Stabilization of Linear Systems over a non-ideal Channel

Francesca Parise e Laura Dal Col

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Control over a communication network

- We want to stabilize a possibly unstable system across a communication channel (NCS).

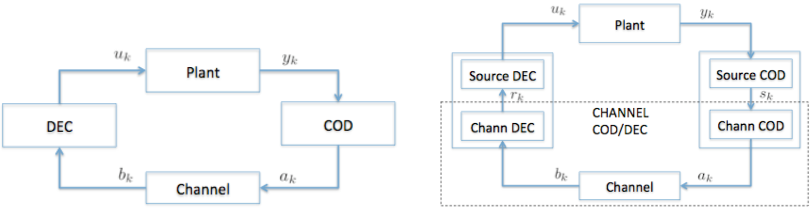


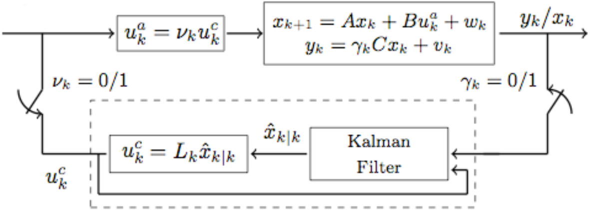
Figure: Schemes of control system over a communication channel

- Design of the optimal control and estimation scheme to achieve stability and optimal performance.
- control theory → "perfect" signals
  communication theory → stationary and stable sources
- Decoupling of COD/DEC in Source and Channel Blocks.

Review of state of art

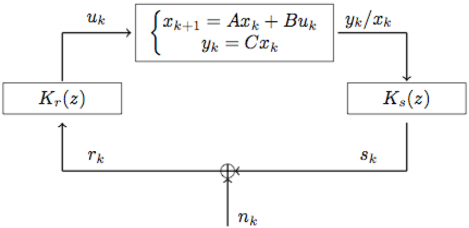
- NCS with lossy channels [1]
- NCS with SNR-limited channels [2] and [3]
- NCS with rate-limited channels [4]
- NCS with limited informations [5]

Article [1]: Foundations of control and estimation over lossy networks



- **channel model**: packet drop modeled by i.i.d Bernoulli r.v. γ_k and ν_k. No packet delay or quantization noise.
- **objective**: closed-loop stability.
- **technique**: LQG approach.
- **results**: Critical threshold for the arrival probabilities $\bar{\gamma}$ and $\bar{\nu}$ that depends on the unstable eigenvalues of the system.

Article [2]: Feedback stabilization over signal-to-noise ratio constrained channels



- **channel model**: the quantization noise is an additive zero-mean white Gaussian noise with variance σ_n^2. No packet drop or delay.
- **objective**: closed-loop stability s.t.

$$||s||_{Pow} < \mathcal{P}_d \quad (1)$$

- **technique**: spectral analysis of TF.

Let $T(z) := \frac{s_k}{n_k}$, then $||s||_{Pow} = ||T(z)||^2_{\mathcal{H}_2}\sigma_n^2$ and the constraint (1) become:

$$||T(z)||^2_{\mathcal{H}_2} < \frac{\mathcal{P}_d}{\sigma_n^2} \quad (2)$$

The minimum SNR that ensures stabilty is the smallest value of $||T(z)||^2_{\mathcal{H}_2}$ that satisfies (2).

results

- state feedback:

$$\frac{\mathcal{P}_d}{\sigma_n^2} > \left(\prod_{i=1}^{m} \lambda_i^u\right) - 1 := SNR_{min}$$

  where $\{\lambda_i^u, i = 1, \ldots, m\}$ are the unstable eigenvalues of the system.

- output feedback:

$$\frac{\mathcal{P}_d}{\sigma_n^2} > \left(\prod_{i=1}^{m} \lambda_i^u\right) - 1 + \eta + \delta := SNR_{min}$$

  where the coefficients η and δ depends on the non-minimum phase zeros and the relative degree of $T(z)$.

Article [3]: Minimum Variance Control Over a Gaussian Communication Channel

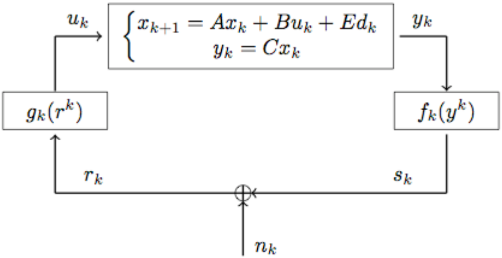


Figure: Feedback control over a SNR-constrained communication link

- **channel model**:the quantization noise is an additive zero-mean white Gaussian noise with variance σ_n^2. No packet drop or delay.

- **objective**: optimal coding and decoding sequences $f_k(y^k)$ and $g_k(r^k)$ design that minimize the index the fixed time $k = N+1$:

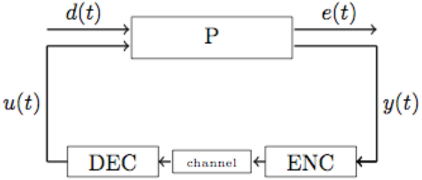
$$J_{N+1}^{opt} := \inf_{f_k, g_k, k=0,\dots,N} \mathbb{E}\left[||y_{N+1}||^2\right] \tag{3}$$

subject to:

$$\mathbb{E}\left[||s_k||^2\right] \leq P$$

- **technique**: LQG “cheap” control.
- **results**: no separation principle. The optimal control at the last-time step is linear function of the estimated-state. A lower bound on (3) is derived. Only suboptimal strategies for the infinite-horizon.

Article [4]: A Framework for Control System Design Subject to Average Data-Rate Constraints



- **channel model**: Rate-distorted channel. No noise, error or delay .
- **objective**:minimal average data-rate that allows closed-loop mean square stability.
- **results**: In the article [6] for a noisy model the average data-rate $\mathcal{R}$ that ensure stability is:

$$\mathcal{R} > \sum_{i=1}^{n_p} log_2 |p_i| \tag{4}$$

Focussing on a particular class of source- coding schemes a lower bound for $\mathcal{R}$ is derived, which is at most 1.254 bits per sample away from the absolute minimum rate (4).

Article [5]: Stabilization of linear systems with limited information

- **objective**:stabilization of LTI systems using only a finite number of measurements and control levels
- **results**: Let $f : \mathcal{X} \to \mathcal{U}$ be a suitable quantizer for a given LTI system, the measure of the coarseness of the quantizer is expressed by its density:

$$\eta_f := \lim_{\varepsilon \to 0} \sup \frac{\# f[\varepsilon]}{-ln\varepsilon}$$

where $\# f[\varepsilon]$ is number of levels that f takes in $[\varepsilon, 1/\varepsilon] \quad 0 < \varepsilon \leq 1$
A quantizer f is said to be coarsest for the system if it has the smallest quantization density. In this paper it's shown that the coarsest quantizer that quadratically stabilizes the plant follows a logarithmic law.

Our contribution

- Revision of the work [2] using LQG approach
  - SNR-constrained state feedback stabilization
  - SNR-constrained output feedback stabilization
- Channel Model
- Adaptive quantizer
  - two methods to compute the variance of the quantization error
- A first application for the adaptive quantizer
- Other implemented schemes

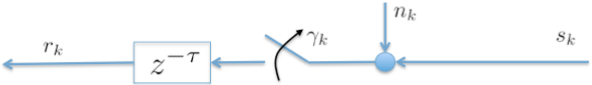
Problem formulation

- **plant model** SISO LTI-system with state equations:

$$\begin{cases} x_{t+1} = Ax_t + Bu_t + w_t \\ y_t = Cx_t + v_t \end{cases}$$

where $v_t \sim \mathcal{N}(0, R)$, $w_t \sim \mathcal{N}(0, Q)$, $x_0 \sim \mathcal{N}(0, P_0)$, $w_t \perp v_t \perp x_0$. (A, B) and (A, Q) controllable, (A, C) and (A, W) observable, $R > 0$.

- **channel model**: AWGN, packet drop, delay and quantization noise.



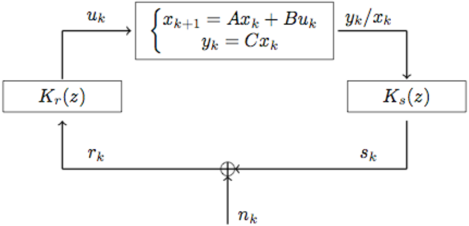
- **tecnique**: LQG optimal control with performance index [1]:

$$J = \lim_{t \to +\infty} \mathbb{E}[x_t^T W x_t + u_t^T U u_t]$$

[1] the plant control input is available at the Kalman estimator site

Revision on the literature work [2]

Starting from the work [2] we applied the LQG approach in order to characterize the closed-loop system stability under the constrain (1). This technique provides the same results as the transfer function analysis.



$$T(z) = \frac{K_r(z)K_s(z)G(z)}{1 + K_r(z)K_s(z)G(z)} = \frac{K(z)G(z)}{1 + K(z)G(z)}$$

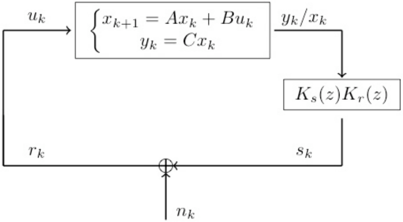


Figure: Schema 1

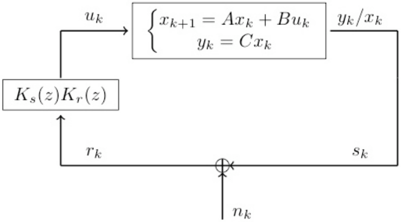
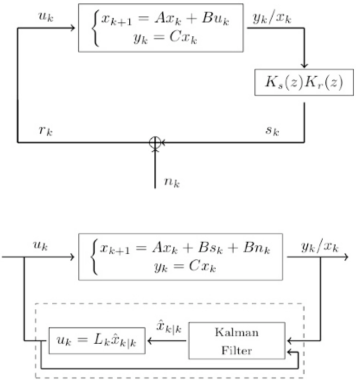


Figure: Schema 2

Scheme 1: SNR-constrained output feedback stabilization



- **index cost:**
  $J = \lim_{k\to+\infty} \mathbb{E}_{n_k} \left[x_k^\top W x_k + s_k^\top U s_k\right]$
  we take $W = CC^\top$, $U = \rho \in [0, +\infty)$

$$\Downarrow$$

$$J = \lim_{k\to+\infty} \rho\mathbb{E}_{n_k} \left[\frac{1}{\rho}||y_k||^2 + ||s_k||^2\right]$$

$$\Downarrow$$

$$\arg\min J = \arg\min \lim_{k\to+\infty} \mathbb{E}_{n_k} \left[\frac{1}{\rho}||y_k||^2 + ||s_k||^2\right]$$

$$\Downarrow \rho \to +\infty$$

$$\min \lim_{k\to+\infty} \mathbb{E}\left[||s_k||^2\right] := \mathcal{P}_{min}$$

- **Kalman Filter:**

$$\mathbf{\hat{x}_{k|k}} = (\mathbf{I} - \mathbf{KC})(\mathbf{A\hat{x}_{k-1|k-1}} + \mathbf{Bu_{k-1}}) + \mathbf{Ky_k}$$

Kalman Filter static Gain:

$$K = PC^\top (CPC^\top)^{-1}$$

$$P = A\left[P - PC^\top (CPC^\top)^{-1} CP\right] A^\top + \sigma_n^2 BB^\top$$

- **LQ controller:**

$$\mathbf{u_k} = \mathbf{L}^* \mathbf{\hat{x}_{k|k}}$$

LQ controller static Gain:

$$L^* = -(B^\top SB + \rho)^{-1} B^\top SA$$

$$S = A^\top SA - A^\top SB(B^\top SB + \rho)^{-1} B^\top SA + W$$

In the case of **state feedback stabilization** Kalman Filter is no longer needed (i.e. it can be replaced with an identity block). The LQ controller equations can be computed as in the previous point.

Examples and Comparisons

We report the results found for a 2-d system characterized by the matrices:

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix} \qquad C^T = B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

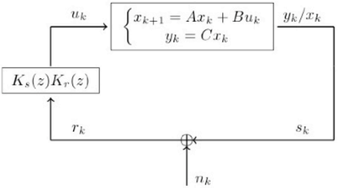
$W = C^T C$, $U = \rho$ and random initial condition, $x_0 \in \mathcal{N}(0, 4I_2)$

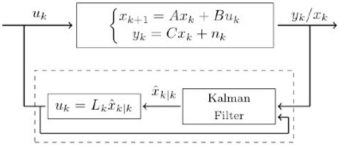
- Theoretical bound: computed from the article [2];
- The norm $||T(z)||^2_{\mathcal{H}_2}$ computed with $\rho = 10^5$ multiplied by σ_n^2;
- Matlab Simulation : expected cost $\mathbb{E}_{n_k}\left[\frac{1}{\rho}||y_k||^2 + ||s_k||^2\right]$ over $N = 50$ realizations, with $\rho = 10^5$.

$\mathcal{P}_{min}$	Theoretical	Transfer Function	Simulation
state	0.071225	0.071225	0.071218
output	0.089289	0.089298	0.089552

Table: $\lambda_1, \lambda_2 = \{1.5, 1.9\}, z = 1.2, \sigma_n = 0.1$

Scheme 2: SNR-constrained output feedback stabilization





- **index cost:**

  $J = \lim_{k\to+\infty} \mathbb{E}_{n_k}\left[x_k^\top W x_k + u_k^\top U u_k\right]$

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$$\Downarrow$$

$$J = \lim_{k\to+\infty} \mathbb{E}_{n_k}\left[||y_k||^2 + \rho||u_k||^2\right]$$

$$\Downarrow$$

$$\arg\min J = \arg\min \lim_{k\to+\infty} \mathbb{E}_{n_k}\left[||y_k||^2 + \rho||u_k||^2\right]$$

$$\Downarrow \rho \to 0$$

$$\min \lim_{k\to+\infty} \mathbb{E}_{n_k}\left[||s_k||^2\right] := \mathcal{P}_{min}$$

For this scheme $s_k = y_k$!!

- **Kalman Filter:**

$$\mathbf{\hat{x}_{k|k}} = (\mathbf{I} - \mathbf{KC})(\mathbf{A\hat{x}_{k-1|k-1}} + \mathbf{Bu_{k-1}}) + \mathbf{Ky_k}$$

Kalman Filter static Gain:

$$K = PC^\top(CPC^\top + \sigma_n^2)^{-1}$$

$$P = A\left[P - PC^\top(CPC^\top + \sigma_n^2)^{-1}CP\right]A^{\top 1}$$

- **LQ controller:**

$$\mathbf{u_k} = \mathbf{L}^*\mathbf{\hat{x}_{k|k}}$$

LQ controller static Gain:

$$L^* = -(B^\top SB + \rho)^{-1}B^\top SA$$

$$S = A^\top SA - A^\top SB(B^\top SB + \rho)^{-1}B^\top SA + W$$

Only the output feedback stabilization can be considered since the vector state can't be send across a scalar channel!

[1] The solution may not be unique, since (A, Q) is not stabilizable

Examples and Comparisons

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$\mathcal{P}_{min}$	Theoretical	Transfer Function	Simulation
output	0.089289	0.089324	0.089318

Table: $\lambda_1, \lambda_2 = \{1.5, 1.9\}, z = 1.2, \sigma_n = 0.1$

Channel Model

Shannon's theorem

Let $\bar{P}$ be the average signal power at the input of the channel and suppose the noise is white thermal noise of power N in the band W. By sufficiently complicated encoding systems it is possible to transmit binary digits at a rate

$$C = Wlog_2\left(1+\frac{\bar{P}}{N}\right)[bit/s]$$

with as small a frequency of errors as desired. It is not possible by any encoding method to send at higher rate and have an arbitrarily low frequency of errors.

Power Constraints:

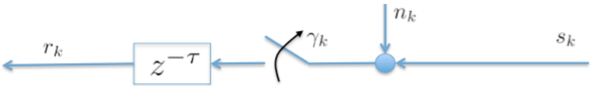
$$\lim_{T\to+\infty}\frac{1}{T}\sum_{t=0}^{T}\mathbb{E}[||s_t||^2] = P_s \leq \bar{P} = N(2^{\frac{C}{W}}-1)$$

SNR Constraints:

$$SNR = \frac{P_s}{N} \leq (2^{\frac{C}{W}}-1) = SNR^*$$

Problem: infinite long decoding delays

- erasure probability ϵ;
- quantization errors σ_n^2;



Parameters:

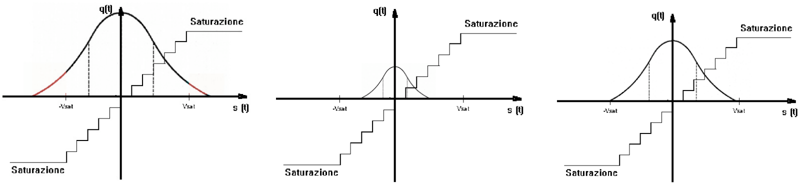
- power constraint: SNR^*
- quantization noise: $n_k \sim \mathcal{N}(0, \sigma_n^2)$;
- packet loss: $\bar{\gamma} = \mathbb{P}[\gamma_t = 1]$;
- delay τ;

Hypotesis:

- uniform quantizer;
- $s(t)$ gaussian;

Adaptive Quantizer

$$SNR = \frac{P_s(t)}{\sigma_n^2(t)} = \frac{P_s}{\sigma_n^2} = SNR^*$$



(a) not adapted (b) not adapted (c) adapted

Equivalent noise in adaptive condition

- Uniform quantizer with a fixed number of bits b:

$$SNR = 3k_f^2 L^2 \qquad k_f = \frac{\sqrt{P_s}}{v_{sat}}$$

- $s(t)$ gaussian $\Rightarrow v_{sat} \geqslant k\sqrt{P_s}$, where usually $k = 3$

$$SNR = 3\frac{P_s}{v_{sat}^2}L^2 \leqslant 3\frac{P_s}{k^2 P_s}L^2 = \frac{3}{k^2}L^2 = SNR^*$$

- We will say that the channel is adapted if $v_{sat} = k\sqrt{P_s}$ and so $SNR = SNR^*$:

$$\frac{P_s}{\sigma_n^2} = SNR^* \Rightarrow \sigma_n^2 = \frac{P_s}{SNR^*}$$

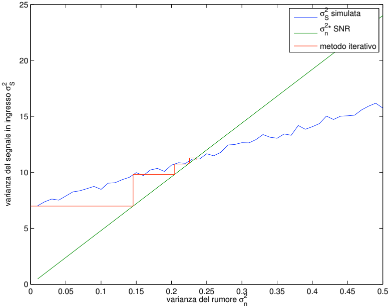
$$\sigma_n^2 = \frac{P_s}{SNR^*} = \frac{f(\sigma_n^2)}{SNR^*}$$

Algorithms: Fixed Point method

$$\sigma_n^2 = \frac{P_s}{SNR*} = \frac{f(\sigma_n^2)}{SNR*}$$

Through graphical analysis:

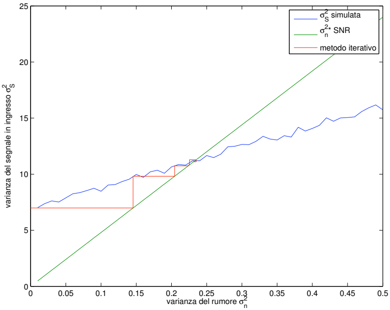
$$y_1 = SNR^*\sigma_n^2 \qquad y_2 = f(\sigma_n^2)$$



Algorithms: Iterative method

Initialize $\sigma_n^2(1) = 0$ and repeat:

1. implement the system with $\sigma_n^2(i)$ and find the power $P_s(i)$;
2. $\sigma_n^2(i+1) = P_s(i)/SNR^*$;



Multidimensional Case

If the input signal is multidimensional:

- allocate b_i bits for component
- if $b_1 = b_2 = b/2$ then:

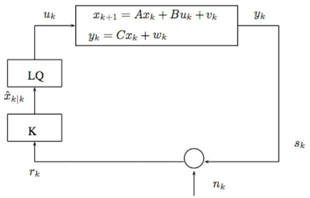
$$SNR_c^* = \frac{3}{k^2} L_c^2 = \frac{3}{k^2} * 2^{2b/2} = \frac{3}{k^2} * 2^b = \frac{SNR^*}{2^b}$$

from which can be derived:

$$\sigma_{(n,i)}^2 = \frac{P_{(s,i)}}{SNR_c^*} \qquad Q_n = Var\left[n(t)\right] = \left[\begin{array}{c|c} \frac{P_{(s,1)}}{SNR_c^*} & 0 \\ \hline 0 & \frac{P_{(s,2)}}{SNR_c^*} \end{array}\right]$$

- iterative method only.

A first model application



System equations

$x(t+1) = Ax(t) + Bu(t) + v(t)$ $Var(v(t)) = Q$

$y(t) = Cx(t) + w(t)$ $Var(w(t)) = R$

Equivalent system

$x(t+1) = Ax(t) + Bu(t) + v(t)$

$y'(t) = Cx(t) + w(t) + n(t)$

adaptive condition: $Var(n(t)) = N = \alpha P_y$ with $\alpha = \frac{1}{SNR^*} < 1$.

Controller

$u(t) = L\hat{x}(t|t)$

Kalman predictor

$\hat{x}(t+1|t) = A\hat{x}(t|t-1) + Bu(t) + G\left[y'(t) - C\hat{x}(t|t-1)\right]$

$e(t+1|t) = (A - GC)e(t|t-1) + v(t) - G(w(t) + n(t))$

Kalman estimator

$\hat{x}(t|t) = \hat{x}(t|t-1) + K\left[y'(t) - C\hat{x}(t|t-1)\right]$

System equation:

$$\begin{bmatrix} \hat{x}(t+1) \\ e(t+1) \end{bmatrix} = \begin{bmatrix} (A+BL) & (A+BL)KC \\ 0 & A(I-KC) \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} v(t) + \\ + \begin{bmatrix} (A+BL)K \\ -AK \end{bmatrix} \begin{bmatrix} w(t) + n(t) \end{bmatrix}$$

System output: $$y(t) = \begin{bmatrix} C & C \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ e(t) \end{bmatrix} + w(t)$$

State variance:

$$P = \bar{A}P\bar{A}' + \begin{bmatrix} 0 \\ I \end{bmatrix} Q \begin{bmatrix} 0 & I \end{bmatrix} + \begin{bmatrix} (A+BL)K \\ -AK \end{bmatrix} \begin{bmatrix} R+N \end{bmatrix} \begin{bmatrix} K'(A+BL)' & -(AK)' \end{bmatrix}$$

Output variance: $$P_y = \begin{bmatrix} C & C \end{bmatrix} P \begin{bmatrix} C \\ C \end{bmatrix} + R \qquad N = \alpha P_y$$

Scalar case: $b = c = 1$

$$P = \bar{A}P\bar{A}' + q\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} (a+l)k \\ -ak \end{bmatrix}(r + \alpha P_y)\begin{bmatrix} (a+l)k & -ak \end{bmatrix}$$

$$P_y = \begin{bmatrix} 1 & 1 \end{bmatrix} P \begin{bmatrix} 1 \\ 1 \end{bmatrix} + r$$

$$P = \bar{A}P\bar{A}' + \alpha\bar{B}P\bar{B}' + \bar{Q}$$

Where:

$$\bar{A} = \begin{bmatrix} (a+l) & (a+l)k \\ 0 & a(1-k) \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} (a+l)k & (a+l)k \\ -ak & -ak \end{bmatrix}$$

$$\bar{Q} = q\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + r(1+\alpha)\begin{bmatrix} (a+l)^2k^2 & -ak^2(a+l) \\ -ak^2(a+l) & (ak)^2 \end{bmatrix}$$

This is solvable using the vectorized form:

$$vecP = \left[I - (\bar{A}\otimes\bar{A} + \alpha\bar{B}\otimes\bar{B})\right]^{-1} vec\bar{Q}$$

Minimization of the cost $J = E\left[y^2\right]$

- $J = p_{11} + 2p_{12} + p_{22} + r = p_{11} + p_{22} + r$

- $p_{11} = (a+l)^2\left[(p_{11} + 2kp_{12} + k^2p_{22}) + \alpha k^2(p_{11} + 2p_{12} + p_{22}) + r(1+\alpha)k^2\right]$

  $p_{12} = (a+l)\left[a(1-k)(p_{12} + kp_{22}) - \alpha ak^2(p_{11} + 2p_{12} + p_{22}) - r(1+\alpha)ak^2\right]$

  $p_{22} = a^2(1-k)^2p_{22} + q + \left[(1+\alpha)r + \alpha(p_{11} + 2p_{12} + p_{22})\right](ak)^2$

- Optimal values:

$$l^* = -a \qquad\qquad k^* = arg\min_k p_{22} = \frac{p_{22}}{(1+\alpha)(p_{22}+r)}$$

  Substituting this value in p_{22} we get:

MARE

$$p_{22} = a^2 p_{22} + q - \frac{1}{1+\alpha} \frac{a^2 p_{22}^2}{p_{22} + r}$$

$$\bar{\gamma} = \frac{1}{1+\alpha} > \gamma_c = 1 - \frac{1}{a^2}$$

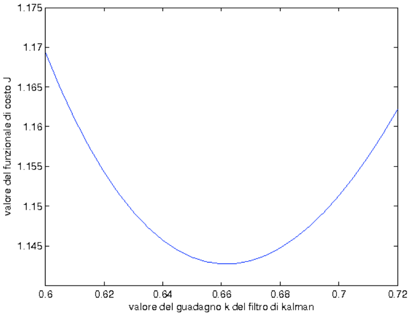
$$\alpha < \frac{1}{a^2 - 1} \quad \Rightarrow \quad SNR^* > a^2 - 1$$

Computation of k^*

1) Exhaustive search

$$vecP = \left[I - (\bar{A} \otimes \bar{A} + \alpha \bar{B} \otimes \bar{B})\right]^{-1} vec\bar{Q}$$

$$J = Var(y(t)) = P_y = \sum_i [vecP]_i + r$$



2) First iterative method

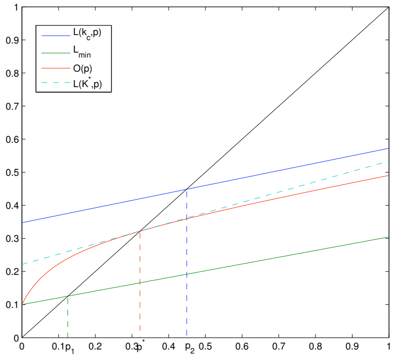
$$p_{22}(t+1) = \left[a^2(1-k(t))^2 + \alpha(ak(t))^2\right] p_{22}(t) + q + (1+\alpha)r(ak(t))^2$$

Let $p_{22}(1) = 0$ and iterate until convergence:

1. $k(t) = arg\min_k p_{22}(t+1) = \frac{p_{22}(t)}{(1+\alpha)(p_{22}(t)+r)}$
2. $p_{22}(t+1) = \left[a^2(1-k(t))^2 + \alpha(ak(t))^2\right] p_{22}(t) + q + (1+\alpha)r(ak(t))^2$

Sketch of Proof

- $k^* = arg\min_k p$, s.t. $p = \mathcal{L}(k,p) = a^2(1-k)^2 p + q + [(1+\alpha)r + \alpha p](ak)^2$
- $\phi(p) = \min_k \mathcal{L}(k,p) = a^2 p + q - \frac{1}{1+\alpha}\frac{a^2p^2}{p+r}$
- $\phi(p)$ has a fixed point $p^* = \phi(p^*)$:



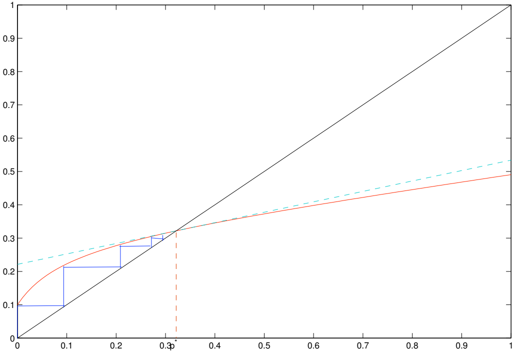
- $k^*(p^*)$ is the desired value since for any other value of k:

$$\tilde{p} = \mathcal{L}(k, \tilde{p}) > \phi(\tilde{p})$$

  which implies $\tilde{p} > p^*$

- the algorithm proposed converges to p^* and so that to k^*

$$p_{22}(t+1) = \phi(p_{22}(t))$$



3) Second iterative method

Let $\sigma_n^2 = 0$ and repeat until convergence:

1. solve the ARE with $\sigma_n^2(k)$;
2. compute $P_y(k)$:

$$P = \bar{A}P\bar{A}' + q \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} + (r + \sigma_n^2(k)) \begin{bmatrix} 0 & 0 \\ 0 & (ak)^2 \end{bmatrix}$$

   in vectorized form:

$$vecP = \left[\bar{A} \otimes \bar{A}\right] vecP + vec\bar{Q}(k)$$

3. let $\sigma_n^2(k+1) = \alpha P_y(k)$

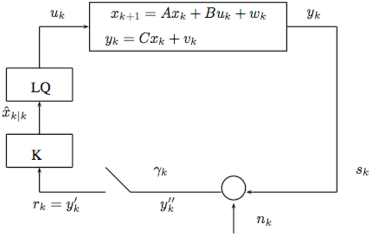
Analysis with other cost functionals

- $J = E\left[y^2 + \rho u^2\right]$

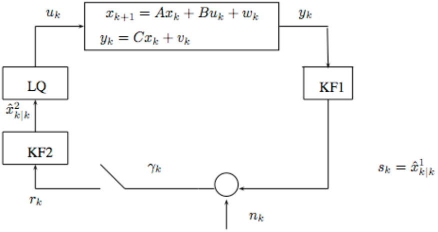
- $J = p_{11}(1 + \rho l^2(\alpha k^2 + 1)) + 2p_{12}(1 + \rho l^2 k(\alpha k + 1)) +$
  $+ p_{22}(1 + \rho l^2 k^2(\alpha + 1)) + R(1 + (1 + \alpha)\rho l^2 k^2)$

- no separation principle;

Packet loss: proposed schemes

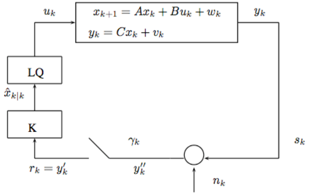


(a) Scheme 3



(b) Scheme 5

Scheme 3



System equations

$x_{k+1} = Ax_k + Bu_k + w_k$

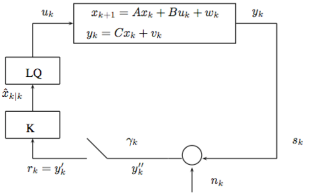
$s_k = y_k = Cx_k + v_k$

LQ controller

Since the separation principle is valid we can use the classical LQG approach $u_k = L\hat{x}_{k|k}$:

$$L = -(B'SB + U)^{-1}B'SA \qquad (ARE) : S = A'SA + W - A'SB(B'SB + U)^{-1}B'SA$$

Scheme 3



Kalman Filter

- Equivalent System

$$x_{k+1} = Ax_k + Bu_k + w_k$$
$$y'_k = \gamma_k y''_k = \gamma_k (Cx_k + v''_k)$$

where $v''_k = v_k + n_k$ and $R^{eq} = R + N$

- $e_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k} = A(I - \gamma_k KC)e_{k|k-1} + w_k - \gamma_k AKv''_k$

$$P_{k+1|k} = A(I - \gamma_k KC)P_{k|k-1}(I - \gamma_k KC)'A' + Q + \gamma_k^2 AKR^{eq}K'A'$$

- $\bar{P}_{k+1|k} = \bar{\gamma}A(I - KC)\bar{P}_{k|k-1}(I - KC)'A' + (1 - \bar{\gamma})A\bar{P}_{k|k-1}A' + Q + \bar{\gamma}AKR^{eq}K'A'$

$$\bar{P}_{k+1|k} = \bar{\gamma} A(I - KC)\bar{P}_{k|k-1}(I - KC)'A' + (1-\bar{\gamma})A\bar{P}_{k|k-1}A' + Q + \bar{\gamma} AKR^{eq}K'A'$$

Define the operators:

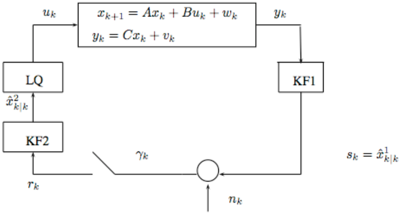
$$\mathcal{L}_{\bar{\gamma}}(K, P) = \bar{\gamma} A(I - KC)P(I - KC)'A' + (1-\bar{\gamma})APA' + Q + \bar{\gamma} AKR^{eq}K'A'$$

$$\Phi_{\bar{\gamma}}(P) = APA' + Q - \bar{\gamma} APC'(CPC' + R^{eq})^{-1}CPA'$$

Proposition

Let (A, C) be observable, $(A, Q^{1/2})$ controllable and $R^{eq} > 0$ then:
(a) if A is unstable and $\bar{\gamma} < \gamma_c$ then there is no kalman filter with costant gain that is also stable;
(b) if instead $\bar{\gamma} > \gamma_c$ the optimal costant gain Kalman filter exist and the corrispondent gain is $\bar{K} = \bar{P}C'(C\bar{P}C' + R^{eq})^{-1}$ where $\bar{P}$ is the solution of the MARE $\bar{P} = \Phi_{\bar{\gamma}}(\bar{P})$.

Scheme 5



System equations

$x_{k+1} = Ax_k + Bu_k + w_k$

$y_k = Cx_k + v_k$

$s_k = \hat{x}^1_{k|k} \quad r_k = \gamma_k(\hat{x}^1_{k|k} + n_k)$

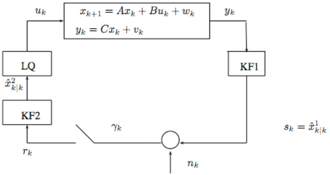
$u_k = L\hat{x}^2_{k|k}$

LQ controller

Since the separation principle is valid we can use the classical LQG approach $u_k = L\hat{x}_{k|k}$:

$$L = -(B'SB + U)^{-1}B'SA \qquad (ARE): S = A'SA + W - A'SB(B'SB + U)^{-1}B'SA$$

Scheme 5



First Kalman Filter

- constant gain $K_{P_1} = P_1 C'(CP_1 C' + R)^{-1}$
- estimate error variance: $P^1_{k|k} = P_1 - P_1 C'(CP_1 C' + R)^{-1} CP_1$

Second Kalman filter

- $x_{k+1} = Ax_k + Bu_k + w_k$
  $r_k = \gamma_k(\hat{x}^1_{k|k} + n_k) = \gamma_k(x_k - e^1_{k|k} + n_k)$
- Scheme 3 with: $C = I$ and $v''_k = n_k - e^1_{k|k} \in \mathcal{N}(0, N + P^1_{k|k})$

$$K_{P_2} = P_2(P_2 + N + P^1_{k|k})^{-1}$$

$$(MARE) \quad P = APA' + Q - \bar{\gamma} AP(P + N + P^1_{k|k})^{-1} PA'$$

Compute the cost functional $J = E\left[y^2\right]$

1. compute the SNR^* corresponding to the available number of *bits*:

$$L = 2^{bit}, \qquad SNR^* = \frac{3}{k^2}L^2 \qquad \text{in the scalar case}$$

$$L_c = 2^{bit/2}, \qquad SNR^* = \frac{3}{k^2}L_c^2 \quad \text{in the multi-dim case}$$

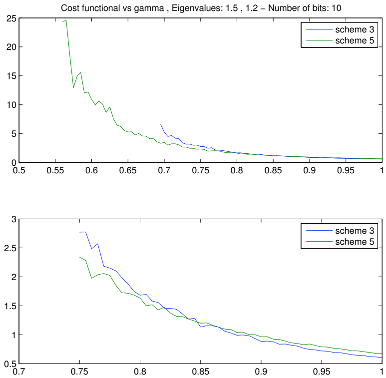
2. compute the error variance in adaptive condition;
3. compute the cost functional, relative to the quantization error just derived.

Systems structure:

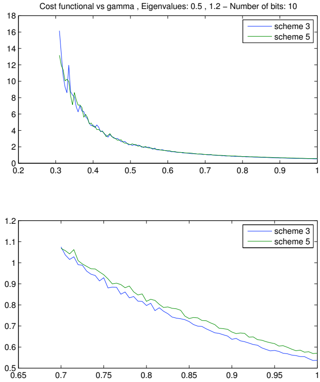
$$A = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix} \qquad C^T = B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$W = C^T C \qquad U = \rho = 1/1000 \qquad Rs = 0.1 \qquad Qs = 0.1 I_2$$

Simulations with packet loss

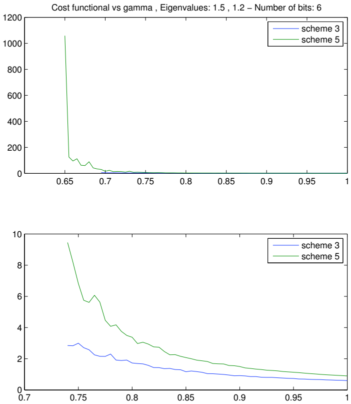


(a) $\Lambda_1 = \{1.5, 1.2\}$ and 10 bits

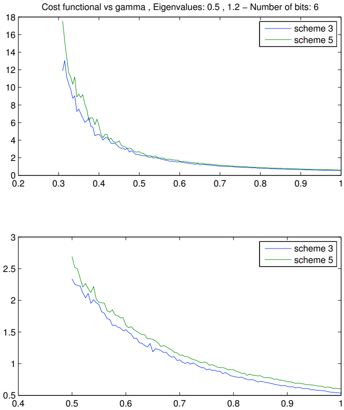


(b) $\Lambda_2 = \{0.5, 1.2\}$ and 10 bits

Simulations with packet loss



(a) $\Lambda_1 = \{1.5, 1.2\}$ and 6 bits



(b) $\Lambda_2 = \{0.5, 1.2\}$ and 6 bits

Conclusions

- LQG approach:
  - results of [3] as corollary;
- Channel model:
  - new model;
  - adaptive quantizer;
  - multidimensional case;
- A first application with cost functional:
  - $J = E\left[y^2\right]$
  - $J = E\left[y^2 + \rho u^2\right]$
- Channel with packet loss;

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The End

Equivalence between TFs in the scalar case: sketch of the proof (1)

Scheme 1

- Kalman filter Gain

$$P = APA' + Q - APC'(CPC' + R)^{-1}CPA' = a^2 p + \sigma^2 b^2 - \frac{a^2 p^2 c^2}{c^2 p} \Rightarrow p = \sigma^2 b^2$$

$$K = PC'(CPC' + R)^{-1} = \frac{pc}{c^2 p} = \frac{1}{c}$$

- LQ controller Gain

$$S = A'SA + W - A'SB'(B'SB + \rho)^{-1}B'SA = a^2 s + c^2 - \frac{a^2 s^2 b^2}{sb^2 + \rho}$$

$$\rho = \frac{sb^2 c^2 - s^2 b^2}{(1 - a^2)s - c^2}$$

$$L = -(B'SB + \rho)^{-1}B'SA = -\frac{(1 - a^2)bas^2 - c^2 bsa}{b^2(1 - a^2)s^2 - b^2 s^2 - b^2 sc^2 + sb^2 c^2} \longrightarrow \frac{(1 - a^2)}{ab} \quad s \to +\infty$$

Equivalence between TFs in the scalar case: sketch of the proof (2)

- Regulator TF:

$$R(z) := \frac{u_k}{y_k} = \frac{(1-a^2)}{ab}\frac{1}{c}$$

- Signal-Noise TF:

$$T(z) = \frac{u_k}{n_k} = \frac{\frac{1-a^2}{a}\frac{1}{z-a}}{1-\frac{1-a^2}{a(z-a)}} n_k = \frac{1-a^2}{az-1}$$

Scheme 2

- Kalman filter Gain

$$P = APA' + Q - APC'(CPC'+R)^{-1}CPA' = a^2 p - \frac{a^2p^2c^2}{c^2p+\sigma^2} \Rightarrow \begin{cases} p = 0 & \text{if} |a| < 1 \\ p = \frac{1}{c^2}(a^2-1) & \text{if } |a| \geq 1 \end{cases}$$

$$K = \frac{pc}{c^2p+\sigma^2} = \frac{1}{c}\frac{a^2-1}{a^2}$$

Equivalence between TFs in the scalar case: sketch of the proof (3)

- LQ controller Gain

$$s = a^2 s + c^2 - \frac{a^2 s^2 b^2}{sb^2 + \rho} \longrightarrow c^2 \quad \text{for} \quad \rho \to 0$$

$$L = -\frac{bsa}{sb^2 + \rho} \to -\frac{bc^2 a}{c^2 b^2} = -\frac{a}{b}$$

- Regulator TF:

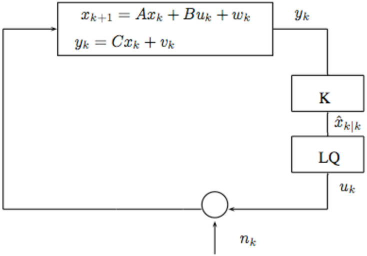
$$R(z) := \frac{u_k}{y'_k} = -\frac{(a^2 - 1)}{abc}$$

- Signal-Noise TF:

$$T(z) := \frac{y_k}{n_k} = \frac{1 - a^2}{az - 1}$$

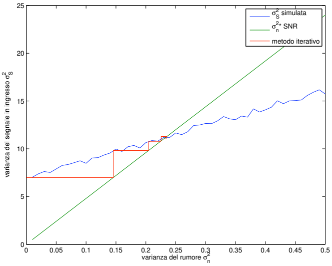
Even if the two Schemes are **different** they share the same TF $T(z)$ from n_k to s_k!!

Scheme 4

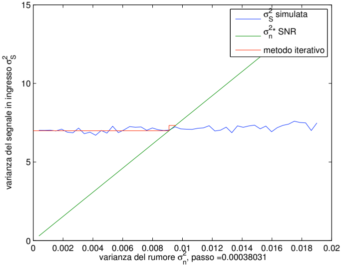


Simulations without packet loss: Convergence

Scheme 3



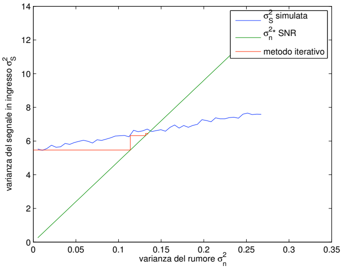
(a) $\Lambda = \{1.5, 2\}$ and 2 bits



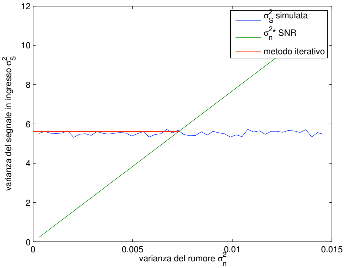
(b) $\Lambda = \{1.5, 2\}$ and 4 bits

Simulations without packet loss: Convergence

Scheme 4



(a) $\Lambda = \{1.5, 2\}$ and 2 bits



(b) $\Lambda = \{1.5, 2\}$ and 4 bits

Simulations without packet loss: Cost vs Bits

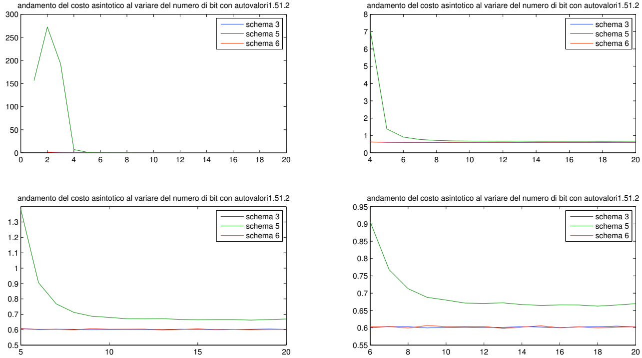


Figure: Unstable system with $\Lambda = \{1.5, 1.2\}$

Simulations without packet loss: Cost vs Bits

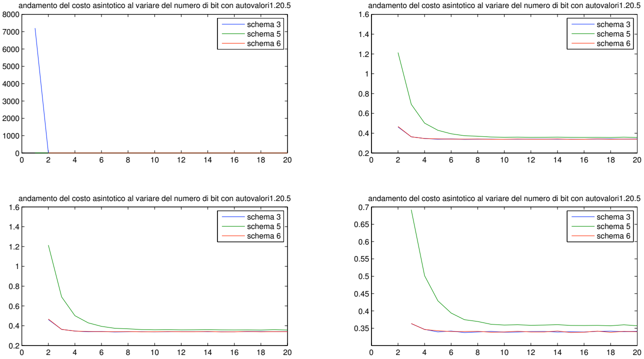


Figure: Unstable system with $\Lambda = \{0.5, 1.2\}$