Rs = 0.1 \qquad Qs = 0.1 I_2$$



(a) $\Lambda = \{1.5, 2\}$ and 2 bits

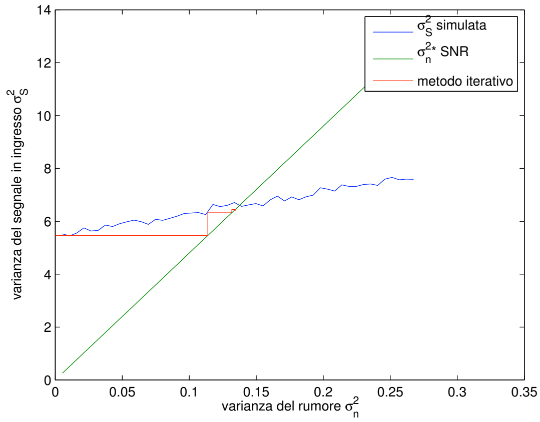


(b) $\Lambda = \{1.5, 2\}$ and 4 bits

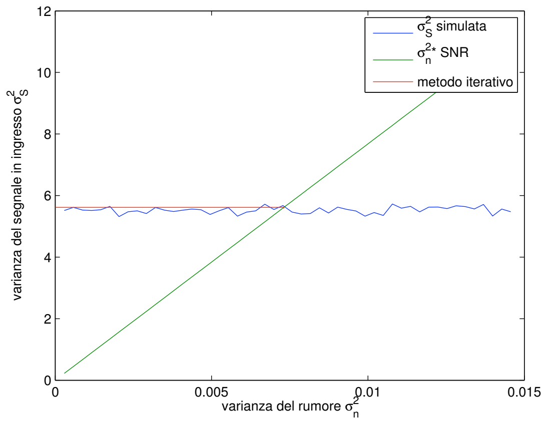


(c) $\Lambda = \{1.5, 1.5\}$ and 3 bits

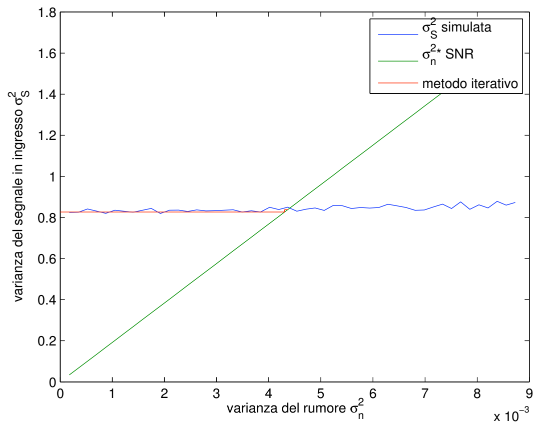
Figura 15: Check on the convergence for scheme 3



(a) $\Lambda = \{1.5, 2\}$ and 2 bits



(b) $\Lambda = \{1.5, 2\}$ and 4 bits



(c) $\Lambda = \{1.5, 1.5\}$ and 3 bits

Figura 16: Check on the convergence for scheme 4

Secondly we want to compare the cost functional of the three schemes, under the hypothesis that there is no packet loss, i.e. $\bar{\gamma} = 1$, when the number of available bits changes. The aim of this comparison is to find which is the best approach, i.e. which signal must be sent in the channel, in order to obtain the best performance. Figures 17 and 18 show some results.

7.3.2 Matlab analysis with packet loss

Finally we are interested in simulating what happens if also packet drop is considered. We already noticed that under this circumstances scheme 4 is not operable. The following figure seems to suggest that there is no an optimal scheme for each value of $\bar{\gamma}$ but the optimal choice depends on the packet loss, on the number of bits available and on the system. In particular we think that a measure of the instability of the systems like that of 7 could be relevant also in this context.

More in particular we implemented two different system one fully unstable $\Lambda_1 = \{1.5, 1.2\}$ and the other one with only one unstable eigenvalue $\Lambda_2 = \{0.5, 1.2\}$. For each system we tested our model with two different channel of 6 and 10 bits respectively. Figure 19 show that if we use a 10 bits channel with system 1 there is a critical value such that if $\bar{\gamma}$ is greater than this value the best is scheme 3, otherwise scheme 5 becomes better. This fact is not true if we use instead a 6 bits channel, figure 21: in this case scheme 3 seems to be always a better choice. This might be because the equivalent noise in adaptive condition is always greater if we use a channel with a lower number of bits. The same situation arises if we use system 2, which is less “unstable : in this case, both with 6 and 10 bits, scheme 3 has a better behaviour. Finally notice that since the two MARE used in Scheme 3 and 5 are different, the critical probability for $\bar{\gamma}$ is different, i.e for some values of $\bar{\gamma}$ the cost functional is finite with scheme 5 but not with scheme 3. See for example the range $\bar{\gamma} \in (0.55, 0.7)$ in figure 19, for these values of $\bar{\gamma}$ only Scheme 5 can be used.

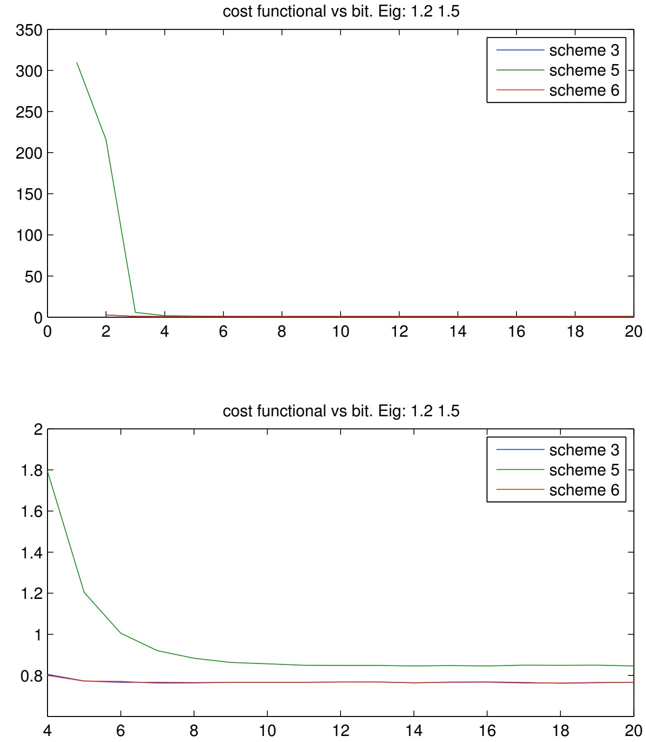


Figura 17: Unstable system with $\Lambda = \{1.5, 1.2\}$

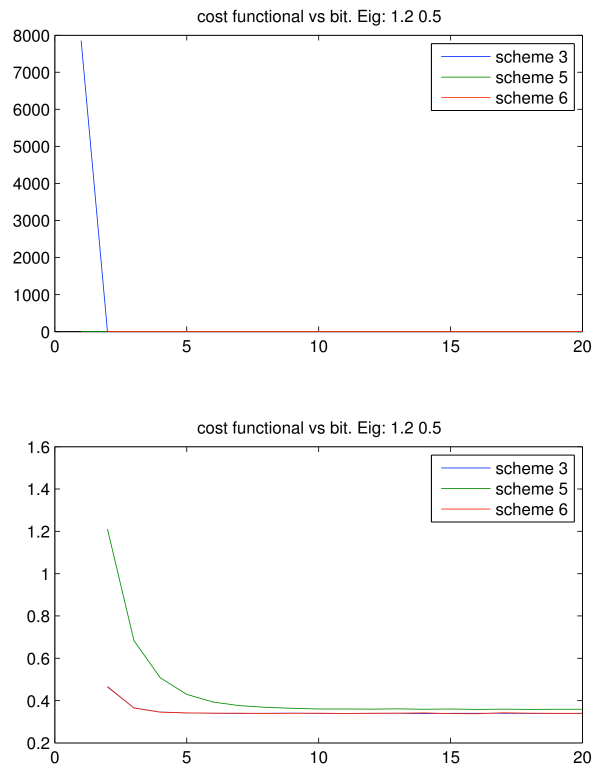


Figura 18: Unstable system with $\Lambda = \{0.5, 1.2\}$

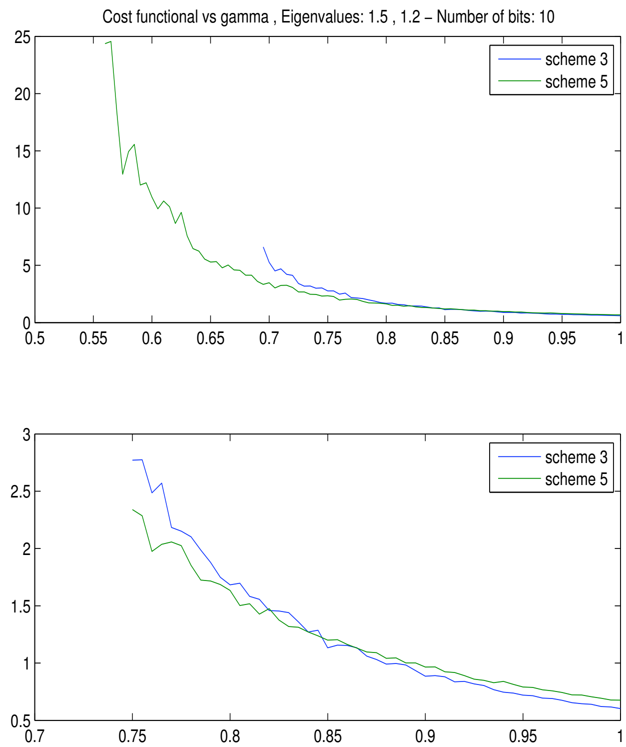


Figura 19: $\Lambda_1 = \{1.5, 1.2\}$ and 10 bits

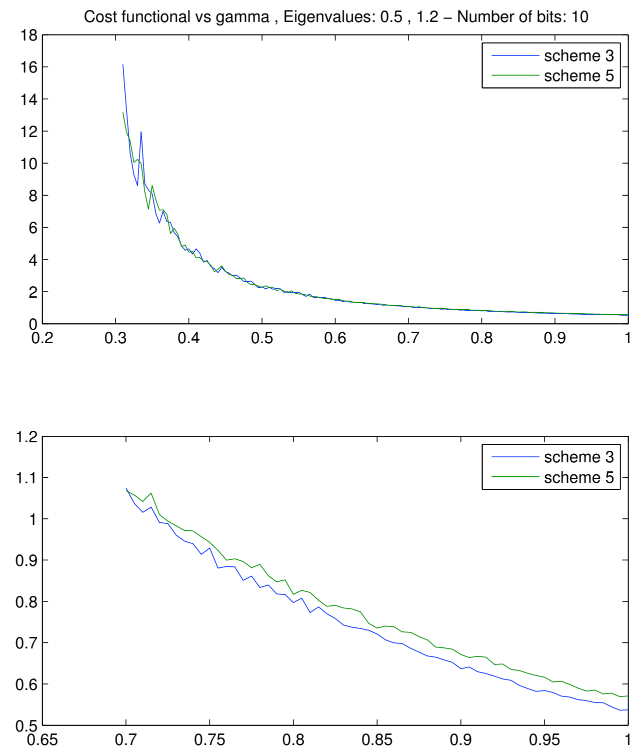


Figura 20: $\Lambda_2 = \{0.5, 1.2\}$ and 10 bits

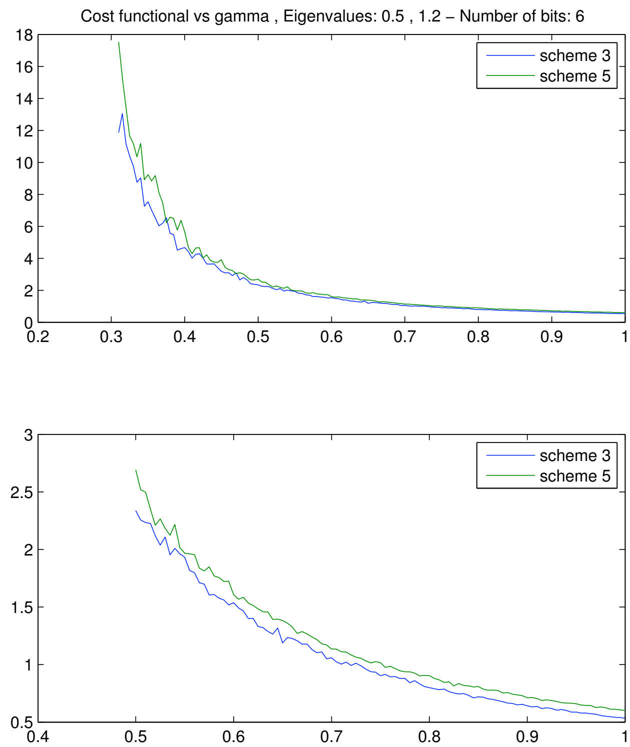


Figura 22: $\Lambda_2 = \{0.5, 1.2\}$ and 6 bits

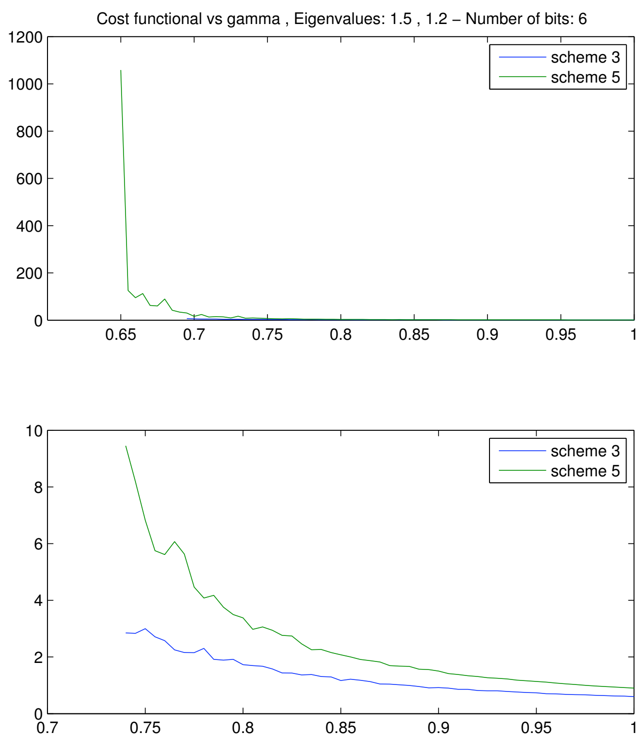


Figura 21: $\Lambda_1 = \{1.5, 1.2\}$ and 6 bits

8 Conclusions

In this paper we have discussed the problem of stabilizing a possibly unstable system across a communication channel (e.g. , Networked Controlled System), where the plant is modeled as a discrete time LTI dynamical system subject to additive measurement and process noise. In literature several possible approaches to deal with this problem have been taken into account; in this paper we have focussed on the LQG context. This technique has tourned out to be very useful since it allows to find the optimal scheme deriving, at the same time, the results of a previous work, [1], as corollary.

The main result of this paper is the channel model derived in section 5. At the best of our knowledge, this is the first model that includes power constraint, quantization noise and packet loss in the same framework. We have also proposed the use of a uniform adaptive quantizer and derived several algorithms to find the quantization noise variance in adaptive condition. It is important to remark that the proposed model is suitable also for the multidimensional case.

As an application we have studied a simpler scheme, were only power constraint and quantization noise were taken into account. In particular we have discussed both the case of cost functional $J = E\left[y^2\right]$, i.e. to minimize the input signal variance, and the general case $J = E\left[y^2 + \rho u^2\right]$. In the first case we proved that the optimal controller gain is that of the standard LQG approach and we proposed several methods to compute the optimal Kalman gain. Moreover, with our channel model, the bounds for stabilizability are the same of [1] have been derived. In the latter case, i.e. $J = E\left[y^2 + \rho u^2\right]$, we showed through simulations that the separation principle doesn't hold anymore, since the optimal controller gain is different from that of the LQG approach. Notice that, if this is true, the proposed scheme could be only sub-optimal. It would be interesting for the future to demonstrate this experimental evidence, for example using the standard dynamic programming approach.

Finally, we have proposed three different schemes to stabilize the plant when also packet loss is present. Simulations suggest that there is no an optimal scheme but the optimal choice depends on the packet loss $\bar{\gamma}$, on the number of bits available (SNR^*) and on the system. In particular we think that a measure of the instability of the systems like that of 7 could be relevant also in this context.

A key open problem not addressed in this work is how to incorporate the delay in our channel model, this problem has been addressed in [4]. For the future it would be interesting to extend those results to our channel model.
This work is simply a first step in this interesting arena, since many results are supported only by simulations, however we think that it clearly highlights the effectiveness of the LQG approach and can be useful as starting point for future developments.

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Appendix

A Proof of the equivalence between Scheme 1 and 2 in the scalar case

A.1 Scheme 1: Transfer Function computation Using the Index Cost

In relation to the Scheme 1 we want to compute the transfer function from n_k to u_k using the index cost: $J = \sum x'_k W x_k + \rho u_k^2 = \sum x'_k c' c x_k + \rho u_k^2$ and we take the limit $\rho \to \infty$. In such a way u_k^2 is minimized (this problem is known in literature as "cheap" optimal control). The state system at the regulator side is:

$$x_{k+1} = Ax_k + Bu_k + Bn_k$$

$$y_k = Cx_k$$

so the output and the process noises variances are $R = 0$ e $Q = \sigma^2 BB'$, respectively.

Kalman Filter Gain

The solution of the following Algebraic Riccati Equation has to be found:

$$P = APA' + Q - APC'(CPC' + R)^{-1}CPA'$$

For SISO systems becomes:

$$p = a^2 p + \sigma^2 b^2 - \frac{a^2 p^2 c^2}{c^2 p} \Rightarrow p = \sigma^2 b^2$$

and the Kalman gain is:

$$K = PC'(CPC' + R)^{-1} = \frac{pc}{c^2 p} = \frac{1}{c}$$

LQ controller gain

The solution of the following Algebraic Riccati Equation has to be found:

$$S = A'SA + W - A'SB'(B'SB + \rho)^{-1}B'SA$$

For SISO systems becomes:

$$s = a^2 s + c^2 - \frac{a^2 s^2 b^2}{sb^2 + \rho}$$

From the simulations it tuns out that s increases as ρ, so we can't say that the last term approaches to zero as $\rho \to \infty$. So we get:

$$(1 - a^2)s - c^2 = -\frac{a^2 s^2 b^2}{sb^2 + \rho}$$

$$s^2 b^2 - sb^2 c^2 + \rho[(1 - a^2)s - c^2] = 0$$

$$\rho = \frac{sb^2 c^2 - s^2 b^2}{(1 - a^2)s - c^2}$$

Let L be the LQ filter:

$$L = -(B'SB+\rho)^{-1}B'SA = -\frac{bsa}{(b^2 s + \rho)} = -\frac{bsa}{b^2 s + \frac{sb^2c^2 - s^2b^2}{(1-a^2)s - c^2}} =$$

$$= -\frac{bsa[(1 - a^2)s - c^2]}{b^2(1 - a^2)s^2 - b^2 sc^2 + sb^2 c^2 - s^2 b^2} =$$

$$= -\frac{(1 - a^2)bas^2 - c^2 bsa}{b^2(1 - a^2)s^2 - b^2 s^2 - b^2 sc^2 + sb^2 c^2}$$

And we get:

$$lim_{s\to\infty} L = -\frac{(1 - a^2)ba}{b^2(1 - a^2) - b^2} = \frac{(1 - a^2)ba}{a^2 b^2} = \frac{(1 - a^2)}{ab}$$

Regulator Transfer Function

For the regulator the following equations holds:

$$\hat{x}_{k|k} = F\hat{x}_{k-1|k-1} + Ky_k$$

$$u_k = LF\hat{x}_{k-1|k-1} + LKy_k$$

let $F = (I - KC)(A + BL)$, the Transfer Function is:

$$u_k = [LF(zI - F)^{-1}K + LK]y_k$$

Substituting the former values we get:

$$F = (1 - \frac{1}{c}c)(a + b\frac{(1 - a^2)}{ab}) = 0$$

and then:

$$u_k = \frac{(1 - a^2)}{ab}\frac{1}{c}y_k$$

Noise-Signal Trasfer Function

Finally we get the Transfer Function from the noise n_k to the signal u_k:

$$u_k = \frac{(1 - a^2)}{ab}\frac{1}{c}y_k = \frac{(1 - a^2)}{ab}\frac{1}{c}c(z - a)^{-1}b(n_k + u_k)$$

$$u_k = \frac{\frac{1-a^2}{a}\frac{1}{z-a}}{1 - \frac{1-a^2}{a(z-a)}}n_k = \frac{1 - a^2}{az - 1}n_k$$

A.2 Scheme 2: Transfer Function computation Using the Index Cost

In relation to the Scheme 2 we want to compute the transfer function from n_k to y_k using the index cost: $J = \sum x'_k W x_k + \rho u_k^2 = \sum x'_k c' c x_k + \rho u_k^2$ and we take the limit $\rho \to 0$. In such a way y_k^2 is minimized (this problem is known in literature as "expansive" optimal control).

The state system at the regulator side is:

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + n_k$$

and the variances of the process and ouput noises are $Q = 0$ e $R = \sigma^2$, respectively.

Kalman Filter Gain

The solution of the following Algebraic Riccati Equation has to be found:

$$P = APA' + Q - APC'(CPC' + R)^{-1}CPA'$$

For SISO systems becomes:

$$p = a^2p - \frac{a^2p^2c^2}{c^2p + \sigma^2}$$

If $|a| < 1$ the only positive solution is $p = 0$, on the contrary we get: $1 - a^2 = -\frac{a^2pc^2}{c^2p+\sigma^2} \Rightarrow p = \frac{1}{c^2}(a^2-1)\sigma^2$. And the gain becomes

$$K = \frac{pc}{c^2p + \sigma^2} = \frac{\frac{1}{c}(a^2-1)\sigma^2}{(a^2-1)\sigma^2 + \sigma^2} = \frac{1}{c}\frac{a^2-1}{a^2}$$

LQ controller gain

As we did for the Scheme 1:

$$s = a^2s + c^2 - \frac{a^2s^2b^2}{sb^2 + \rho}$$

Taking the limit $\rho \to 0$ we get $s = c^2$. The gain is:

$$L = -\frac{bsa}{sb^2 + \rho} \to -\frac{bc^2a}{c^2b^2} = -\frac{a}{b}$$

Regulator Transfer Function

For the regulator the following equations holds:

$$\hat{x}_{k|k} = F\hat{x}_{k-1|k-1} + Ky'_k$$

$$u_k = LF\hat{x}_{k-1|k-1} + LKy'_k$$

with $F = (I - KC)(A + BL)$ and then the Transfer Function is:

$$u_k = [LF(zI - F)^{-1}K + LK]y'_k$$

Substituting the former values we get::

$$F = (1 - \frac{1}{c}\frac{(a^2-1)}{a^2}c)(a - b\frac{a}{b}) = 0$$

and then:

$$u_k = -\frac{a}{b}\frac{1}{c}\frac{(a^2-1)}{a^2}y'_k = -\frac{(a^2-1)}{abc}y'_k$$

Noise-Signal Trasfer Function

Finally we get the Transfer Function from the noise n_k to the signal y_k:

$$y_k = c(z-a)^{-1}bu_k = -c(z-a)^{-1}b\frac{(a^2-1)}{abc}[y_k + n_k]$$

$$y_k = -\frac{\frac{(a^2-1)}{a(z-a)}}{1 + \frac{(a^2-1)}{a(z-a)}}n_k = -\frac{a^2-1}{az-1}n_k = \frac{1-a^2}{az-1}n_k$$

B Section 6

B.1 Equivalent system for the variance matrix P

Let consider the equation:

$$P = \bar{A}P\bar{A}^\top + \alpha\bar{B}P\bar{B}^\top + \bar{Q}$$

$$\bar{A}P\bar{A}^\top = \begin{bmatrix} (a+l) & (a+l)k \\ 0 & a(1-k) \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} (a+l) & 0 \\ (a+l)k & a(1-k) \end{bmatrix}$$

$$= \begin{bmatrix} (a+l)^2(p_{11} + 2kp_{12} + k^2p_{22}) & a(1-k)(a+l)(p_{12} + kp_{22}) \\ a(1-k)(a+l)(p_{12} + kp_{22}) & a^2(1-k)^2p_{22} \end{bmatrix}$$

$$\bar{B}P\bar{B}^\top = \begin{bmatrix} (a+l)k & (a+l)k \\ -ak & -ak \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} (a+l)k & -ak \\ (a+l)k & -ak \end{bmatrix}$$

$$= \begin{bmatrix} (a+l)^2k^2(p_{11} + 2p_{12} + p_{22}) & -ak^2(a+l)(p_{11} + 2p_{12} + p_{22}) \\ -ak^2(a+l)(p_{11} + 2p_{12} + p_{22}) & a^2k^2(p_{11} + 2p_{12} + p_{22}) \end{bmatrix}$$

$$\bar{Q} = q\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + r(1+\alpha)\begin{bmatrix} (a+l)^2k^2 & -ak^2(a+l) \\ -ak^2(a+l) & (ak)^2 \end{bmatrix}$$

And we get:

$$p_{11} = (a+l)^2(p_{11} + 2kp_{12} + k^2p_{22}) + \\ + \alpha(a+l)^2k^2(p_{11} + 2kp_{12} + k^2p_{22}) + rk^2(a+l)^2(1+\alpha)$$

$$p_{22} = a^2(1-k)^2p_{22} + \alpha a^2k^2(p_{11} + 2p_{12} + p_{22}) + \\ + q + ra^2k^2(1+\alpha)$$

$$p_{12} = a(1-k)(a+l)(p_{12} + kp_{22}) - \alpha ak^2(a+l)(p_{11} + 2p_{12} + p_{22}) - \\ - r(1+\alpha)ak^2(a+l)$$

B.2 Cost functional: general case

$$J_t = \mathbb{E}[y(t)^2 + \rho u(t)^2] = Var(y(t)) + \rho Var(u(t))$$

We define the state vector $\begin{bmatrix} \hat{x}(t|t-1) \\ e(t|t-1) \end{bmatrix}$ with $e(t|t-1) := x(t) - \hat{x}(t|t-1)$.

The LQ controller output $u(t)$ follows the equation:

$$\begin{aligned} u(t) &= l\hat{x}(t|t-1) = l(I - kc)\hat{x}(t|t-1) + lky'(t) \\ &= l(1-kc)\hat{x}(t|t-1) + lk(cx(t) + w(t) + n(t)) \\ &= l\hat{x}(t|t-1) + lkce(t|t-1) + lk(w(t) + n(t)) \\ &= l\begin{bmatrix} 1 & kc \end{bmatrix}\begin{bmatrix} \hat{x}(t|t-1) \\ e(t|t-1) \end{bmatrix} + lk(w(t) + n(t)) \end{aligned}$$

The plant output $y(t)$ follows the equation:

$$\begin{aligned} y(t) &= cx(t) + w(t) \\ &= c(\hat{x}(t|t-1) + e(t|t-1)) + w(t) \\ &= c \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}(t|t-1) \\ e(t|t-1) \end{bmatrix} + w(t) \end{aligned}$$

Let R be the Covariance of $w(t)$ and N the Covariance of the channel quantization noise $n(t)$ (under adaptive conditions we have $Var(n(t)) := N = \alpha Var(y)$) and let P be the covariance of the vector state. Then,

$$Var(u(t)) = l^2 \begin{bmatrix} 1 & kc \end{bmatrix} P \begin{bmatrix} 1 \\ kc \end{bmatrix} + l^2k^2(R + \alpha Var(y))$$

$$Var(y(t)) = c^2 \begin{bmatrix} 1 & 1 \end{bmatrix} P \begin{bmatrix} 1 \\ 1 \end{bmatrix} + R$$

And the index cost becomes:

$$\begin{aligned} J_t &= Var(y(t)) + \rho l^2 \begin{bmatrix} 1 & kc \end{bmatrix} P \begin{bmatrix} 1 \\ kc \end{bmatrix} + \rho l^2k^2(R + \alpha Var(y)) \\ &= (1 + \alpha\rho l^2k^2)Var(y(t)) + \rho l^2 \begin{bmatrix} 1 & kc \end{bmatrix} P \begin{bmatrix} 1 \\ kc \end{bmatrix} + \rho l^2k^2R \end{aligned}$$

We impose $b = c = 1$ and we get:

$$J_t = (1 + \alpha\rho l^2k^2) \begin{bmatrix} 1 & 1 \end{bmatrix} P \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \rho l^2 \begin{bmatrix} 1 & k \end{bmatrix} P \begin{bmatrix} 1 \\ k \end{bmatrix} + \\ + R(1 + (1 + \alpha)\rho l^2k^2) \quad (21)$$

Let $P := \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, then we get:

$$\begin{aligned} J_t &= (1 + \alpha\rho l^2k^2)(p_{11} + 2p_{12} + p_{22}) + \\ &\quad + \rho l^2 \begin{bmatrix} 1 & k \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 1 \\ k \end{bmatrix} + R(1 + (1 + \alpha)\rho l^2k^2) \\ &= (1 + \alpha\rho l^2k^2)(p_{11} + 2p_{12} + p_{22}) + \rho l^2(p_{11} + 2kp_{12} + k^2p_{22}) + \\ &\quad + R(1 + (1 + \alpha)\rho l^2k^2) \\ &= p_{11}(1 + \rho l^2(\alpha k^2 + 1)) + 2p_{12}(1 + \rho l^2k(\alpha k + 1)) + \\ &\quad + p_{22}(1 + \rho l^2k^2(\alpha + 1)) + R(1 + (1 + \alpha)\rho l^2k^2) \end{aligned}$$

B.3 Proof of the 1^{st} iterative method

We want to show that $k(\infty)$, derived as in section 6, is the value that minimize p_{22}.

Proof The argument is similar to that used in the Kalman filter context.

- Let $p_{22} = p$. The goal is to find:

$$k^* = arg \min_k p, \quad \text{s.t.} \quad p = \mathcal{L}(k, p)$$

where

$$\mathcal{L}(k, p) = a^2(1 - k)^2p + q + [(1 + \alpha)r + \alpha p] (ak)^2$$

Consider the following operator:

$$\phi(p) = \min_k \mathcal{L}(k, p)$$

We have found in section 6 that:

$$k^*(p) = arg \min_k \mathcal{L}(k, p) = \frac{p}{(1 + \alpha)(p + r)}$$

Given that:

$$\phi(p) = a^2p + q - \frac{1}{1 + \alpha}\frac{a^2p^2}{p + r}$$

- we want to show that $\phi(p)$ has a fixed point $p^* = \phi(p^*)$. In oder to do that consider $k_c = \frac{1}{(1+\alpha)}$, we have already shown that for this value of k the problem $p_2 = \mathcal{L}(k_c, p_2)$ has a solution. Moreover $\phi(p) \leqslant \mathcal{L}(k_c, p)$ by definition. It easy to prove that:

$$\phi(p) \geqslant \mathcal{L}_{min}(p) = \frac{\alpha}{1 + \alpha}a^2p + q$$

And clearly, since $\mathcal{L}_{min}(p)$ is linear with angular coefficient smaller than 1, exists p_1 such that $p_1 = \mathcal{L}_{min}(p_1)$. Given that $\phi(p)$ is bounded between two functions with fixed point, this implies that exist p^* such that $p^* = \phi(p^*)$. It easy to prove that $\phi(p)$ in monotonically increasing from which follows the unicity of p^* and the bound $p_1 < p^* < p_2$. See figure 23.

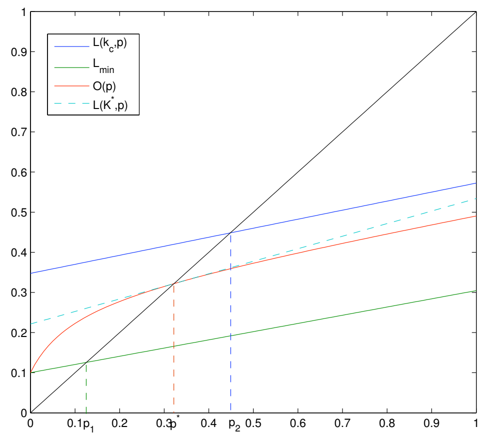

Figura 23: Relations between $\mathcal{L}_{min}(p)$, $\mathcal{L}(k, p)$ and $\phi(p)$

- $k^*(p^*)$ is the desired value since for any other value of k:

$$\tilde{p} = \mathcal{L}(k, \tilde{p}) > \phi(\tilde{p})$$

which implies $\tilde{p} > p^*$. Notice that $\mathcal{L}(k^*(p^*), p)$ is tangent to $\phi(p)$ in p^*.

- the algorithm proposed converges to p^* and so that to k^*, see figure 24.

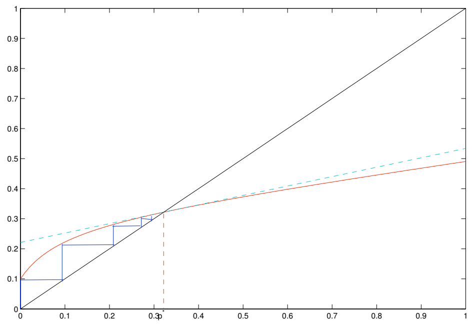

Figura 24: Iterative method

C A first application using the predictor instead of the estimator

We now want to apply the channel model derived in the previous section to the scheme of figure 13. This is the same scheme of figure 8 in which process and measurement noise, $v(t)$ and $w(t)$, are added. The main difference relies in the channel model: with respect to Braslavsky context, in which the quantization noise was fixed, here instead we want to use an adaptive quantizer so that $SNR = SNR^*$. In other words the scheme is the same used in section 4.2 but the model of the channel is that described before, without considering the packet loss or the delay. This non idealities will be considered in a second moment. The equation of the system are:

- controller

$$u(t) = L\hat{x}(t|t-1)$$

- system

$$x(t+1) = Ax(t) + Bu(t) + v(t)$$

$$y(t) = Cx(t) + w(t)$$

where $Var(v(t)) = Q$ e $Var(w(t)) = R$. Let the variance of the output $y(t)$ be P_y.

- Equivalent system at the Kalman filter site

$$x(t+1) = Ax(t) + Bu(t) + v(t)$$

$$y'(t) = Cx(t) + w(t) + n(t)$$

where in adaptive condition $Var(n(t)) = N = \alpha P_y$ con $\alpha = \frac{1}{SNR^*} < 1$.

- Kalman predictor

$$\hat{x}(t+1|t) = A\hat{x}(t|t-1) + Bu(t) + G\left[y'(t) - C\hat{x}(t|t-1)\right]$$

so:

$$e(t+1|t) = x(t+1) - \hat{x}(t+1|t) = Ax(t) + Bu(t) +$$

$$+v(t) - A\hat{x}(t|t-1) - Bu(t) - G\left[y'(t) - C\hat{x}(t|t-1)\right] =$$

$$= Ae(t|t-1) + v(t) - G\left[Ce(t|t-1) + w(t) + n(t)\right] =$$

$$= (A - GC)e(t|t-1) + v(t) - G(w(t) + n(t))$$

Substituing the input given by the controller in the system equation:

$$x(t+1) = Ax(t) + BL\hat{x}(t|t-1) + v(t) =$$

$$= Ax(t) + BL\left[x(t) - e(t|t-1)\right] + v(t) =$$

$$= (A + BL)x(t) - BLe(t|t-1) + v(t)$$

it follows that the equation of the feedback loop system are:

$$\begin{bmatrix} x(t+1) \\ e(t+1) \end{bmatrix} = \begin{bmatrix} A+BL & -BL \\ 0 & A-GC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} +$$

$$+ \begin{bmatrix} I \\ I \end{bmatrix} v(t) + \begin{bmatrix} 0 \\ -G \end{bmatrix} \begin{bmatrix} w(t) + n(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + w(t)$$

let $e(t) = e(t|t-1)$. Let P be the matrix variance of the state and:

$$\bar{A} = \begin{bmatrix} A+BL & -BL \\ 0 & A-GC \end{bmatrix}$$

since $\begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$, $v(t)$ e $[w(t) + n(t)]$ are uncorrelated, it follows:

$$P = \bar{A}P\bar{A}' + \begin{bmatrix} I \\ I \end{bmatrix} Q \begin{bmatrix} I & I \end{bmatrix} + \begin{bmatrix} 0 \\ -G \end{bmatrix} \begin{bmatrix} R+N \end{bmatrix} \begin{bmatrix} 0 & -G' \end{bmatrix}$$

$$N = \alpha P_y \quad P_y = \begin{bmatrix} C & 0 \end{bmatrix} P \begin{bmatrix} C \\ 0 \end{bmatrix} + R$$

In the scalar case with $b = c = 1$ the equations become:

$$\begin{bmatrix} x(t+1) \\ e(t+1) \end{bmatrix} = \begin{bmatrix} a+l & -l \\ 0 & a-g \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v(t) +$$

$$+ \begin{bmatrix} 0 \\ -g \end{bmatrix} \begin{bmatrix} w(t) + n(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + w(t)$$

With variance:

$$P = \bar{A}P\bar{A}' + q \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} + (r + \alpha P_y) \begin{bmatrix} 0 \\ -g \end{bmatrix} \begin{bmatrix} 0 & -g \end{bmatrix} =$$

$$= \bar{A}P\bar{A}' + q \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + r \begin{bmatrix} 0 & 0 \\ 0 & g^2 \end{bmatrix} + \begin{bmatrix} 0 \\ -g \end{bmatrix} \alpha P_y \begin{bmatrix} 0 & -g \end{bmatrix}$$

$$P_y = \begin{bmatrix} 1 & 0 \end{bmatrix} P \begin{bmatrix} 1 \\ 0 \end{bmatrix} + r$$

Substituting the second equation in the first one we get:

$$P = \bar{A}P\bar{A}' + q \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + r \begin{bmatrix} 0 & 0 \\ 0 & g^2 \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 \\ -g \end{bmatrix} \left\{\alpha \begin{bmatrix} 1 & 0 \end{bmatrix} P \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha r\right\} \begin{bmatrix} 0 & -g \end{bmatrix} =$$

$$= \bar{A}P\bar{A}' + q \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + r \begin{bmatrix} 0 & 0 \\ 0 & g^2 \end{bmatrix} +$$

$$+\alpha \begin{bmatrix} 0 \\ -g \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} P \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -g \end{bmatrix} + \alpha r \begin{bmatrix} 0 & 0 \\ 0 & g^2 \end{bmatrix} =$$

$$= \bar{A}P\bar{A}' + q \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (1+\alpha) r \begin{bmatrix} 0 & 0 \\ 0 & g^2 \end{bmatrix} +$$

$$+\alpha \begin{bmatrix} 0 & 0 \\ -g & 0 \end{bmatrix} P \begin{bmatrix} 0 & -g \\ 0 & 0 \end{bmatrix} =$$

$$= \bar{A}P\bar{A}' + \alpha\bar{B}P\bar{B}' + \bar{Q}$$

Where:

$$\bar{B} = \begin{bmatrix} 0 & 0 \\ -g & 0 \end{bmatrix}$$

$$\bar{Q} = q\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (1+\alpha)r\begin{bmatrix} 0 & 0 \\ 0 & g^2 \end{bmatrix}$$

this is solvable using the vectorized form:

$$vecP = \left[\bar{A}\otimes\bar{A} + \alpha\bar{B}\otimes\bar{B}\right] vecP + vec\bar{Q}$$

$$vecP = \left[I - (\bar{A}\otimes\bar{A} + \alpha\bar{B}\otimes\bar{B})\right]^{-1} vec\bar{Q} \qquad (22)$$

and $J = Var(y(t)) = P_y = [vecP]_1 + r$.

Notice that this equation is solvable if and only if the difference equation:

$$vecP(k+1) = \left[\bar{A}\otimes\bar{A} + \alpha\bar{B}\otimes\bar{B}\right] vecP(k) + vec\bar{Q}$$

converges and this happens iff the matrix $\left[\bar{A}\otimes\bar{A} + \alpha\bar{B}\otimes\bar{B}\right]$ is stable. If $\alpha = 0$, i.e without quantization error, this condition is equivalent to the stability of $\bar{A}$. In this case $\bar{A}$ is a triangular matrix and so it is immediate to find out the two constraints:

$$|a+l| < 1 \Rightarrow -a-1 < l < -a+1$$

$$|g-a| < 1 \Rightarrow a-1 < g < a+1$$

however these are necessary but not sufficient conditions when $\alpha \neq 0$.

The equation $P = \bar{A}P\bar{A}' + \alpha\bar{B}P\bar{B}' + \bar{Q}$ can be written as:

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} a+l & -l \\ 0 & a-g \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} a+l & 0 \\ -l & a-g \end{bmatrix} +$$

$$+\alpha\begin{bmatrix} 0 & 0 \\ -g & 0 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & -g \\ 0 & 0 \end{bmatrix} + \bar{Q}$$

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} (a+l)p_{11} - lp_{12} & (a+l)p_{12} - lp_{22} \\ (a-g)p_{12} & (a-g)p_{22} \end{bmatrix}\begin{bmatrix} a+l & 0 \\ -l & a-g \end{bmatrix}$$

$$+\alpha\begin{bmatrix} 0 & 0 \\ -gp_{11} & -gp_{12} \end{bmatrix}\begin{bmatrix} 0 & -g \\ 0 & 0 \end{bmatrix} + \bar{Q} =$$

$$= \begin{bmatrix} (a+l)^2p_{11} - 2l(a+l)p_{12} + l^2p_{22} & (a+l)(a-g)p_{12} - (a-g)lp_{22} \\ (a+l)(a-g)p_{12} - (a-g)lp_{22} & (a-g)^2p_{22} \end{bmatrix}$$

$$+\alpha\begin{bmatrix} 0 & 0 \\ 0 & g^2p_{11} \end{bmatrix} + \begin{bmatrix} q & q \\ q & q+(1+\alpha)rg^2 \end{bmatrix}$$

this is equivalent to the following system:

$$\begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} (a+l)^2 & -2l(a+l) & l^2 \\ 0 & (a+l)(a-g) & -(a-g)l \\ \alpha g^2 & 0 & (a-g)^2 \end{bmatrix}\begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} +$$

$$+\begin{bmatrix} q \\ q \\ q+(1+\alpha)rg^2 \end{bmatrix}$$

For each value of α, this algebraic equation has a solution iff at least a couple (l,g) exists for which the matrix is stable. It would be interesting, for a future work, to find some conditions on α in order to guarantee the stability of this matrix.

Analysis with the LQG controller gain $l = -a$

From the simulations it seems that the minimum value of $J = E[y^2] = p_{11} + r$ is always obtained when $l = -a$. This value is the controller gain that can be obtained with the standard LQG approach, see Appendix. With this constraint the previous algebraic equation becomes:

$$\begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 & a^2 \\ 0 & 0 & a(a-g) \\ \alpha g^2 & 0 & (a-g)^2 \end{bmatrix}\begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} + \begin{bmatrix} q \\ q \\ q+(1+\alpha)rg^2 \end{bmatrix}$$

And so:

$$p_{11} = a^2p_{22} + q$$

$$p_{12} = a(a-g)p_{22} + q$$

$$p_{22} = \alpha g^2p_{11} + (a-g)^2p_{22} + q + (1+\alpha)rg^2$$

Substituting the first equation in the third one we get :

$$p_{22} = \alpha g^2a^2p_{22} + \alpha g^2q + (a-g)^2p_{22} + q + (1+\alpha)rg^2$$

and so:

$$p_{22} = \left[\alpha g^2a^2 + (a-g)^2\right]p_{22} + \alpha g^2q + q + (1+\alpha)rg^2$$

Existence of the solution

The algebraic equation is solvable iff the difference equation:

$$p_{22}(k+1) = \left[\alpha g^2a^2 + (a-g)^2\right]p_{22}(k) + \alpha g^2q + q + (1+\alpha)rg^2$$

converges and this happens iff there is at least a value of g such that $|\alpha g^2a^2 + (a-g)^2| = \alpha g^2a^2 + (a-g)^2 < 1$. Notice that the constraint can be written as:

$$\alpha g^2a^2 < 1 - (a-g)^2$$

Since the first term is positive for each value of g also the term on the right should be positive. From $1-(a-g)^2 > 0$ we get $a-1 < g < a+1$. This is the same necessary condition that we get in the general case. Even with $l = -a$ this is a necessary condition for the stability but it is sufficient only if $\alpha = 0$.

In order to find a necessary and sufficient condition we can write the constraint as:

$$\alpha < \frac{1-(a-g)^2}{g^2a^2} = f_a(g)$$

and study the behaviour of $f_a(g)$. We just find out that it must be $g \in (a-1, a+1)$, i.e. this is the only interval where $f_a(g)$ is positive. Imposing the first derivative equal to zero

$$\frac{df_a(g)}{dg} = \frac{2(a-g)g^2a^2 - 2ga^2\left[1-(a-g)^2\right]}{g^4a^4} = 0$$

we get two stationary points: $g_1 = 0$ and $g_2 = \frac{a^2-1}{a}$. If the system is unstable, i.e. $|a| > 1$, only the second point fits the interval $(a-1, a+1)$ and so it must be a maximum point. The corresponding value of $f_a(g)$ is:

$$f_a(g_2) = \frac{1}{(a^2-1)a^2}$$

Given that, the constrained problem has a solution (in particular we can choose $g = g_2$) iff $\alpha \leqslant f_a(g_2) = \frac{1}{(a^2-1)a^2}$, in fact:

- if $\alpha < f_a(g_2)$ there is an interval of values centered in g_2 that allows the feasibility of the constraint;
- if $\alpha > f_a(g_2)$ the problem has no solution since even with the value of g that maximize $f_a(g)$ we can't satisfy the constraint.

To summarize the problem has a solution iff :

$$\alpha < \frac{1}{(a^2-1)a^2} \Rightarrow SNR^* > (a^2-1)a^2$$

This is a constraint on the minimum value of SNR^* needed to guarantee the stabilization of the plant. Notice that this constraint is more conservative than that obtained in section 4, that is $a^2 - 1$.

Minimization of the cost functional

If the problem is solvable, i.e. $\alpha < \frac{1}{(a^2-1)a^2}$, there is an interval of values such that the feedback loop system is stable. We now want to find the one that minimizes the cost functional $J = E\left[y^2\right] = p_{11} + r$. There are three possible way of proceding:

1. Exhaustive search
   Given a grid of values g, for each point the cost function is calculated solving equation 22 where $l = -a$. The desired value of g is that one corresponding to the minimum value of J;
2. First iterative method
   In order to minimize $J = p_{11}+r$, since $p_{11} = a^2p_{22}+q$, we have to minimize p_{22}. Consider the recursive equation:

   $$p_{22}(k+1) = \left[\alpha g(k)^2a^2 + (a-g(k))^2\right] p_{22}(k) + \\ +\alpha g(k)^2 q + q + (1+\alpha) r g(k)^2$$

   where $g(k) = arg\min_g p_{22}(k+1)$. Let $p_{22}(1) = 0$ and iterate until convergence:

   (a) $g(k) = arg\min_g p_{22}(k+1)$
   (b) $p_{22}(k+1) = \left[\alpha g(k)^2a^2 + (a-g(k))^2\right] p_{22}(k) + \alpha g(k)^2 q + q + (1+\alpha) r g(k)^2$

   Then $g(\infty)$ is the desired value.
3. Second iterative method
   Let $\sigma_n^2 = 0$ and repeat until convergence:

   (a) solve the ARE to find out the gain of the Kalman filter with input noise $\sigma_n^2(k)$:

   $$\bar{P} = ARE(a, 1, r+\sigma_n^2(k), q), \quad g = a\bar{P}(\bar{P}+r+\sigma_n^2(k))^{-1}$$

   (b) compute the output variance $P_y(k)$. This can be done in two different ways. One approach is to implement the system and compute the sampling variance of $y(t)$, otherwise it can be calculated with a procedure similar to that previously described. If the variance of the input noise is given and it is independent from the input signal we have:

   $$P = \bar{A}P\bar{A}' + q\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} + \\ +(r+\sigma_n^2(k))\begin{bmatrix} 0 \\ -g \end{bmatrix}\begin{bmatrix} 0 & -g \end{bmatrix} = \bar{A}P\bar{A}' + \bar{Q}(k)$$

   in vectorized form:

   $$vecP = \left[\bar{A} \otimes \bar{A}\right] vecP + vec\bar{Q}(k)$$

   (c) let $\sigma_n^2(k+1) = \alpha P_y(k)$

   The desired gain is that of the Kalman filter when the noise variance is fixed $\sigma_n^2(\infty)$. Notice that this is the iterative scheme proposed in section 5.

The simulations confirm that these three methods are equivalent, i.e. they all give the same value of g.

Analysis with other cost functionals

All the previous work refer to the cost functional $J = E\left[y^2\right]$, in this case the simulations show that the optimal value for the controller gain is that of the standard LQG approach and the value of the Kalman filter gain is that of a system where the quantization error is calculated with the iterative procedure described in section 5.

These results don't seem to be correct if we use a functional $E\left[y^2 + \rho u^2\right]$ where the cost depends not only on the output but also on the control. In this scenario the simulations show that the optimal value for the controller gain is different from that of the standard LQG. This is an evidence that in this context the separation principle is not valid. Notice that in this case it is not guaranteed that the choice of a scheme where the controller and the predictor are independent is optimal, whatever the values of their gains are. A future direction of research could be to investigate more this case to demonstrate analytically that the separation principle is violated and to study the properties of the cost functional . In this setting even the unicity of the minimum is not guaranteed so that also the proposed iterative algorithms may fail.

To summarize we have shown that the simple channel model developed in section 5, even without packet loss or delay, can lead to very interesting and difficult situations where it is not clear which is the optimal configuration and even if this is unique.