Control Laboratory:	**a.a. 2015/2016**

Lecture 6 — 15th March

Instructor: Luca Schenato Scribes: Nicola Bastianello, Matthias Pezzutto, Enrico Picotti

6.1 Control design in frequency domain

This design method can be used only for systems with no poles on right-half of the complex plane, i.e. with $\Re[p_i] \leq 0$.

6.1.1 Analysis of second order system $W_{II}(s)$

In the frequency domain a second order system can be described as

$$W_{II}(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where $0 \leq \xi \leq 1$ is the *damping coefficient* and ω_n is the *natural frequency*.
From this description we can plot the poles in the complex plane:

$$\Im[p_1] = -\Im[p_2] = \omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\Re[p_1] = \Re[p_2] = \sigma = \xi\omega_n$$

$$\sin\theta = \xi$$

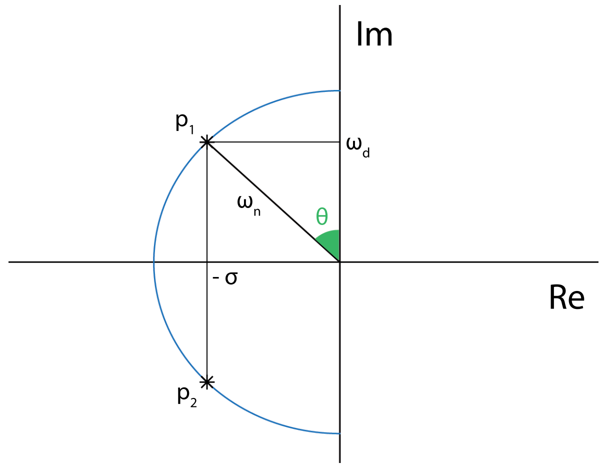

Figura 6.1. Poles in the complex plane

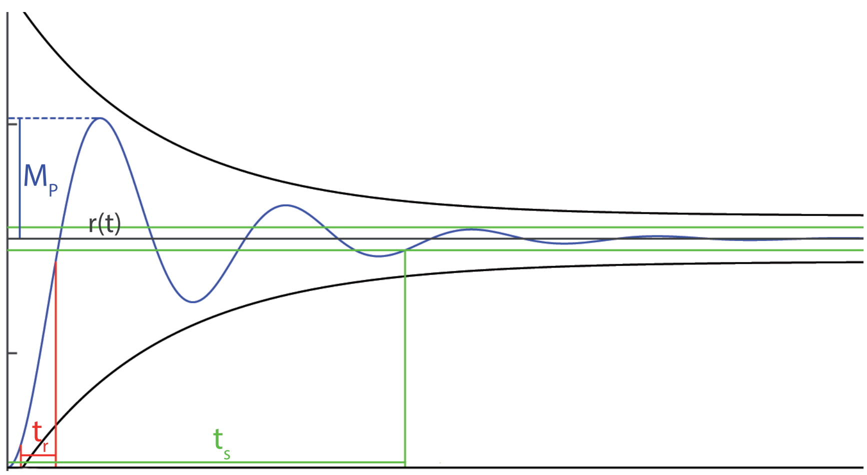

Figura 6.2. Step response of a second order system with raising time (t_r), settling time ($t_{s,1\%}$) and overshoot (M_p)

For this kind of systems the step response looks like 6.2, where we can define:

- t_r, the time that the step response takes to change from 10% to 90% of the reference signal, called *raising time*;
- $t_{s,1\%}$, the time that the step response takes to enter and remain within $\pm$ 1% of the reference signal, called *settling time*;
- M_p, the the maximum peak value of the step response measured from the reference signal, called *overshoot*.

6.1.2 Relationship between the time domain specs and the frequency domain parameters

The relationship between the raising time, the settling time and the overshoot with the natural frequency and the damping coefficient are:

- $t_r = \frac{1.8}{\omega_n}$
- $t_{s,1\%} = \frac{4.6}{\xi\omega_n}$
- $M_p = e^{-\pi \frac{\xi}{\sqrt{1-\xi^2}}}$

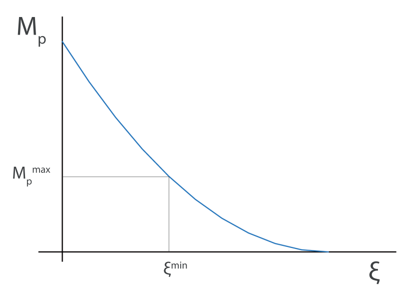

Figura 6.3. Overshoot M_p as a function of damping coefficient ξ

Given a set of required specifications $t_r^{max}, t_{s,1\%}^{max}$ and M_p^{max}, we can use these formulas to derive an area in the complex plane where to place the poles in order to satisfy the required specifications:

- $t_r \leq t_r^{max} \Rightarrow \omega_n \geq \frac{1.8}{t_r^{max}} = \omega^{min}$
- $t_{s,1\%} \leq t_{s,1\%}^{max} \Rightarrow \sigma \geq \frac{4.6}{t_{s,1\%}^{max}} = \sigma^{min}$
- $M_p \leq M_p^{max} \Rightarrow \xi \geq \xi^{min}$ (as we can inferred from Fig. 6.3) $\Rightarrow \theta \geq \theta^{min}$

By considering all these conditions simultaneously, we can identify the *performance area* as the orange area in figure 6.4. We can see that:

- all points outside the circle of radius ω_n^{min} have natural frequency grater than ω_n^{min} and so the systems is sufficiently fast;
- all points at the left at vertical line in $-\sigma_{min}$ have real part in absolute value greater than $-\sigma_{min}$ and then the settling time is small enough;

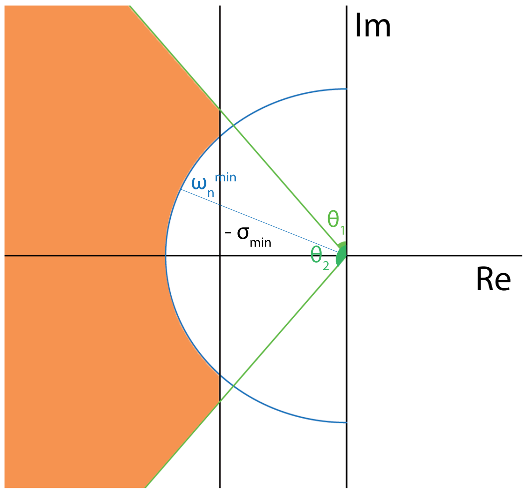

Figura 6.4. Area where we can place the poles to obtain the desired raising time, settling time and overshoot

- all points in the cone between $-\theta^{min}$ and θ^{min} satisfy the conditions on overshoot.

The position of the poles changes the *resonance peak* M_r and *resonance frequency* ω_r (see figure 6.5). In fact the resonance frequency is $\omega_r = \omega_n\sqrt{1-2\xi^2}$ and so, if we increase ω_n, we move the peak of resonance to the right. The height of the peak depends on ξ. If $\xi \geq \frac{\sqrt{2}}{2}$ there is no resonant peak, while the peak M_r tends to infinity as ξ tends to 0.

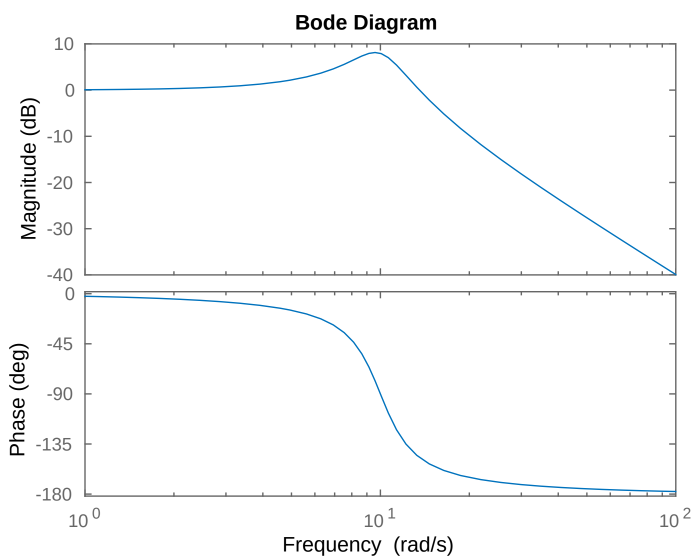

Figura 6.5. Bode plot of second order system

6.1.3 From conditions on closed-loop transfer function to conditions on open-loop transfer function

The second order system $W_{II}(s)$ is the system that is obtained from the feedback on system $G(s)$, the open-loop system, which includes the plant and the controller.

$$W_{II}(s) = \frac{G_{II}(s)}{1 + G_{II}(s)} \Rightarrow G_{II}(s) = \frac{\omega_n^2}{s(s + 2\xi\omega_n)}$$

The conditions on ω_n and ξ of system $W_{II}(s)$ can be expressed as condition on the *phase margin* φ_m and *crossing frequency* ω_c of the system $G_{II}(s)$. Using a change of variable $s \to s' = \frac{s}{2\xi\omega_n} \Rightarrow s = 2\xi\omega_n s'$ we obtain

$$G_{II}(s') = \frac{\omega_n^2}{2\xi^2\omega_n s'(s'+1)} = \frac{1}{2\xi^2 s'(s'+1)}$$

Nyquist plots of $G_{II}(s)$ and $G_{II}(s')$ are the same, so φ_m is the same for both transfer functions. The relationship between φ_m and ξ is

$$\varphi_m = \operatorname{atan}\left(\frac{2\xi}{\sqrt{1+4\xi^4} - 2\xi^2}\right)$$

as depicted in Fig. 6.6.

For what concern the crossing coefficient we will consider the approximation $\omega_c \simeq \omega_n$.

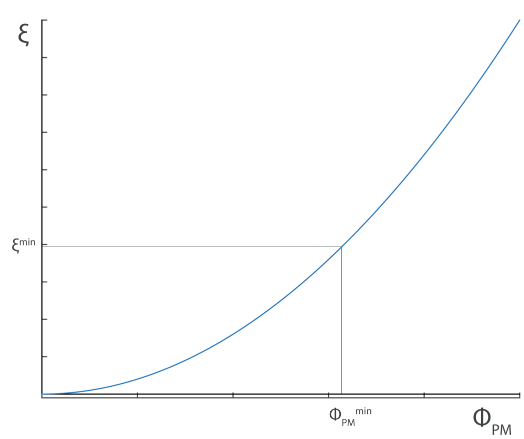

Figura 6.6. Damping coefficient ξ as a function of phase margin φ_m

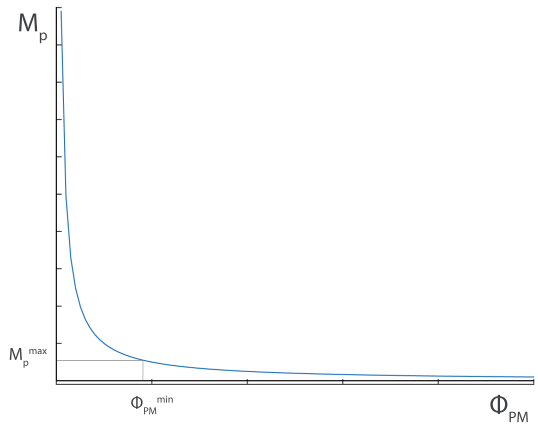

Figura 6.7. Approximate dependence of overshoot M_p as a function of the phase margin φ_m

6.1.4 Conclusions

The damping coefficient ξ as a function of φ_m of $G_{II}(s)$ is monotonically increasing, so we have to achieve a sufficiently great φ_m in order to get ξ and $t_{s,1\%}$ for the closed-loop system that satisfy the specifics.
As we have seen, raising time and natural frequency satisfy $t_r = \frac{1.8}{\omega_n}$ and, assuming $\omega_n = \omega_c$, $t_r = \frac{1.8}{\omega_c}$; therefore the bigger is the crossing frequency the faster is the response of closed-loop system.
For the overshoot, we know that M_p is monotonically decreasing with ξ and ξ is monotonically increasing as a function of φ_m of $G_{II}(s)$, therefore M_p is monotonically decreasing with φ_m (see picture 6.7). As so the larger the phase margin is, the smaller the overshoot is.

6.2 Design of a PID controller

The open-loop transfer function is $G(s) = C(s)P(s)$ and we want $C(s)$ such that ω_c and φ_m of $G(s)$ are as desired.
The PID controller has a proportional part, an integral part and a derivative part (see picture 6.8); the transfer function is $C(s) = K_P + \frac{K_I}{s} + K_D s$.
The design's purpose is to determine the gains (K_P, K_I, K_D) such that the closed-loop system satisfies the required specifics $(t_r, t_{s,1\%}, M_p)$, from which we can derive, as written before, $(\omega_c^*, \varphi_m^*)$. In order to change the ω_c we will change the magnitude of C(s) while in order to change φ_m we will change the phase of C(s). Magnitude and phase of $G(s)$ con be expressed as:

- $|G(j\omega)|_{dB} = |C(j\omega)|_{dB}|P(j\omega)|_{dB}$
- $\underline{/G(j\omega)} = \underline{/C(j\omega)} + \underline{/P(j\omega)}$

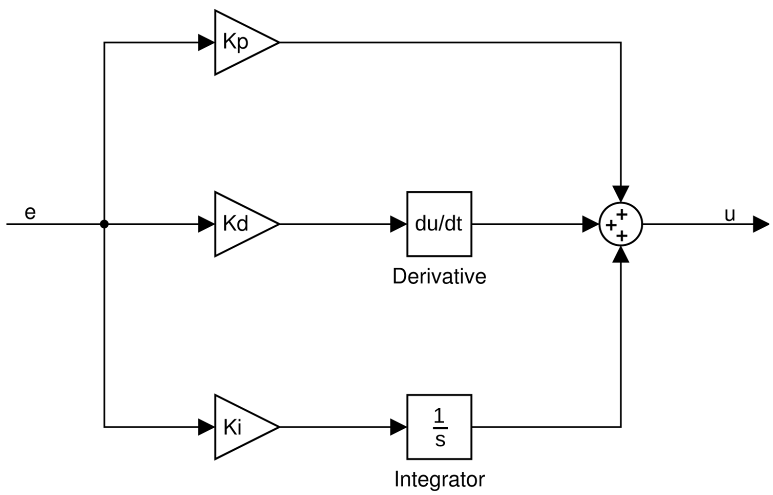

Figura 6.8. PID's scheme

The three components of the PID are:

- C(s) $= K_P$ (see figure 6.9);
- C(s) $= \frac{K_I}{s}$ (see figure 6.10);
- C(s) $= K_D s$ (see figure 6.11);

Each component of a PID modifies in a different way the magnitude and the phase of the open-loop system. In the following table we can see the pros ant the cons for using one of the components.

	PROS	CONS
Proportional	- choose ω_c	- no control on φ_m
Integral	- guarantees zero steady state error - removes possible constant input disturbances	- reduces φ_m
Derivative	- augment φ_m	- amplifies measurement noise

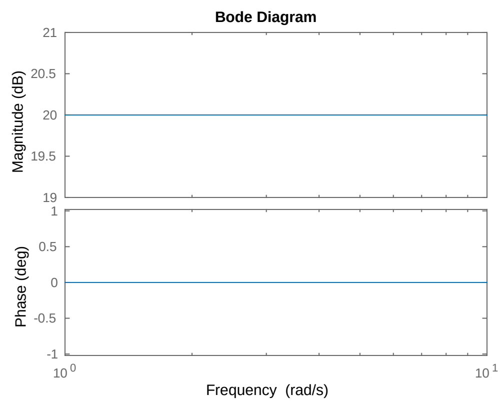

Figura 6.9. Bode plot of a proportional controller

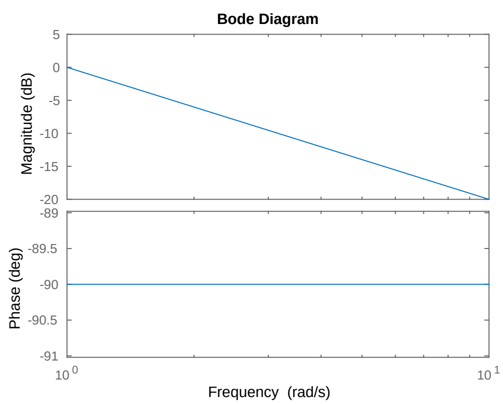

Figura 6.10. Bode plot of a integrative controller

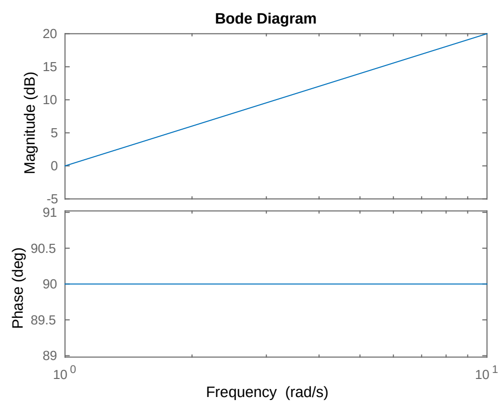

Figura 6.11. Bode plot of a derivative controller