

## Lezione 9 — 30 March

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What is left to do is how to design the low pass pole  $\tau_L$  for the derivative term. In practice we need to design a proper transfer function, so we add a high frequency pole to the derivative term. To understand the role of  $\tau_L$ , consider a simple PD controller which has a transfer function given by

$$C_{PD}(s) = K_P + \frac{K_D s}{1 + \tau_L s} = K_P \left( 1 + \frac{\tau_D s}{1 + \tau_L s} \right) = K_P \left( \frac{1 + (\tau_D + \tau_L) s}{1 + \tau_L s} \right)$$

which is approximated by (assuming  $\tau_L \ll \tau_D$ ):

$$C_{PD}(s) \simeq K_P \left( \frac{1 + \tau_D s}{1 + \tau_L s} \right) =: C'_{PD}(s)$$

Now, plotting the Bode diagram of our transfer function we can notice the differences between the transfer function with and without the high frequency pole. As we can see in figure 9.1, before the crossing frequency the two plots are almost the same, they differ only in high frequency. If we move  $\log_{10} \frac{1}{\tau_L}$  to the right side sufficiently far away from  $\tau_D$  and  $\omega_c$ , then the differences between the ideal transfer function and minimal in range defined by the closed loop bandwidth. The phase margin of the open loop transfer function  $G'(s) = P(s)C'_{PD}(s)$  at the desired crossing frequency, it's approximately equal to the phase margin obtained by considering the real transfer function  $G(s) = P(s)C_{PD}(s)$  :

$$\varphi_{PM}^G(\omega_c^*) \cong \varphi_{PM}^{G'}(\omega_c^*)$$

so, now what we want is that

$$\frac{1}{\tau_L} \gg \omega_c^* \text{ or, equally } \tau_L \ll \frac{1}{\omega_c^*}$$

Typically the range of values used for  $\tau_L$  are:

$$\frac{1}{10\omega_c^*} < \tau_L < \frac{1}{4\omega_c^*}$$

where  $\omega_c^*$  is the desired crossing frequency. Note that we cannot put the pole too far away since the controller will amplify any output noise present at high frequency.

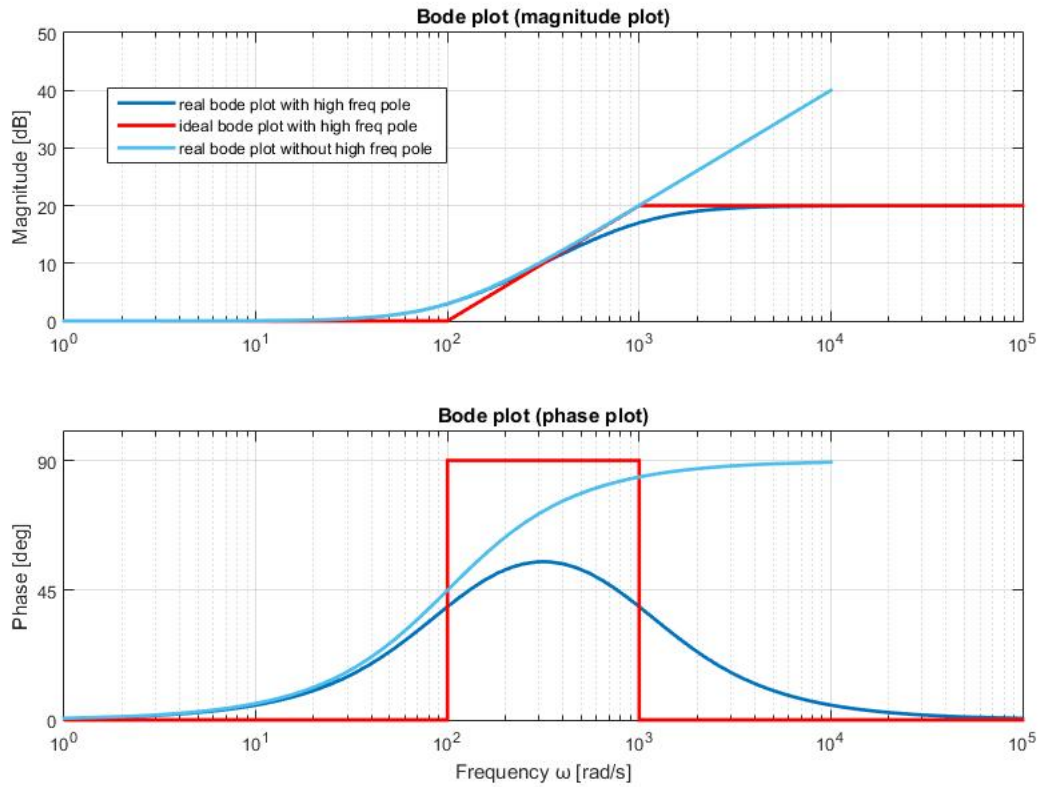


Figure 9.1: Bode Plot for non-proper PD (light blue) and proper PD controller (dark blue)

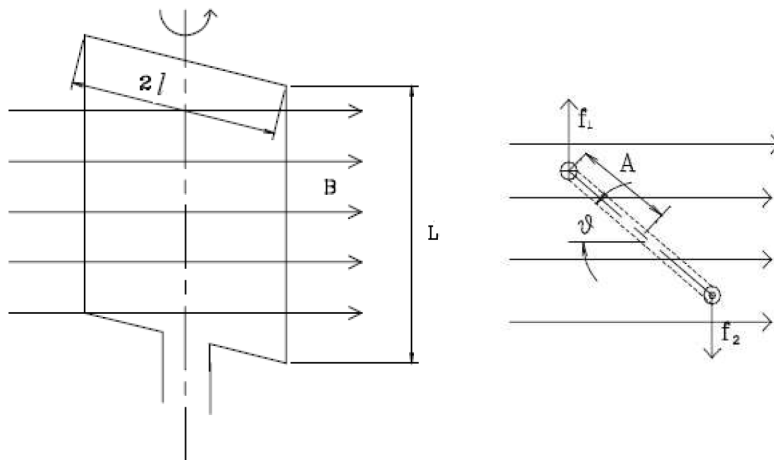
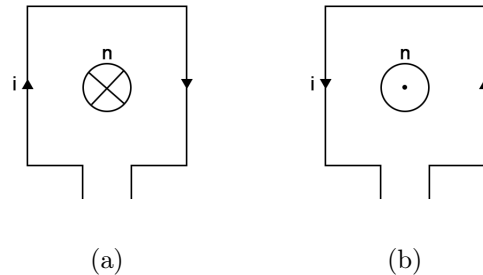


Figure 9.2: Wire coil from different viewpoints

Figura 9.3: Directions of  $\vec{n}$ 

## 9.1 DC MOTOR MODELLING

In a DC Motor there are basically 2 components: the rotor and the stator. The rotor is a component rolled up with electric wire that spins thanks to the current flowing in it, whereas the stator is a fixed component and it is basically composed of a big magnet which creates the magnetic field. As we can see in figure 9.2, the magnetic field given by the magnet on the stator and the current flow through the wire produce a force  $F$  that spins the rotor clockwise, and if the current flow is inverted the force will be opposite and it will make the rotor spin anticlockwise. The force is given by:

$$\vec{F} = L\vec{i} \times \vec{B} \quad \Rightarrow \quad \tau = 2LiB \sin \alpha = AiB \sin \alpha$$

where  $A = 2lL$  is the area of the coil and  $\tau$  is the total torque.

The maximum force is then obtained when  $\alpha = 90^\circ$ . This force creates a rotating moment that is equal to the force times the distance from the rotation point times the angle between the point where the force is applied.

When a coil is rotating inside a magnetic field due to a current flow, this creates an effect known as the Back-Electromotive Force (BEMF), a voltage that basically counter acts the motion of the current. In this way when the motor is spinning at high velocity it will consume less power.

The Back-Electromotive Force is given by:

$$u_e(t) = -\frac{d}{dt}\Phi(t)$$

where  $\Phi(t)$  is the magnetic flux:

$$\Phi(\alpha(t)) = A\vec{n} \cdot \vec{B} = AB(-\cos \alpha(t))$$

and  $\vec{n}$  is a vector perpendicular to the area with the direction given by the current flow as in figure 9.3.

If  $\alpha(t)$  is constant there is no BEMF, because the derivative of a constant is 0. In reality we have  $\Phi(t)$  which is a function of time and  $\alpha(t)$  itself is a function of time. So the BEMF

is given by:

$$u_e(t) = -AB(-\sin \alpha) \frac{d\alpha(t)}{dt} = AB\omega(t) \sin \alpha(t)$$

where  $\omega(t)$  is the angular velocity of the motor.

Now suppose we are making the motor rotating at constant velocity in the magnetic field, i.e.  $\omega(t) = \omega$ , then we have:

$$\tau(t) = ABi \sin(\omega t)$$

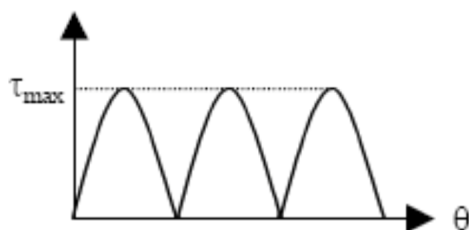


Figura 9.4: Motor torque

When  $\tau$  starts going negative, we reverse the current flux using a commutator in order to obtain a torque that is always positive, as in figure 9.4, but we would like to have a constant torque. The solution of this problem is to put multiple coils, as actually happens in DC motors. The current flows through the coil that is passing on the commutator. We are trying to push current only when we are on the maximum but since we have a finite number of coils, the result will be like in figure 9.5, where we can observe a current ripple.

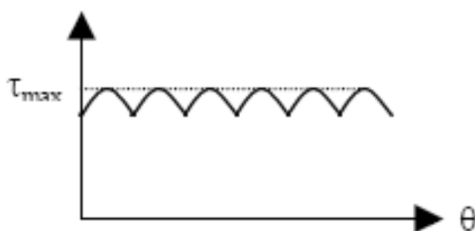


Figura 9.5: Current ripple

If  $\alpha = 90^\circ \rightarrow \sin \alpha = 1$  and we can write the expression for the mechanical interaction

$$\tau \simeq k_t i$$

where  $k_t$  is the torque constant. In addition we can consider the electrical interaction:

$$u_e = k_e \omega$$

where  $k_e$  is the electric constant.

So, we can find 2 effects of having a DC motor:

- in order to have a torque we need to apply a voltage;
- there is an effect (the BEMF) that reduce the current flow when the motor is spinning.

## DC Motor scheme

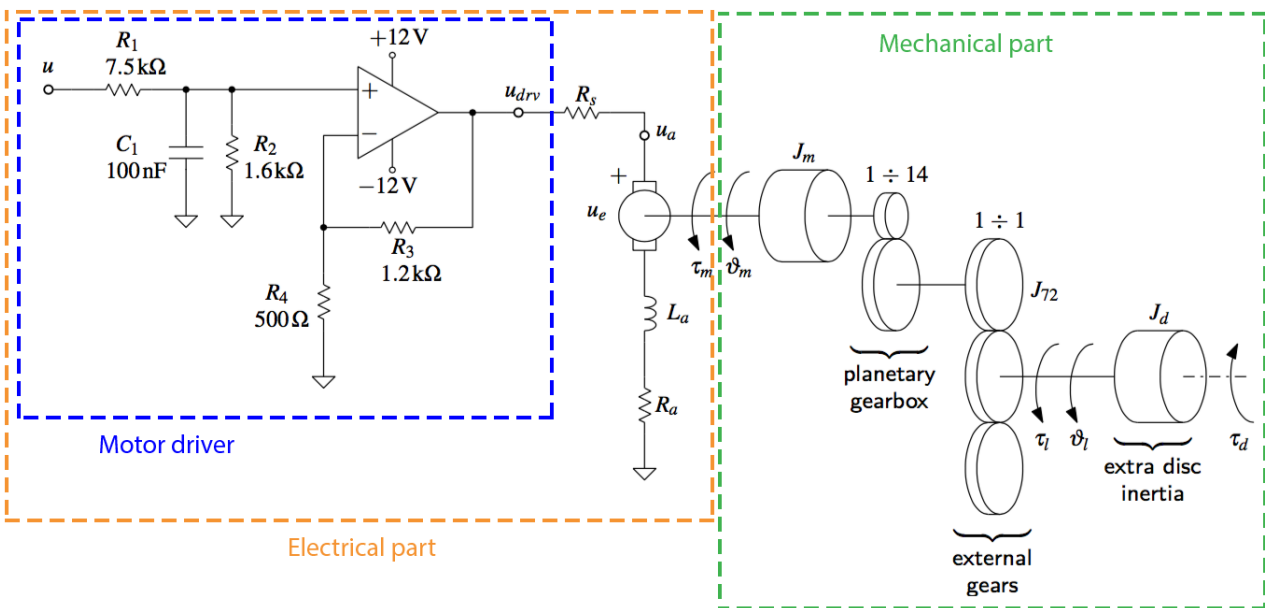


Figure 9.6: DC Motor scheme

As we can see in figure 9.6, we have a voltage to apply to the driver,  $u(t)$ , which we can regulate. Summarizing, the electrical and the mechanical part are related by these two expressions:

$$u_e = k_e \omega_m \quad (9.1)$$

$$\tau_m = k_t i_a \quad (9.2)$$

In the motor design:

- $R_s$  is the shunt resistance;
- $L_a$  is the armature inductance;
- $R_a$  is the armature resistance;
- $i_a$  is the current going through the armature;
- $J_m$  is the motor inertia;

- $J_1$  is the load inertia.

$J_1$  is the result of two terms:

$$J_1 = J_d + 3J_{72}$$

as we consider the load made up of a mass with inertia  $J_d$  and of three 72-tooth gears, each with inertia  $J_{72}$ . Of the three wheels just mentioned, the one in the middle is where we actually measure the angle or the velocity, the one on top is necessary to transmit the rotation, and the last one is connected to the encoder, that is the sensor count the number of rotations of the load. When we buy a motor, we will find all the values for  $R$ ,  $C$ ,  $L_a$ ,  $k_e$ ,  $k_t$ , etc... in the data sheet.

Connected to the motor, on the right, there is a Planetary Gearbox, basically composed of a smaller and a bigger wheel, which transmits the rotation from the motor to the load with a ratio  $1 \div N$ . In our case  $N = 14$ , that means that to have 1 rotation at the load, the motor has to do 14 rotations. In formula:

$$\omega_m = N\omega_1. \quad (9.3)$$

## 9.2 How to find the transfer function $P(s)$

What we want now is to find the transfer function  $P(s)$  from the voltage applied,  $u(t)$ , to the angular velocity of the load,  $\omega_1$ :

$$\omega_1(t) = P(s)u(t)$$

We also define:

$\theta_1$ : the load angle. If we derive it, we obtain:  $\dot{\theta}_1 = \omega_1$

$\theta_m$ : the motor angle. As before:  $\dot{\theta}_m = \omega_m$

We can decompose the problem and try to derive the total transfer function  $P(s)$  as the product of different transfer functions. So let's define first the transfer function from the voltage at the input to the voltage at the output of the driver:

$$u_{\text{drv}}(t) = P_{\text{drv}}(s)u(t) \quad (9.4)$$

Assuming to have an ideal amplifier, we get that  $P_{\text{drv}}(s)$  is a first order transfer function with the following expression:

$$P_{\text{drv}}(s) = \frac{K_{\text{drv}}}{1 + T_{\text{drv}}s}$$

where  $K_{\text{drv}}$  and  $T_{\text{drv}}$  can be obtained by the values of resistances and capacity.

The next step is to find the transfer function from the voltage of the driver to the angular velocity of the load:

$$\omega_1 = P'(s)u_{\text{drv}}(t) \quad (9.5)$$

If we substitute (9.4) in (9.5), we get the overall transfer function:

$$\omega_1(t) = P_{\text{drv}}(s)P'(s)u(t)$$

and we can define:

$$P(s) = P_{\text{drv}}(s)P'(s)$$

Now we need to compute  $P'(s)$ .

### 9.2.1 Model of the electrical part

From the dynamics of the electrical part we have:

$$L_a \frac{di_a}{dt} + R_a i_a(t) + R_s i_a(t) = u_{\text{drv}}(t) - u_e(t) \quad (9.6)$$

and we can see that the effect of the Back-Electromotive Force is to reduce the voltage applied. If we substitute (9.1) in (9.6) and if we gather  $R_a$  with  $R_s$  in order to define the equivalent resistance  $R_{\text{eq}} = R_a + R_s$ , we get:

$$L_a \frac{di_a}{dt} + R_{\text{eq}} i_a(t) = u_{\text{drv}}(t) - k_e \omega_m(t) \quad (9.7)$$

### 9.2.2 Model of the mechanical part

If we have a torque  $\tau_1$  at the load, and since there is a  $1 \div N$  gearbox,  $\tau_1$  is transferred as it is at the external gears. For Newton's Law, there is a counter torque  $\tau'_1$  at the motor output. In order to compute  $\tau'_1$ , we have to assume perfect power transmission, that means that the power applied by the motor is transferred with no loss to the load:

$$P_W^m = \tau'_1 \omega_m \quad (9.8)$$

$$P_W^l = \tau_1 \omega_l \quad (9.9)$$

So, from the equivalence of (9.8) and (9.9), and from (9.3) we obtain the expression for  $\tau'_1$ :

$$\tau'_1 = \frac{1}{N} \tau_1 \quad (9.10)$$

from which we see that the torque applied by the motor is  $N$  times smaller than the torque transferred to the load.

We can now write the mechanical equation for the motor:

$$J_m \dot{\omega}_m = -B_m \omega_m + \tau_m - \tau'_1 \quad (9.11)$$

where  $B_m \omega_m$  is the friction at the motor, proportional to velocity. By the substitution of (9.10) and (9.2) in (9.11), we get:

$$J_m \dot{\omega}_m = -B_m \omega_m + k_t i_a - \frac{1}{N} \tau_1 \quad (9.12)$$

With regard to the load the equation is:

$$J_1\dot{\omega}_1 = -B_1\omega_1 + \tau_1 - \tau_d \quad (9.13)$$

where similarly as before  $B_1\omega_1$  is the friction at the load, and  $\tau_d$  is a counter torque generated from an external disturbance.

From equations (9.7), (9.12) and (9.13), by the application of the Laplace Transform, we will derive an expression for  $P'(s)$  and then for  $P(s)$ .