Control Laboratory:	**a.a. 2015/2016**

Lecture 7 — 22 March

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7.1 PID design: effects of integrator

After an analysis on the transfer function $P(S)$, it is necessary to make a choice for the controller $C(S)$ that will be used. PID controllers are flexible and useful, but they require that all poles p_i of $P(S)$ have $\mathbb{R}e[p_i] \leq 0$, otherwise the controller might not even guarantee stability. The PID transfer function can be written as

$$C(s) = K_P + \frac{K_I}{s} + \frac{K_D s}{1 + \tau_L s},$$

so basically PID design corresponds to setting parameters K_P, K_I, K_D and τ_L, according to systems specs and frequency domain analysis. The high frequency pole determined by the parameter τ_L is added to the derivative action to be physically implementable.

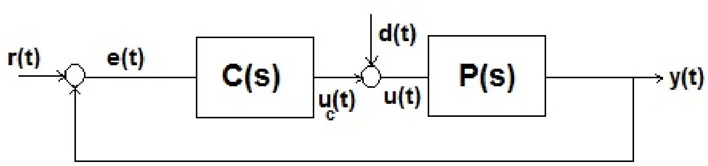

Figura 7.1. Closed loop system with disturbance addition

We want to analyze the effect of the integrator on the closed loop system. As said on previous lectures, integrator guarantees no steady state error and rejects disturbances. Re-writing the transfer function,

$$\begin{aligned} C(s) &= \frac{sK_P(1 + \tau_L s) + K_I(1 + \tau_L s) + K_D s^2}{s(1 + \tau_L s)} \\ &= \frac{(K_D + \tau_L K_P)s^2 + (K_I \tau_L + K_P)s + K_I}{s(1 + \tau_L s)} = \\ &= \frac{1}{s} \frac{(K_D + \tau_L K_P)s^2 + (K_I \tau_L + K_P)s + K_I}{(1 + \tau_L s)} \end{aligned}$$

we separate $C'(s)$ obtaining:

$$C(s) = \frac{1}{s}C^{'}(s).$$

At frequency s=0 , $C^{'}(0) = K_I \neq 0$. We assume also that $P(s)$ has no zeros at the origin s=0.
We add now a disturbance signal $d(t)$ to the closed-loop model, assuming $d(t) = d$ constant (figure 7.1).

The output signal $y(t)$ is a linear combination of two terms:

$$y(t) = W(s)r(t) + W'(s)d(t)$$

On first part of expression we find the usual transfer function,

$$W(s) = \frac{G(s)}{1 + G(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{C^{'}(s)P(s)}{s + C'(s)P(s)},$$

and final value theorem provides DC-gain $W(0)$:

$$W(0) = \frac{C^{'}(0)P(0)}{s + C'(0)P(0)} = 1$$

Under the assumption that $W(s)$ is stable, for a finite reference signal this system should provide a finite output. A gain equal to one guarantees the absence of steady state error, which is one of the goals that we want to reach.

The second part of $y(t)$ contains the transfer function referred to the disturbance, $W'(s)$. In order to find its expression, we identify two different components of $u(t)$, one is provided by controller ($u_c(t)$) and the other is $d(t)$:

$$\begin{cases} y(t) = P(s)u(t) \\ u(t) = u_c(t) + d(t) \\ u_c(t) = C(s)e(t) \\ e(t) = -y(t) \end{cases}$$

Solving this system we get:

$$u_c(t) = -C(s)y(t)$$

and

$$u(t) = d(t) - C(s)y(t)$$

So the previous expression becomes:

$$y(t) = P(s)d(t) - P(s)C(s)y(t) =$$

$$= \frac{P(s)}{1 + P(s)C(s)}d(t) = \frac{sP(s)}{s + P(s)C'(s)}d(t) =$$

$$= W^{'}(s)d(t)$$

The only difference between $W(s)$ and $W'(s)$ is on zeros, so knowing that $W(s)$ is stable ensures that $W'(s)$ is stable too. Using again the final value theorem, we can compute disturbance gain:

$$W^{'}(0) = \frac{0}{0 + P(0)C'(0)} = 0,$$

so perfect disturbance rejection is guaranteed.

To make a comparison, we can consider a transfer function for a controller where K_I=0:

$$C(S) = K_P + \frac{K_D s}{1 + \tau_L s}.$$

At frequency 0, this controller has DC-gain $C(0)$=K_P and the corresponding closed-loop system has the equation:

$$W(0) = \frac{C(0)P(0)}{1 + C(0)P(0)}.$$

We can notice that:

- if $P(s)$ has at least one pole in $s = 0 \Rightarrow P(0) = \infty \Rightarrow W(0) = 1$.

- if $P(0) \neq \infty = \alpha \Rightarrow$ the steady state value is $W(0) = \frac{\alpha K_P}{1+\alpha K_P} < 1$, and the bigger is αK_P, the closer $W(0)$ is to 1;

For disturbance rejection, we can see that in both cases it is possible to reduce noise but not to completely eliminate it, in fact:

$$W^{'}(0) = \frac{P(0)}{1 + P(0)C(0)} = \begin{cases} \frac{P(0)}{1+K_P P(0)} \neq 0 & \text{If } P(0) \neq \{0, \infty\} \\ \frac{1}{K_P} \neq 0 & \text{If } P(0) = \infty \end{cases}$$

So this confirms that the only way to completely eliminate disturbances is to use a controller with an integrative action.

7.2 Choosing terms and parameters for PID

There are typically 5 possible configurations for PID: I, P, PI, PD and full PID with all actions. Bode diagrams of $C(s)$, figure 7.2, show directly how a variation on parameters can affect system performances.

Choosing a purely integrative controller, 7.2 a), with transfer function $C(s) = \frac{K_I}{s}$, the only degree of freedom is the parameter K_I. Phase of $C(s)$ in this case is -90°, and K_I can only push up or down the straight line that represents magnitude.

If $C(s)$ has just a proportional term, 7.2 b), transfer function is $C_P(s) = K_P$, the only degree of freedom is K_P and the straight line on magnitude can be moved up or down, but phase is always equal to zero.

In the case of PI, 7.2 c), transfer function of controller is $C_{PI}(s) = \frac{K_P + K_I}{s} = \frac{K_I(1+\tau_I s)}{s}$. Using both integrative and proportional action, $C(s)$ has two parameters to be set, which are K_I and $\tau_I = \frac{K_P}{K_I}$ (referred as to *reset time* and expressed in seconds). In particular, τ_I determines break point for the curve on the magnitude graph and it gives control of phase too, while K_I adjusts the value of magnitude for the system.

For a PD controller, 7.2 d), there is transfer function $C_{PD}(s) = K_P + K_D s = K_P(1+\tau_D s)$, where $\tau_D = \frac{K_D}{K_P}$ is a parameter that represents the *derivative time*. Again τ_D sets the breakpoint of magnitude curve and the phase diagram, while K_P pushes up or down the magnitude of $C(j\omega)$.

The complete PID controller, 7.2 e), has transfer function

$$\begin{aligned} C_{PID}(s) &= \frac{K_D s^2 + K_P s + K_I}{s} = \frac{K_I(1 + \frac{K_P}{K_I}s + \frac{K_D K_P}{K_I K_P}s^2)}{s} \\ &= \frac{K_I(1 + \tau_I s + \tau_D \tau_I s^2)}{s} \simeq \frac{K_I(1 + (\tau_I + \tau_D)s + \tau_D \tau_I s^2)}{s} \simeq \\ &\simeq \frac{K_I(1+\tau_I s)(1+\tau_D s)}{s} \end{aligned}$$

where the approximation is valid if $\tau_D \ll \tau_I$. This solution offers three degrees of freedom: with K_I it is possible to set the magnitude, and with τ_I and τ_D it is possible to adjust the breakpoints for regulation of both magnitude and phase.

During a control design, it is important to find out what kind of controller is more appropriate for the situation. In particular, we have some requirements on closed-loop "final" system that we want to convert them into constraints on open-loop system. These requirements are t_r, raising time, t_s, settling time, that give a constraint on crossing frequency ω_c, and overshoot M_p that determines the phase margin φ_m. We have freedom to design $C(s)$ as long as these specifications are satisfied.

The transfer function of open-loop system is $G(s) = C(s)P(s)$, where $P(s)$ is known. Therefore the magnitude and phase of $G(j\omega)$can be written as follows:

$$|G(j\omega)|_{dB} = |C(j\omega)|_{dB} + |P(j\omega)|_{dB}$$

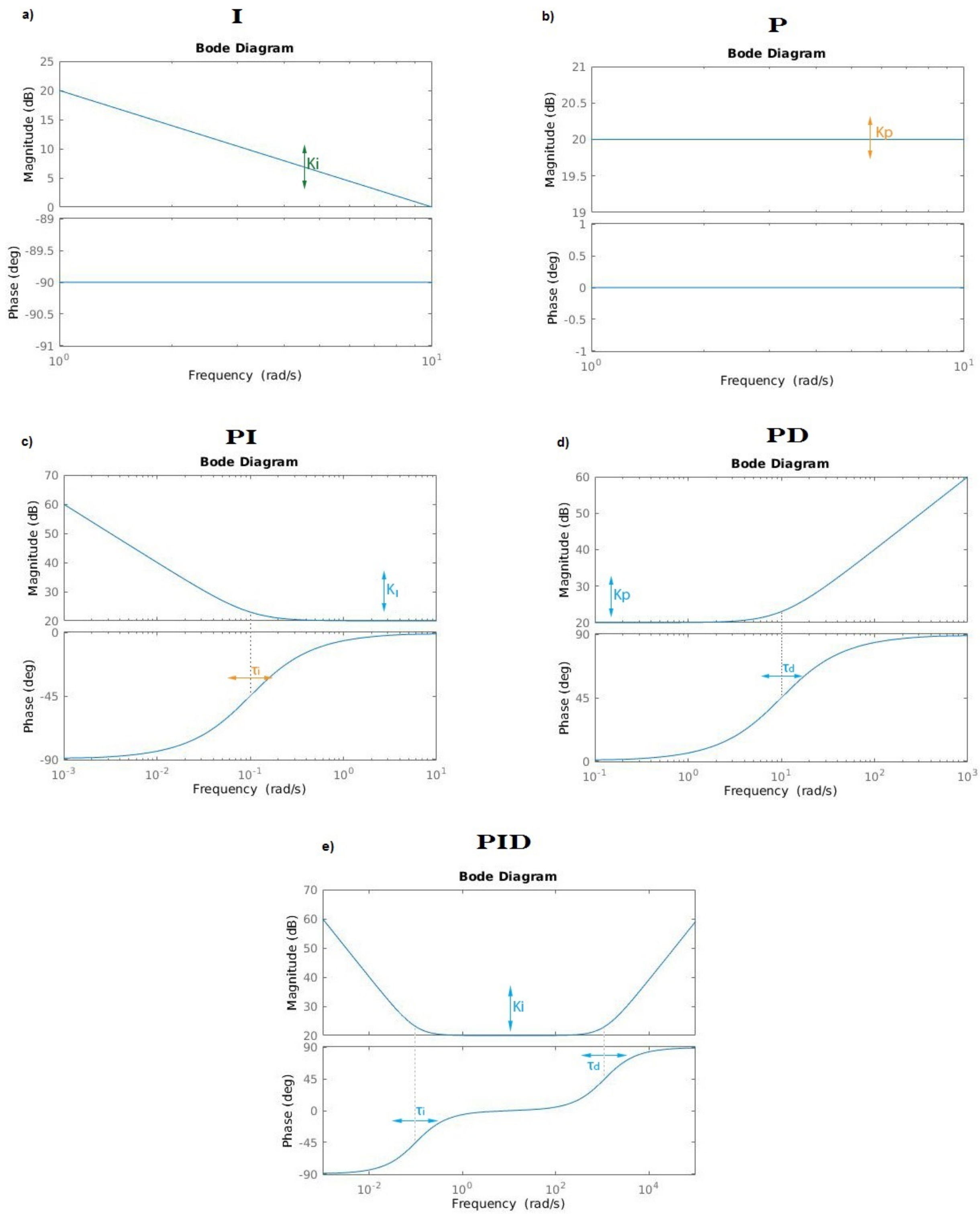

Figura 7.2. Bode diagrams of controllers

$$\angle G(j\omega) = \angle C(j\omega) + \angle P(j\omega)$$

Therefore in order to have a desired crossing frequency ω_c^* and phase margin φ_c^* for the open-loop transfer function $G(s)$ can be mathematically written $|G(j\omega_c^*)| = 1$, and $\varphi_m^G(\omega_c^*) = \varphi_m^*$, where $\varphi_m^G(\omega)$ is defined as follows:

$$\varphi_m^G(\omega) = 180^o + \angle G(j\omega)$$

We can consider for example a transfer function

$$P(s) = \frac{10}{(s+1)^3},$$

whose Bode diagrams (figure 7.3) give monotonically decreasing curves both for magnitude

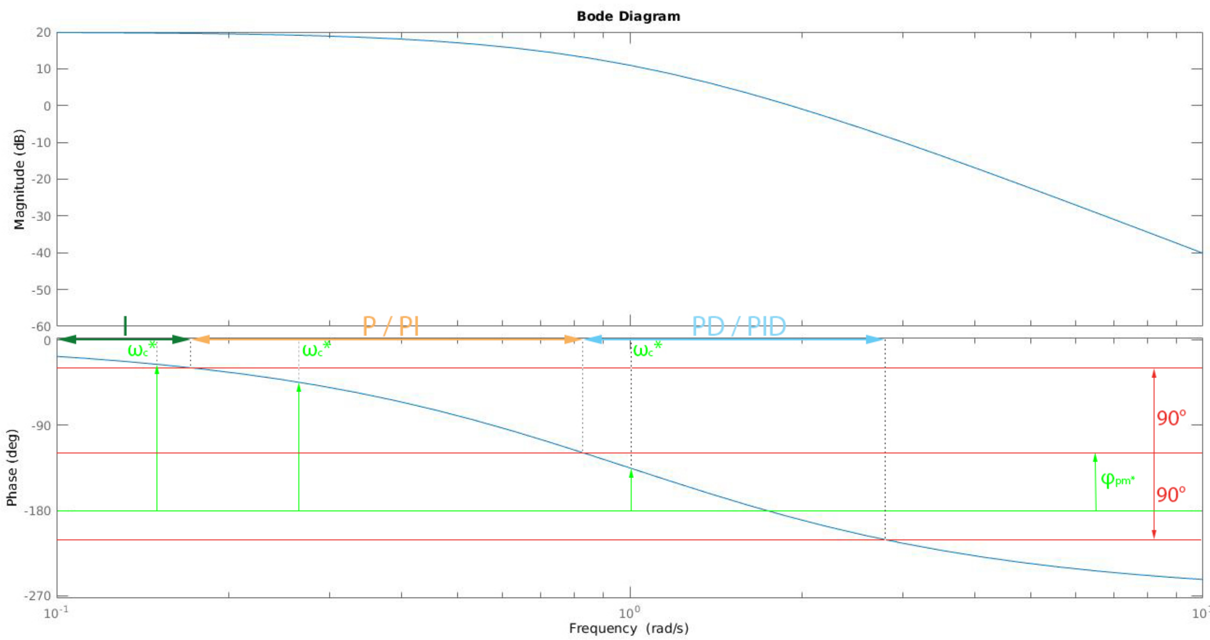

Figura 7.3. Bode diagrams of P(s)

and for phase. As a design requirement, we suppose to want a phase margin $\varphi_m^* = 60°$, and crossing frequency ω_c^* derived from the constraint on the settling time: $\omega_c^* = \frac{4.6}{t_s^*}$. The problem is to find a type of controller that can work at an admissible ω_c but that guarantees also phase margin specification. We can proceed by drawing a straight line on phase diagram which correspond to -180°(green line in Figure 7.3), and then draw a line corresponding to -180°+φ_m^* (pink line in Figure 7.3).

We suppose now to use an Integral controller, and we know that it introduces a change on phase in open loop system of -90°. So we can represent $\angle G(j\omega) = \angle C(j\omega) + \angle P(j\omega)$ simply considering same phase of $P(j\omega)$ but shifting up pink line of 90°: we obtain the red line on top, that represents phase margin constraint for our I controller. The point where

phase curve (blue) crosses the red line is an upper bound for ω_c^*: so Integral controller is suitable for the application if and only if system can work at a certain $\omega_c^* < \omega_c^I$, or in other words if phase curve is above the red line.
In case this solution is not sufficient to satisfy the specification on the phase margin, we have to resort to P or PI controllers. Proportional control leaves the phase intact, i.e. $\angle G(j\omega) = \angle P(j\omega)$, while a PI controller can modulate the phase of the open-loop phase between zero and $+90^o$, i.e. by properly choosing K_I and K_P we can have $\angle P(j\omega) - 90^0 \leq \angle G(j\omega) \leq \angle P(j\omega)$, which can be achieved if $\omega_c^I \leq \omega_c^* < \omega_c^P$, where ω_c^P is the frequency for which $P(j\omega_c^P) = \varphi^* - 180^0$.

Finally, to extend bounds of acceptable cross frequencies, we can apply a PD or PID controller: they both shift up phase $\angle G(j\omega)$ so their effect is representable on diagram with the lower red line, i.e. we can choose the parameters K_P, K_D and possibly K_I to have $\angle P(j\omega) \leq \angle G(j\omega) \leq \angle P(j\omega) + 90^0$, , which can be achieved if $\omega_c^P \leq \omega_c^* < \omega_c^D$, where ω_c^D is the frequency for which $P(j\omega_c^D) = \varphi^* - 270^0$. Summarizing we have

$$\begin{aligned} \omega_c^* < \omega_c^I &\quad \text{I} \\ \omega_c^I \leq \omega_c^* < \omega_c^P &\quad \text{P,PI} \\ \omega_c^P \leq \omega_c^* < \omega_c^D &\quad \text{PD,PID} \\ \omega_c^* \geq \omega_c^D &\quad \text{not feasible} \end{aligned}$$