Control Laboratory:	**a.a. 2015/2016**

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8.1 PID Design

The closed loop specifications are typically given in time domain terms with respect to some specific input signals (step, ramp, sinusoid). A typical request is that the closed-loop systems has a forced output response to a reference step input with a settling time t_s (or rising time t_r) smaller than a certain critical value t_s^* (or t_r^* in case of rising time), an overshoot s smaller than a certain critical value s^* and a steady state error e smaller than a certain critical value e^*:

$$\begin{aligned} t_s &\leq t_s^* & &\textit{settling time} \\ s &\leq s^* & &\textit{overshoot}\ [\%]\,(s = M_p \cdot 100) \\ e &\leq e^* & &\textit{steady state error}\ [\%] \end{aligned}$$

The purpose of this chapter consists on designing the PID control parameters that satisfy the requested specifications

$$t_r^*, s^*, e^* \Rightarrow (K_P, K_I, K_D, \tau_L).$$

Given a specific transfer function $P(s)$ that describes the system's plant, we define $G(s)$ as the transfer function of the open-loop control system (where $C(s)$ represents the controller's transfer function)

$$G(s) = P(s)C(s),$$

and W(s) for the closed-loop one,

$$W(s) = \frac{G(s)}{1 + G(s)}.$$

Under the assumption that the closed-loop system $W(s)$ is BIBO stable, we are going to map the time domain specifications t_s, s, e into the frequency domain of a II order system, using the following approximations:

$$t_s \simeq \frac{4,6}{\omega_c} \Rightarrow \omega_c \geq \omega_c^* = \frac{4,6}{t_s^*} \tag{8.1}$$

$$\varphi_m^G \geq \varphi_m^* \tag{8.2}$$

The Final Value Theorem is used to obtain the steady state error in term of percentage with respect to the desired value as follows:

$$e = \frac{r(\infty) - y(\infty)}{r(\infty)} \cdot 100 \quad [\%] \tag{8.3}$$

In the interesting case of a step function as input $r(t) = \mathbf{1}(t)$, the steady state error assumes this particular form:

$$e(\infty) = (1 - y(\infty)) \cdot 100 = |1 - W(0)| \cdot 100 \quad [\%], \tag{8.4}$$

where $y(\infty) = W(0)r(\infty)$.

From now on, the equivalent phase margin of $P(s)$, at the crossing frequency ω_c^*, will be written as $\varphi_m^P(\omega_c^*)$, whose definition is:

$$\varphi_m^P(\omega_c^*) := 180° + \angle P(j\omega_c^*) \tag{8.5}$$

In the next sections, we will refer our analysis to a specific transfer function $P(s) = \frac{1}{(s+1)^3}$, showing how the three terms P, I and D are related to the conditions mentioned at the beginning of the chapter. (see Figure 8.1).

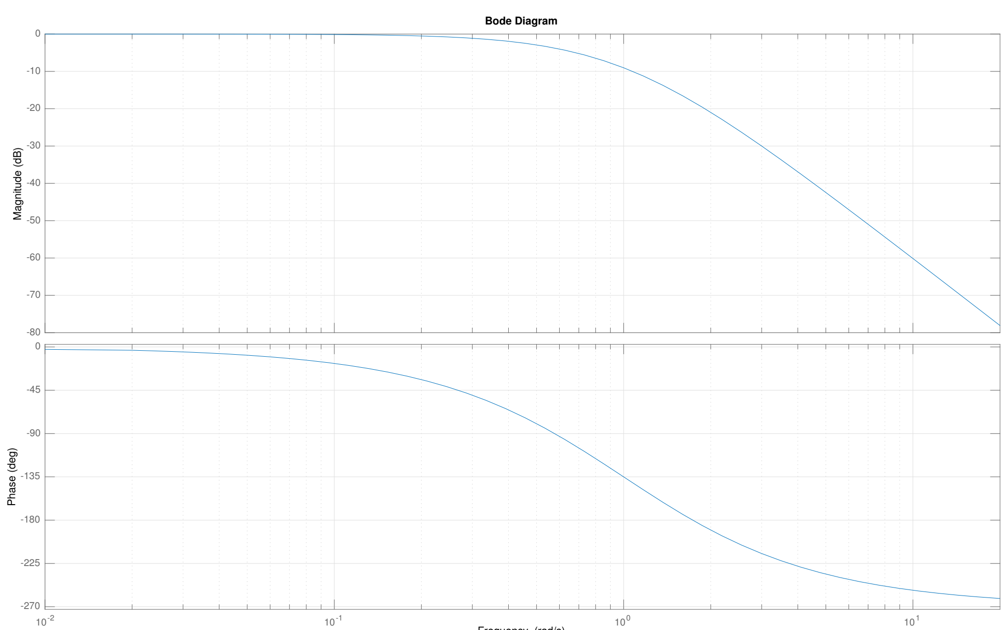

Figure 8.1. Bode plot of $P(s) = \frac{1}{(s+1)^3}$

In order to properly design a PID controller, we are going to fulfil these two constraints:

$$\begin{cases} \varphi_m^G(j\omega_c^*) \geq \varphi_m^* \\ |G(j\omega_c^*)| = 1 \end{cases} \tag{8.6}$$

8.1.1 Integrator Controller

This section refers to the application of the integral term (I) only:

$$C(s) = \frac{K_I}{s},$$

whose magnitude and phase are:

$$|C(j\omega)| = \frac{K_I}{|\omega|}, \quad \angle C(j\omega) = -90° \quad \forall\omega.$$

Where $\varphi_m^P(\omega_c^*) \geq \varphi_m^* + 90°$, the requirements on the phase margin is satisfied:

$$\varphi_m^G(\omega_c^*) = \varphi_m^P(\omega_c^*) - 90° \geq \varphi_m^* \Rightarrow s \leq s^*.$$

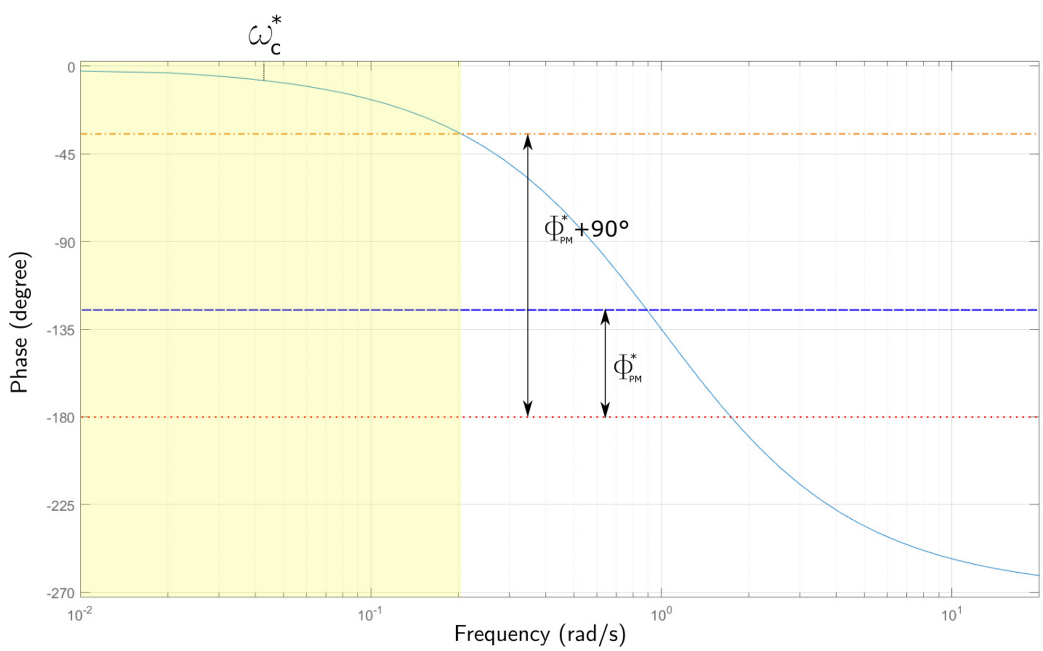

Figure 8.2. Integrator controller: phase margin analysis

In Figure (8.2), we are highlighting the area where all the specifics are fulfilled even by reducing the phase margin by 90°, due to the integrator contribution.
As shown in the equation (8.6), we are designing K_I to obtain the desired magnitude $|G(j\omega_c^*)| = 1$:

$$|G(j\omega_c^*)| = |P(j\omega_c^*)|\,|C(j\omega_c^*)| = |P(j\omega_c^*)|\,\frac{K_I}{\omega_c^*} = 1 \;\Rightarrow\; K_I = \frac{\omega_c^*}{|P(j\omega_c^*)|}. \tag{8.7}$$

According to the relation $t_s = \frac{4{,}6}{\omega_c}$, the I controller are expected to guarantee the settling time condition since:

$$\omega_c^G = \omega_c^* \;\Rightarrow\; t_s = t_s^*.$$

Finally, it has to be noticed that by adding an integral term (I), the requirement on the steady state error is satisfied since $e = 0$, as long as $P(0) \neq 0$.

8.1.2 Proportional Controller

Figure 8.2 underlines a fundamental aspect: when the phase margin of $P(s)$ is smaller than $\varphi_m^* + 90°$, we can no longer use an integrative controller; the constraint on the phase margin of $G(s)$ would not be satisfied anymore.
A Proportional term should be used instead:

$$C(s) = K_P$$

writing magnitude and phase as follows,

$$|C(j\omega)| = K_P, \quad \angle C(j\omega) = 0° \quad \forall\omega \quad \Rightarrow \quad \varphi_m^G(\omega_c^*) = m_\varphi^P(\omega_c^*).$$

As we can see in Figure (8.3), when ω_c^* belongs to the underlined gap, the inequality is valid:

$$\varphi_m^* \leq \varphi_m^P(\omega_c^*) \leq \varphi_m^* + 90°, \tag{8.8}$$

The (8.8) points out:

$$m_\varphi^G(\omega_c^*) \geq \varphi_m^*.$$

As shown in the equations (8.6), we are designing K_P to obtain the desired magnitude $|G(j\omega_c^*)| = 1$:

$$1 = |G(j\omega_c^*)| = |P(j\omega_c^*)|\,|C(j\omega_c^*)| = |P(j\omega_c^*)|\,K_P \;\Rightarrow K_P = \frac{1}{|P(j\omega_c^*)|} \tag{8.9}$$

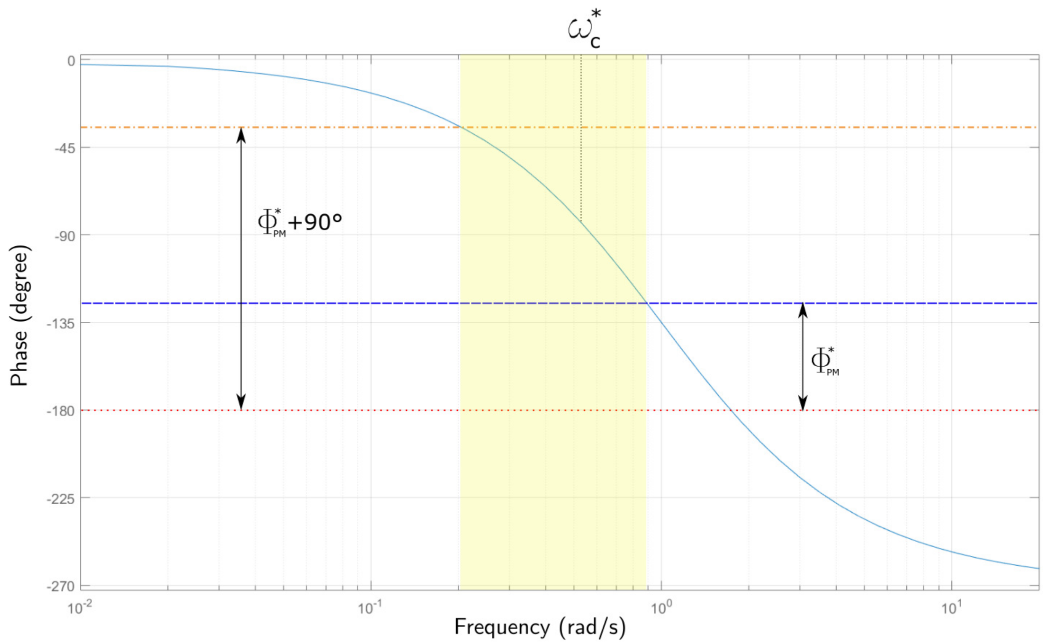

Figure 8.3. Proportional controller: phase margin analysis

The absence of Integrator term makes us wonder about the behaviour of the steady state error,

$$e = (1 - W(0)) \cdot 100 = \left(1 - \frac{C(0)P(0)}{1 + C(0)P(0)}\right) \cdot 100 = \left(\frac{1}{1 + C(0)P(0)}\right) \cdot 100 \leq e^*$$

$$e = \frac{100}{1 + K_P P(0)} = \frac{100}{1 + \dfrac{P(0)}{|P(j\omega_c^*)|}} \overset{?}{\leq} e^* \quad [\%] \tag{8.10}$$

If this condition is not satisfied, then it is necessary to add also the integral term (I), and re-design the controller as described in the following paragraph.

8.1.3 Proportional and Integrator Controller

PI controller has to be used in order to satisfy the constraint on the steady state error, when the proportional contribution is not enough. The integrator term implies a phase margin reduction, thus an overshoot rising:

$$C(s) = K_P + \frac{K_I}{s}$$

In the frequency domain, on the condition $s = j\omega$:

$$C(j\omega) = K_P + \frac{K_I}{j\omega} = K_P - \frac{K_I}{\omega} j.$$

At the crossing frequency ω_c^*,

$$C(j\omega_c^*) = ae^{j\alpha} = a\cos\alpha + ja\sin\alpha \;\Rightarrow$$

$$\Rightarrow Re[C(j\omega_c^*)] = a\cos\alpha = K_P,\;\; Im[C(j\omega_c^*)] = a\sin\alpha = -\frac{K_I}{\omega_c^*}.$$

In order to guarantee the requested steady state error, we're modifying the conditions (8.6), as follows:

$$\begin{cases} \varphi_m^G = \varphi_m^* & (1) \\ |G(j\omega_c^*)| = 1 & (2) \end{cases} \tag{8.11}$$

Thus, (1) gives a,

$$|C(j\omega_c^*)|\,|P(j\omega_c^*)| = 1 \;\Rightarrow\; a = |C(j\omega_c^*)| = \frac{1}{|P(j\omega_c^*)|},$$

(2) gives α,

$$\varphi_m^* - 180^\circ = \angle P(j\omega_c^*) + \angle C(j\omega_c^*),$$
$$\Rightarrow \alpha = \angle C(j\omega_c^*) = \varphi_m^* - 180^\circ - \angle P(j\omega_c^*) = \varphi_m^* - \varphi_m^P(\omega_c^*) < 0.$$

The computed values a and α guarantee the fulfillment of all the specifications:

$$K_P = a\cos\alpha > 0 \tag{8.12}$$

$$K_I = -\omega_c^* a\sin\alpha > 0. \tag{8.13}$$

8.1.4 Proportional and Derivate Controller

A different controller is needed when ω_c^* is such that

$$\varphi_m^* - 90° \leq \varphi_m^P(\omega_c^*) \leq \varphi_m^*,$$

as shown in Figure (8.4).

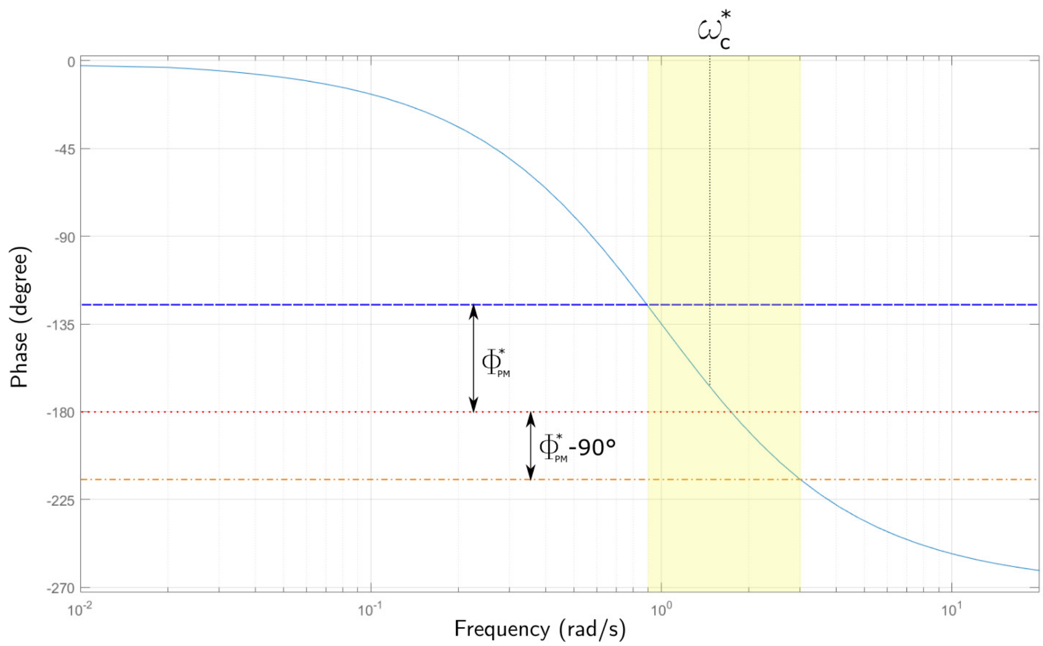

Figure 8.4. Derivative controller: phase margin analysis

A PD controller should be used:

$$C(s) = K_P + K_D s$$

The aspect of this transfer function of being not proper, will be discussed in the further lectures.
In the frequency domain for $s = j\omega$,

$$C(j\omega) = K_P + K_D j\omega$$

At the desired crossing frequency ω_c^* we have:

$$C(j\omega_c^*) = ae^{j\alpha} = a\cos\alpha + ja\sin\alpha \Rightarrow$$

$$\Rightarrow Re[C(j\omega_c^*)] = a\cos\alpha = K_P,\ Im[C(j\omega_c^*)] = a\sin\alpha = K_D\omega_c^*$$

As shown in the equations (8.6), we're computing a and α:

$$a = \frac{1}{|P(j\omega_c^*)|}$$

$$\alpha = \varphi_m^* - 180° - \angle P(j\omega_c^*) = \varphi_m^* - \varphi_m^P(\omega_c^*)$$

Using the relation between the real and imaginary part of the controller $C(s)$ in order to satisfy the requirements in term of rising time and overshoot, we obtain:

$$K_P = a\cos\alpha > 0 \tag{8.14}$$

$$K_D = \frac{a\sin\alpha}{\omega_c^*} > 0 \tag{8.15}$$

As already described for the P controller, a check must be done on the steady state error:

$$e = \frac{100}{1 + P(0)C(0)} = \frac{100}{1 + P(0)K_P} = \frac{100}{1 + \dfrac{P(0)\cos\alpha}{|P(j\omega_c^*)|}} \overset{?}{\leq} e^* \tag{8.16}$$

Otherwise, we need to use also an integrator.

8.1.5 Proportional, Derivative and Integrative Controller

In this section we discuss about the design of PID controller.

$$C(s) = K_P + K_D s + \frac{K_I}{s}$$

In the frequency domain on the condition $s = j\omega$,

$$C(j\omega) = K_P + j\omega K_D + \frac{K_I}{j\omega} = K_P + j\left(\omega K_D - \frac{K_I}{\omega}\right)$$

At the desired crossing frequency ω_c^*,

$$C(j\omega_c^*) = ae^{j\alpha} = a\cos\alpha + ja\sin\alpha \ \Rightarrow$$

$$\Rightarrow \quad Re[C(j\omega_c^*)] = a\cos\alpha = K_P, \quad Im[C(j\omega_c^*)] = a\sin\alpha = \omega K_D - \frac{K_I}{\omega}$$

As shown in the equations (8.11), we compute a and α as:

$$K_P = a\cos\alpha \tag{8.17}$$

$$\omega_c^* K_D - \frac{K_I}{\omega_c^*} = a\sin\alpha \tag{8.18}$$

By adding the integral term (I), the requirement on the steady state error is automatically satisfied since $e = 0$, unless the plant $P(s)$ has zero in the origin. While the previous constraints in the frequency domain uniquely determine the value for proportional gain K_P, the gains K_I and K_D are related by a single equation thus there is an infinite number of possible solutions. In order to determine how to design the parameters K_I and K_D it is convenient to rewrite the transfer function $C(s)$ in terms of other parameters. As so, let us define the following time constants:

$$\tau_I = \frac{K_P}{K_I}, \quad \tau_D = \frac{K_D}{K_P},$$

which are referred as the reset time and derivative time. The PID transfer function can be approximated as

$$C(s) = \frac{K_I}{s}(1 + \tau_I s)(1 + \tau_D s).$$

under the assumption that $\tau_I \gg \tau_D$, enforced by setting:

$$\tau_I = b\tau_D \Rightarrow \frac{K_P}{K_I} = b\frac{K_D}{K_P} \Rightarrow K_D K_I = \frac{K_P^2}{b}, \quad 4 \le b \le 10. \tag{8.19}$$

Now if we multiply both terms of equation (8.18) by $\omega_c^* K_I$, using (8.19):

$$\omega_c^{*2} K_D K_I - K_I^2 = \omega_c^* K_I a \sin\alpha \Rightarrow K_I^2 + (\omega_c^* a \sin\alpha) K_I - \omega_c^{*2}\frac{K_P^2}{b} = 0,$$

whose solutions are:

$$K_I = \frac{-\omega_c^* a \sin\alpha \pm \sqrt{\omega^{*2} a^2 \sin^2\alpha + \frac{4\omega^{*2} K_P^2}{b}}}{2}$$

Due to the condition $\alpha > 0$, we're neglecting the negative solution, thus:

$$K_I = \frac{\omega_c^* a}{2}\left(\sqrt{\sin^2\alpha + \frac{4\cos^2\alpha}{b}} - \sin\alpha\right).$$

Summarizing, the parameters of the PID controller can be obtained as follows:

$K_P = a\cos\alpha$

$K_I = \frac{\omega_c^* a}{2}\left(\sqrt{\sin^2\alpha + \frac{4\cos^2\alpha}{b}} - \sin\alpha\right)$

$K_D = \frac{K_P^2}{bK_I}$

If we further assume that $b = 4$, then the expression for the integral gain simplifies further:

$$b = 4 \quad \Longrightarrow \quad K_I = \frac{\omega_c^* a}{2}(1 - \sin\alpha)$$