Control Laboratory:	**a.a. 2015/2016**
Lecture 4 — 8 March	
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4.1 Zero Order Holder Model

A ZOH is a DAC (Digital to Analog Converter) that has a digital signal $u(k)$ as input, and an analog output $y(t)$ defined by keeping constant $y(t)$ at the value of $u(k)$ in the time interval between $(k-1)T$ and kT.

$$y(t) = u(k) \qquad (k-1)T \leq t < kT \tag{4.1}$$

It is possible to find ZOH's impulse response $y_{ir}(t)$, considering it as a continuous time system (i.e. with an analog signal *u(t)* as input),

$$y(t) = u(kT) \qquad (k-1)T \leq t < kT \tag{4.2}$$

If $u(t) = \delta(t)$, $y(t) = y_{ir}(t)$ is the impulse response of the system. From the impulse response we can evaluate the transfer function of the ZOH,

$$F(s) = \int_{-\infty}^{+\infty} y_{ir}(t) e^{-st} dt = \int_0^T e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_{t=0}^{t=T} = -\frac{1}{s} e^{-sT} + \frac{1}{s} = \frac{1-e^{-sT}}{s} \tag{4.3}$$

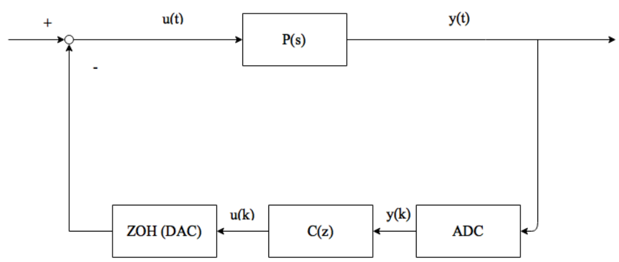
This transfer function is not rational, so it not possible to use it in the modeling of the control system as an linear time invariant system (LTI). As so, it is necessary to find a rational approximation for it. If T is assumed to be small we can approximate $F(s)$ with a rational transfer function:

$$F(s) \simeq \frac{1-(1-sT+\frac{1}{2}s^2T^2)}{s} = \frac{sT-\frac{1}{2}s^2T^2}{s} = T - \frac{1}{2}sT^2 = T\left(1-\frac{1}{2}sT\right) \tag{4.4}$$

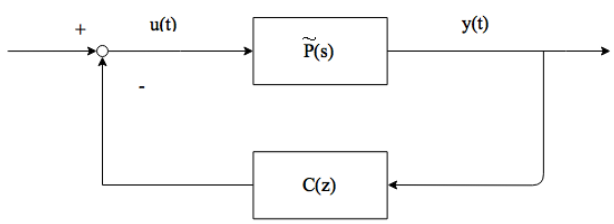
that is rational, but not proper yet. So we use a further approximation, valid for small values of T:

$$F(s) \simeq T\left(\frac{1}{1+\frac{1}{2}sT}\right) \tag{4.5}$$

In the last lecture we approximated the transfer function of a delay in the same way, so it is possible to approximate **ADC** (sensors) and **DAC** (ZOH) with the same transfer function. In conclusion we can see the real control scheme:



in this way:



where

$$\tilde{P}(s) = P(s)\frac{T}{1+\frac{T}{2}s}\frac{T}{1+\frac{T}{2}s} \tag{4.6}$$

obtained by using an approximated LTI model for sensors, actuators and ZOH.

4.2 Feedforward Control

We now consider the problem of tracking a reference signal *r(t)* by properly designing a controller $C(s)$. To achieve this goal, it is possible to use a feedforward control scheme as shown in Fig.4.1.
For example if $P(s) = a$, where $a \in \mathbb{R}$, we can take $C(s) = \frac{1}{a}$, $y(t) = au(t)$, $u(t) = \frac{1}{a}r(t)$ and so $y(t) = r(t)$.

It is important to point that there are some important problems using this control scheme:

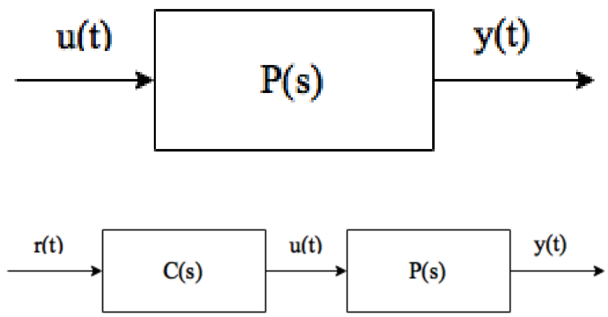


Figura 4.1. Feedforward control

1. **Plant is not perfectly known**
   We do not know the exact structure model of the plant or it might be overly complex, but we only can build a model that approximates the behavior of the real plant. For this reason there is uncertainty on the value of the system's parameters. In the example above, this fact means that we only know a nominal value for the plant a_0 but not the exact value a that describes the actual plant. So it is impossible to perfectly track the reference signal and the output will be $y(t) = \frac{a}{a_0} r(t)$, with $a_0 \neq a$.

2. **Presence of unknown external disturbance**

   If we additionally consider an external unknown input disturbance, as an example $d(t) = d(const.)$

   $$\begin{cases} y(t) = a(u(t) + d) \\ u(t) = \dfrac{1}{a_0} r(t) \end{cases} \qquad \rightarrow \qquad y(t) = \frac{a}{a_0} r(t) + ad \tag{4.7}$$

   where ad is the effect of external disturbance.

So if we use a feedforward control scheme in order to track a reference signal, it is not possible to track it perfectly because of the presence of a steady state error.

4.3 Feedback control

Instead of feedforward control we can use feedback control in reference signal tracking, whose goal is to sense and drive the tracking error $e = r - y$ to 0.

$$e(t) = r(t) - y(t) \rightarrow 0 \tag{4.8}$$

We consider the same case of the previous example, and we build a feedback control scheme using $C(s) = \frac{K}{s}$ (Integral controller), where K is the gain.

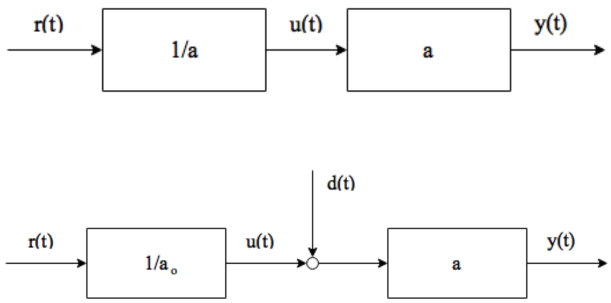


Figura 4.2. Ideal model (top) and more realistic model (bottom).

Defining the Laplace transforms of signals as follow

$$y(t) \to Y(s) \qquad u(t) \to U(s) \qquad e(t) \to E(s) \qquad r(t) \to R(s)$$

$$\begin{cases} Y(s) = aU(s) \\ U(s) = C(s)E(s) \\ E(s) = R(s) - Y(s) \end{cases} \tag{4.9}$$

$$Y(s) = \frac{aK(R(s) - Y(s))}{s}$$

$$(1 + \frac{aK}{s})Y(s) = \frac{aK}{s}R(s)$$

$$Y(s) = \frac{\frac{aK}{s}}{1 + \frac{aK}{s}}$$

$$Y(s) = \frac{1}{1 + \frac{s}{aK}}R(s) \qquad \text{where} \quad P_{cl}(s) = \frac{1}{1 + \frac{s}{aK}}$$

So we can see the whole system as a single dynamical system with transfer function $P_{cl}(s)$ Now we are going to analyze how this control strategy manages the presence of an external disturbance.

$$\begin{cases} Y(s) = a(U(s) + d) \to Y(s) = a(C(s)E(s)) + aC(s)d \\ U(s) = C(s)E(s) \\ E(s) = R(s) - Y(s) \end{cases} \tag{4.10}$$

and so

$$Y(s) = P_{cl}(s)R(s) + \frac{\frac{s}{K}}{1 + \frac{s}{aK}}d \quad where \quad P_d(s) = \frac{\frac{s}{K}}{1 + \frac{s}{aK}}$$

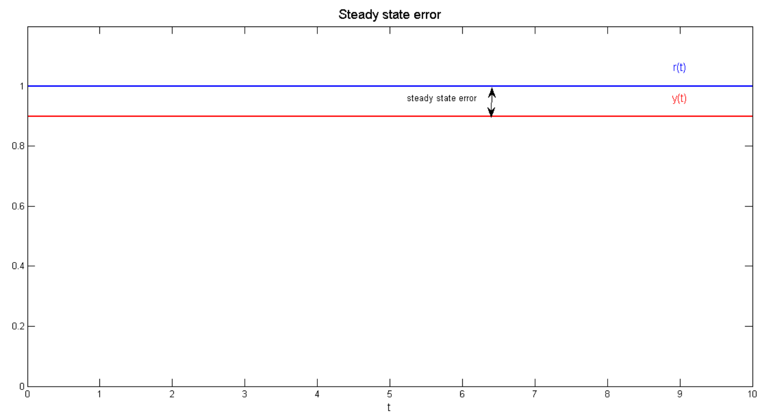
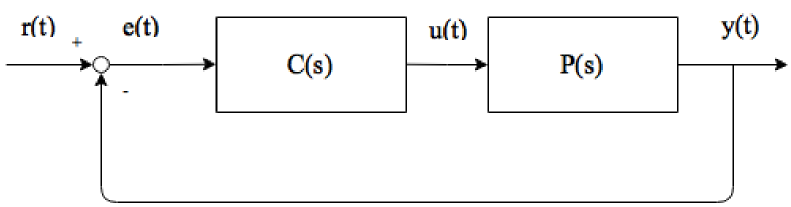
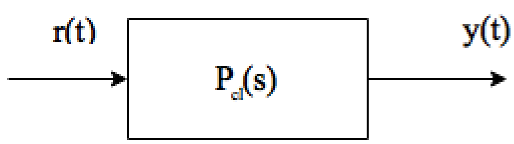
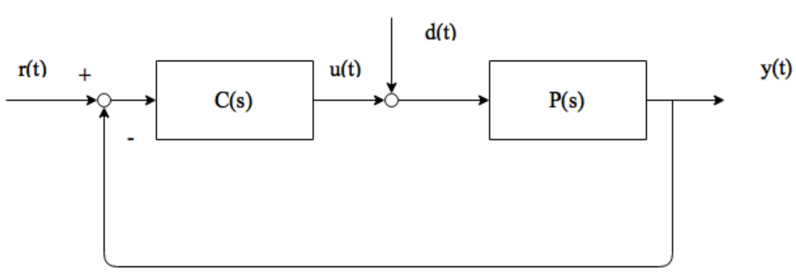


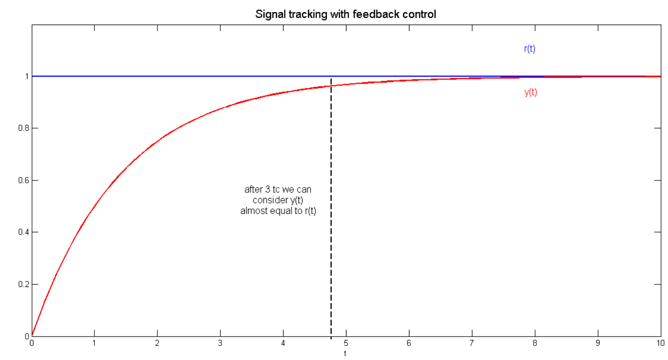
Figura 4.3. Steady State Error tracking a step reference







We assume that $K > 0$ to have stability in the closed-loop transfer function, and now we consider $d(t) = 0$ and we take a step reference in input.



Under the assumption of stable closed loop system, as it is the case if $K > 0$, then using the *final value theorem* we get $y(\infty) = P_{cl}(0) = 1 = r(\infty)$. If we consider $d(t) = d$ (constant), using the same theorem we can always that the DC gain from the inout disturbance and the output is zero since $P_d(0) = 0$. So it is possible to assume $y(t) \approx r(t)$ after a certain amount of time of the order of $3t_c$, where $t_c := \frac{1}{Ka}$.

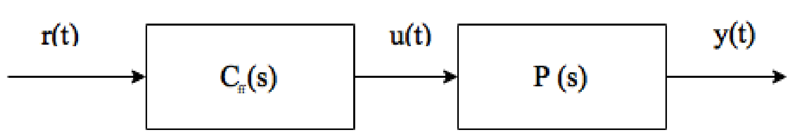
We can compare the two illustrated control strategy:

- *Feedback control* needs time to reach the reference value, introducing delay in this way.
- *Feedforward control* is immediate, it does not introduce any delay, but it might not reach the reference value and input disturbances appears at the output.

Feedback is important to use when there are some uncertainties we can not handle directly.

4.4 Feedforward and Feedback control

It is possible to use the two control schemes together in order to achieve better performances, employing the speed of the feedforward control to have a smaller rising time, and the feedback control to reach stability.

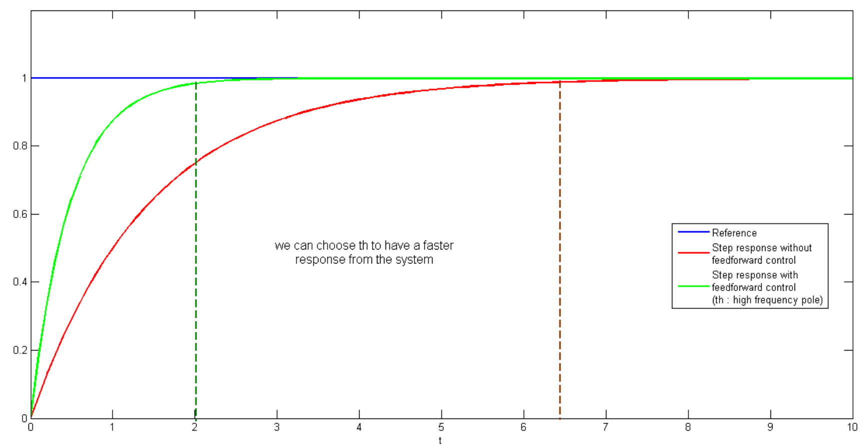


$$Y(s) = P(s)C_{FF}(s)R(s)$$

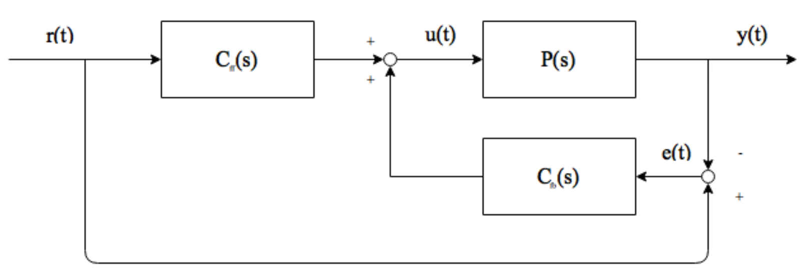
If we have to control a plant with transfer function P(s), first of all we build a feedforward controller $C_{FF}(s) = P(s)^{-1}$, eventually adding high frequncy poles (in this case $(1 + t_H s)$) to make the control transfer function proper.

$$P(s) = \frac{1}{1+s} \quad \rightarrow \quad C(s) = \frac{1+s}{1+t_H s}$$

$$Y(s) = \frac{1}{1+s}\frac{1+s}{1+t_H s}R(s) = \frac{1}{1+t_H s}R(s)$$

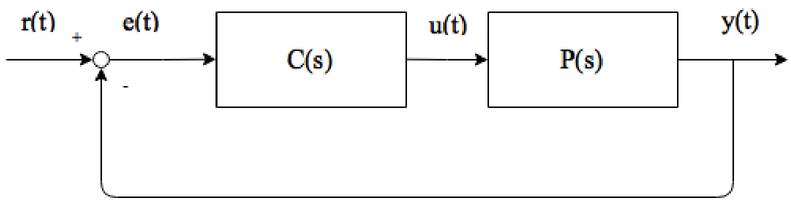


In this way we use the feedforward control to take $y(t)$ near $r(t)$ in very small time lapse and then we manage the error with the proper feedback control. It's important to choose a feedforward control not too much powerful because it may compromise stability.



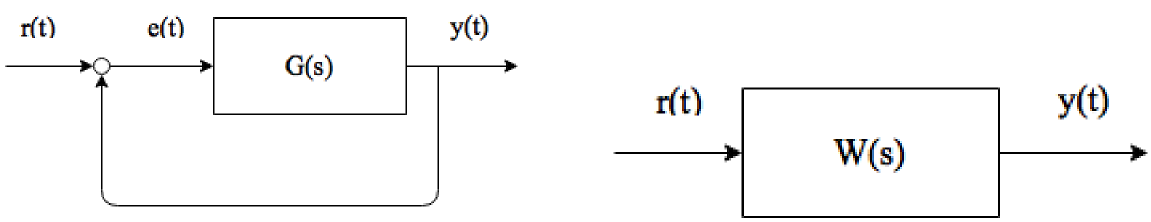
4.5 Frequency domain control design of closed-loop system

One of the most used technique for closed-loop control system design is based on the frequency response. This method offers satisfying results, also when there are uncartainties on parameters and it is simple to handle employing instruments like *Root Locus*, *Nyquist* and *Bode's plot*.



$$G(s) = C(s)P(s) \qquad open - loop \quad system \quad dynamics$$

$$W(s) = \frac{G(s)}{1 + G(s)} \qquad closed - loop \quad system \quad dynamics$$



4.5.1 Root Locus design

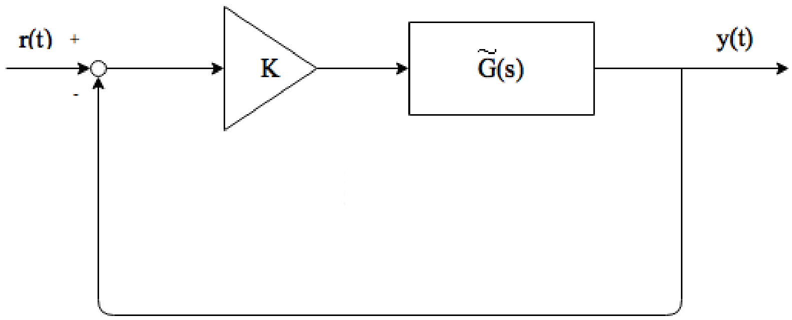
With this method we are going to see how poles of the closed-loop system move when the gain is changing.

$$G(s) = K\tilde{G}(s)$$

where K is the gain and $\tilde{G}(s) = \frac{n(s)}{d(s)}$ a proper rational function.

$$W(s) = \frac{K\tilde{G}(s)}{1 + K\tilde{G}(s)} \rightarrow 1 + K\tilde{G}(s) = 0 \rightarrow d(s) + Kn(s) = 0$$

Depending on the $G(s)$, it could happen that there are values of K, for which the closed-loop system is stable, and other values for which the system is unstable.
K_c is a critical gain value, for this value the locus crosses the imaginary axe (at frequency ω_c) and switch between stability and instability. In correspondence to this gain $1 + KG(jw)$ has two imaginary roots in $\pm j\omega_c$ so the closed-loop system has two imaginary poles that cause instability.



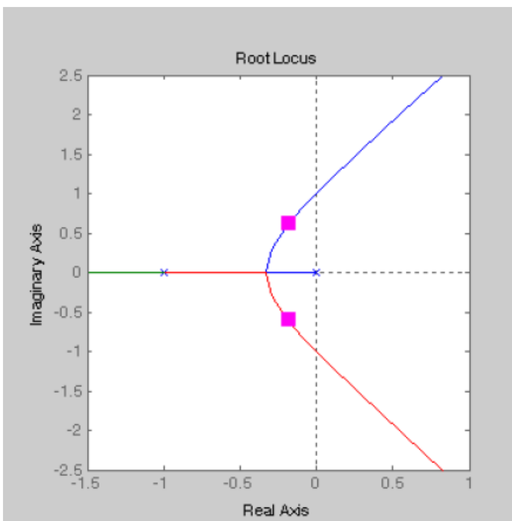


Figura 4.4. Root locus obtained from $\tilde{G}(s) = \frac{s}{s(s+1)^2}$

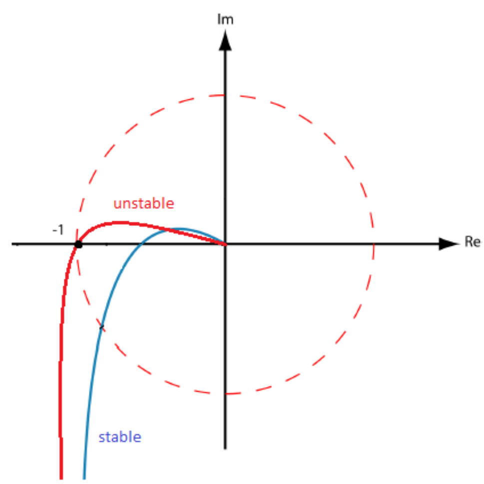


Figura 4.5. Nyquist plot

4.5.2 Nyquist plot

In this diagram we represent $G(j\omega) \in \mathbb{C}$ in function of the frequency ω. For what we have seen before, if $K\tilde{G}(j\omega) = -1$, for that gain there is instability. There is a criterion based on this plot, that says if a closed-loop system is stable or not. This is the Nyquist's stability criterion, and using it, it is possible to declare if a system is stable or not only knowing the number of unstable poles of $G(s)$ and looking at its Nyquist plot.

Stability is very important on the control design, but it is not sufficient, since we have to choose a controller that also provide some performance guarantees for the closed loop system. Nyquist is important also for that purpose, because by looking at it, it is possible to estimate the behavior of the closed-loop system.