Control Laboratory:	**a.a. 2015/2016**

Lezione 3 — 03 March

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3.1 Approximation of stable systems

Given a dynamical system which is assumed to be stable:

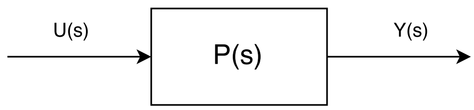

Figure 3.1. Dynamical systems assumed to be associated to a closed loop system

if $P(s) = \frac{n(s)}{d(s)}$, the roots of the denominator $d(s)$ are called poles, and have the property that $\Re[p_i] < 0$. The roots of numerator $n(s)$ are called zeros. Both poles and zeros could be rappresented in the complex plan as seen in figure 3.2

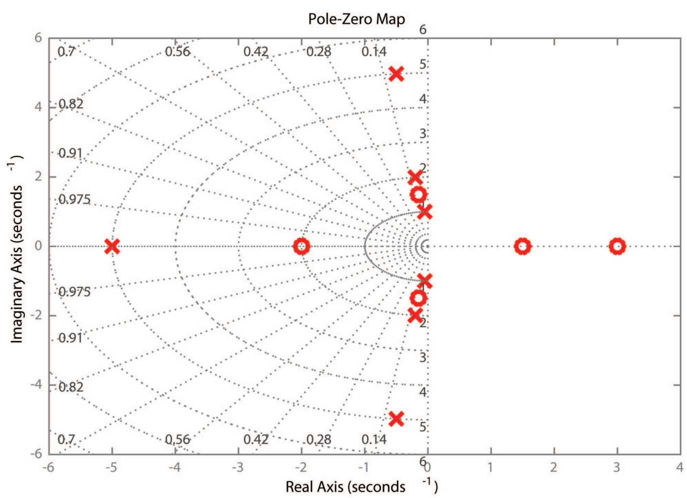

Figure 3.2. Example of zeros (circles) and poles (crosses) in the complex plan

Considering as input $u(t) = \mathbf{1}(t)$, the step function, with Laplace trasformation $\mathcal{L}\{u(t)\} = U(s) = \frac{1}{s}$, then the corrispective force output will be $Y_f(s) = P(s)U(s)$. The inverse Laplace transform of the force output (considering it strictly proper) will be:

$$y_f(t) = \alpha_0 \mathbf{1}(t) + \sum_{i=1}^{n} \alpha_i t^{l_i - 1} e^{p_i t} \tag{3.1}$$

with α_i that depends of both poles and zeros (the coefficients of $n(s)$ and $d(s)$). NB (α_i and $e^{p_i t}$ are complex number, but $y(t) \in \mathbb{R}$)

In a stable system, $\sum_{i=1}^{n} \alpha_i t^{l_i - 1} e^{p_i t} \to 0$ for $t \to \infty$, but all the therms of the series will go to 0 with different rates, depending how negative is p_i in the complex plan: given two poles p_i and p_j if $\mathrm{Re}[p_i] < \mathrm{Re}[p_i]$ then $\alpha_i e^{p_i t}$ will go to zero faster then $\alpha_j e^{p_j t}$. The pole(s) associated with the slowest rate of convergence it is(are) called dominant pole(s). The purpose is to simplify the entire system using the dominant pole approximation.

3.1.1 Dominant pole approximation

The idea behind this approximation is to reduce the system using only the two dominant poles and removing the other poles or zeros of the function. Let us consider the two dominant poles: $p_1 = \bar{p_2} = \sigma + j\omega$ with $\sigma < 0$, then we can write equation (3.1) as:

$$y_f(t) = \alpha_0 1(t) + \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \sum_{i=3}^{n} \alpha_i t^{l_i - 1} e^{p_i t} \simeq \alpha_0 1(t) + \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t}$$

this approximation coincide with a second order system with a transfer function like

$$P_{\mathrm{II}}(s) = \alpha_0 \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where ω_n is the natural frequency, ξ is the damping ratio, and α_0 is the DC gain.

3.1.2 Example: approximation of a transfer function

Given a transfer function in explicit form, it can be written using Bode form:

$$P(s) = K_B \frac{(1 + \tau_1^z s)(1 + 2\xi_2^z \frac{s}{\omega_2^z} + \frac{s}{\omega_2^z}^2)(1 + ...)}{(1 + 2\xi_1^p \frac{s}{\omega_1^p} + \frac{s}{\omega_1^p}^2)(1 + 2\xi_2^p \frac{s}{\omega_2^p} + \frac{s^2}{\omega_2^p}^2)(1 + ...)}$$

considering the dominant pole in the first term of the denominator, and approximating:

$$P(s) \simeq K_E \frac{1}{(1 + 2\xi_1^p \frac{s}{\omega_1^p} + \frac{s}{\omega_1^p}^2)}$$

The rationale behind this approximation is that the components dues to the faster modes will quickly become negligible as compared to the slowest ones. A simulation between a full order system and his dominant poles approximation system, shows that the approximated system in addition to being simpler also has similar performance since the neglected poles and zeros are substantially far away from the dominant poles.

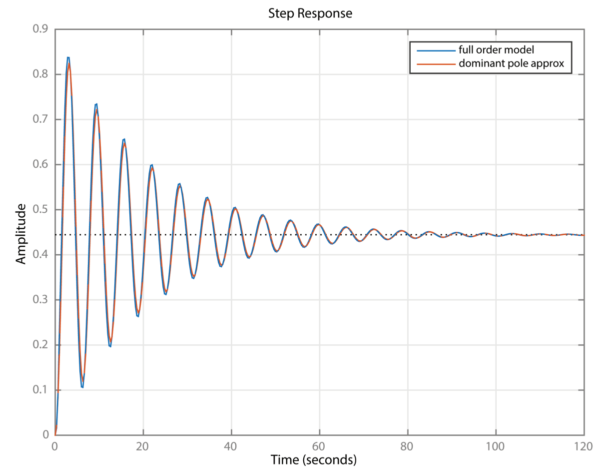

Figure 3.3. Comparison between a full order model and a dominant pole approximation

3.1.3 Frequency representation

In order to simplify the analysis we will consider

$$K_E = 1, \quad 0 \leq \xi \leq 1$$

since K_E simply corresponds to a scaling of the output, and ξ is restricted in order to have stable complex dominant poles. To analyze the second order system performance, it can be useful to plot the transfer function in its frequency domain, by using the Bode plot. Figure 3.4 shows magnitude in dB and phase of the second order transfer function $\mathrm{P}_{II}(j\omega)$. The magnitude is given by:

$$\mid P_{II}(j\omega) \mid = \frac{\omega_n^2}{\mid -\omega^2 + 2j\xi\omega_n + \omega_n^2 \mid} = \frac{\omega_n^2}{\sqrt{(\omega^2 - \omega_n^2)^2 + 4\xi^2\omega_n^2}}$$

By computing its derivative, it is straightforward to show that the magnitude presents a *resonant peak* M_r at the so called *resonant frequency* ω_r for $\xi^2 < 1/2$:

$$M_r \;:=\; \max_{\omega} \mid P_{II}(j\omega) \mid = \frac{1}{2\xi\sqrt{1-\xi^2}}, \qquad \xi^2 < \frac{1}{2} \tag{3.2}$$

$$\omega_r \;:=\; \arg\max_{\omega} \mid P_{II}(j\omega) \mid = \omega_n\sqrt{1-2\xi^2} \tag{3.3}$$

If $\xi^2 \geq 1/2$, then no peak is present and the magnitude is monotonically decreasing. Note that for small ξ, then the resonant frequency is close to the natural frequency:

$$\xi \ll 1 \Rightarrow \omega_r \approx \omega_n, \quad \mid P_{II}(\omega_r) \mid \approx \frac{1}{2\xi} = \mid P_{II}(\omega_n) \mid$$

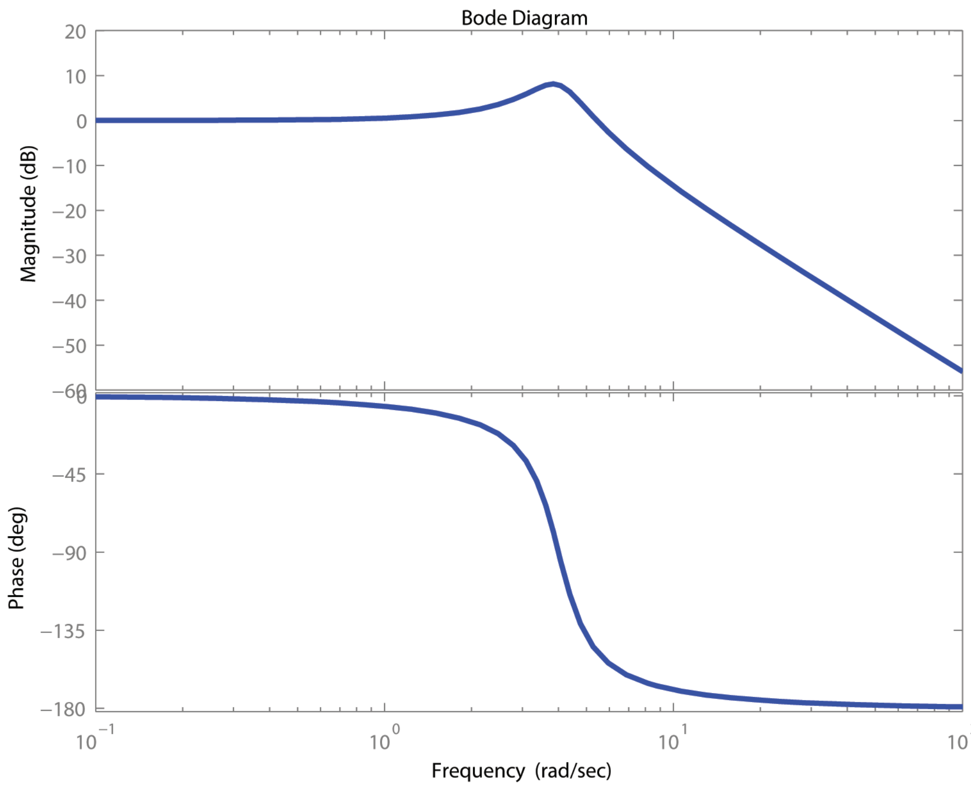

Figure 3.4. Bode plot of $P_{II}(j\omega_n)$

The only thing which is relevant looking at the Bode plot of a second order transfer function is the behavior of the system for values of ω up to a certain frequency. This fact can be explained by referring to two different systems which differ only at the high frequencies and observing that they will react in a similar way when they are connected in close loop.

3.1.4 Step response

Another approach to analyze the performances of a system is through three parameters defined in the time domain, which are raising time, settling time and overshoot.
Considering the input $u(t) = \mathbf{1}(t)$, we can easily compute the forced response $y_f(t)$ of the system using the inverse Laplace tranformation:

$$y_f(t) = \mathcal{L}^{-1}[Y_f(s)] = \mathcal{L}^{-1}[P_{II}(s)U(s)]$$

$$y_f(t) = \mathcal{L}^{-1}\left[\frac{\omega_n^2}{(s^2 + 2\xi\omega_n + \omega_n^2)} \cdot \frac{1}{s}\right] = 1 + [a\sin(\omega_d t) + b\cos(\omega_d t)]\, e^{-\sigma t}$$

where $\sigma = \omega_n \xi$ and $\omega_d = \omega_n\sqrt{1-\xi^2}$.

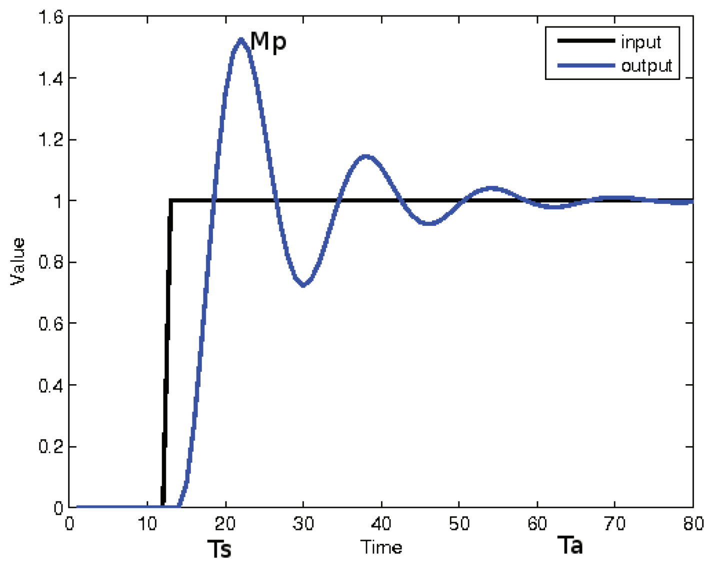

Figure 3.5. Step response of a second order system

In figure 3.5 are shown three parameters that are useful to give a complete description of the system's performances: t_a, t_s and M_p.

- $\mathbf{t}_a$ is called "raising time" and it indicates the time needed to the signal to go from 10% to 90% of its final value. The lowest is the value of t_a and the quickest is the system's response.

- $\mathbf{t}_s$ is called "settling time" and it represents the time spent to have the signal fully confined into a range of $\pm 1\%$ with refence to the final value. A small t_s means few fluctuations near the reference value.

- the last parameter is the "overshoot" $\mathbf{M}_p$, which denotes the maximum percentage of error between the signal and the reference. A large overshoot reflects in a first large deflection of the signal from the final value.

For a second order system we can find a relation between the system parameters (ω_n, ξ) and the performance specifications (t_a, t_s, M_p):

$$t_r \approx \frac{1.8}{\omega_n}, \qquad t_s \approx \frac{4.8}{\omega_n \xi}, \qquad M_p \approx e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

The first two parameters depend mostly on the frequency ω_n, instead M_p depends only on the coefficient ξ. They are all monotonically decreasing functions and in particular M_p, which is defined only for $0 \leq \xi \leq 1$, takes value from 0 to 1 because it represents a percentage.

3.1.5 Modeling of sensors and actuators

Let's consider the model of a plant (P_{phy}) affected by an actuator (A) and measured through a sensor (S) represented in Figure 3.6:

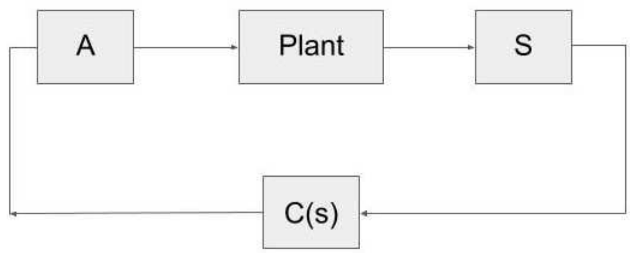

Figure 3.6. Model of a process with actuator (A), plant (P_{phy}) and sensor (S).

u = control signal (voltage, current,etc.)
û= physical dimension affecting the process (force, valve opening, etc.)
y = dimension of interest (temperature, joint position, etc.)
ỹ= y translated dimension (voltage, current, etc.)
A = actuator block
P_{phy} = plant
S = sensor block
The whole feed-forward transfer function is $P(s) = A(s)P_{phy}(s)S(s)$.

Sensors

A sensor or trasducer is a device that receives a physical signal as an inupt (for example the angular position of a motor) and provides an electrical signal as an output according to the input magnitude. The sensor can be modeled as a transfer function that receives continuous signals $y(t)$, and provides descrete quantized signals $y_m(kT)$ according to the sampling time T.

The output signal $y_m \simeq k_s y(kT)$ can assume a finite number of values depending on the quantization width Δ and the slope of the line to approximate k_s. If we consider $k_s = 1$ the quantization error is $|y(kT) - y_m(kT)| \leq \frac{\Delta}{2}$.

The uncertanty introduced by the quantization error is modeled as a quantization noise $V_q(t)$, wich is approximated as a white noise and has the following properties: $E[V_q(t)] = 0$, $E[V_q^2(t)] = \frac{\Delta^2}{12}$.

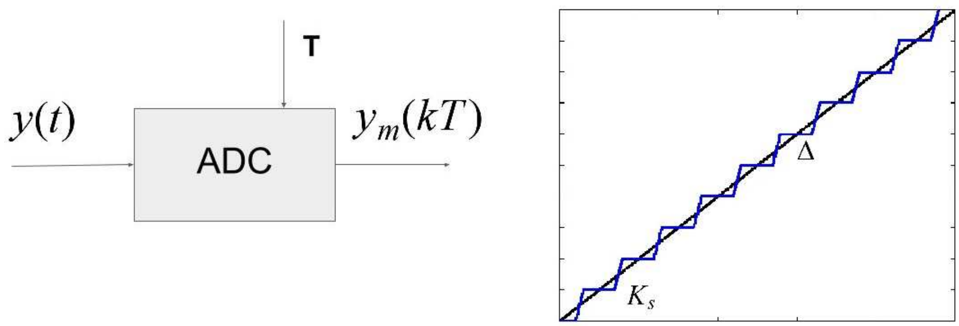

Figure 3.7. Sensor model on the left and quantizer on the right

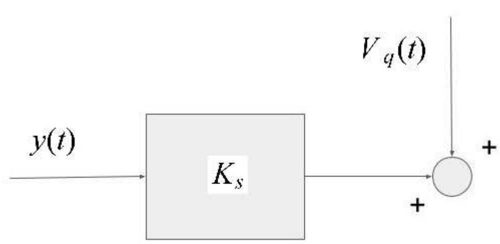

Figure 3.8. Quantization noise

The complete model of the sensor is represented by the following block diagram:

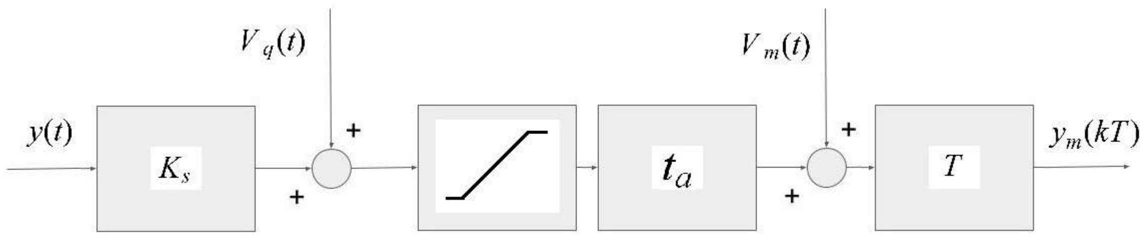

Figure 3.9. Sensor block diagram

The sensor takes some time to reach the reference value and produce the correct output, introducing a measurement delay t_a. A saturation block is placed in the model to bound the input signal, this block is not linear and produces the output

$$y'(t) = \begin{cases} y_{max} & y > y_{max} \\ y & |y| < y_{max} \\ -y_{max} & y < -y_{max} \end{cases}$$

The measurement noise $V_m(t)$ represents the variations in measurement results due to external disturbances and tools lack of precision.

Actuators

The actuator is a device that receives an electrical signal as an inupt and provides a physical action as an output, wich operates on the process.

Figure 3.10. Actuator

The whole model of the actuator is represented by the following block diagram:

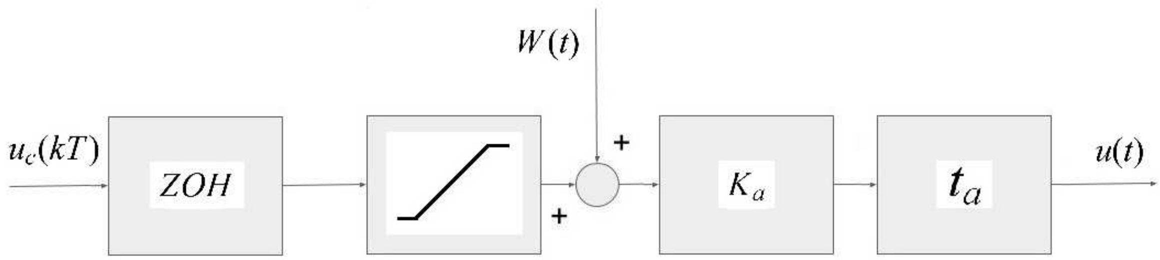

Figure 3.11. Actuator block diagram

This model embodies a zero holder hold, which generates a continuous signal from the samples, a saturation block, extenral disturbances $W(t)$, that are typically constant but unknown, and a delay t_a.

Delay approximation

The described models introduce some delays caused by measurement and data transmission, let's derive the transfer function of a generic delay t_a.

Figure 3.12. Delay

The Laplace transform of a signal with a time delay is the Laplace transform of the very signal multiplied by the exponential of the delay. Concerning the block diagram, let the Laplace transform of the input signal be $Y(s) = \mathcal{L}[y(t)]$, the delay block transfer function is $F(s) = e^{-st_a}$:

$$\mathcal{L}[y(t - t_a)] = e^{-st_a}Y(s)$$

The delay transfer function e^{-st_a} is a non rational function, then it's necessary to aproximate it to a more usable form. To do this we use the Taylor first order expansion:

$$F(s) = e^{-st_a} = \frac{1}{e^{st_a}} \approx \frac{1}{1 + st_a}$$

This approximation is also known as Pade' approximation (0-1) where the two numbers stands for the degrees of the numerator and the denominator. Another used approximation is the Pade' approximation (1-1) wich takes to the following delay transfer function:

$$F(s) = e^{-st_a} = \frac{e^{-\frac{st_a}{2}}}{e^{\frac{st_a}{2}}} \approx \frac{1 - \frac{st_a}{2}}{1 + \frac{st_a}{2}}$$