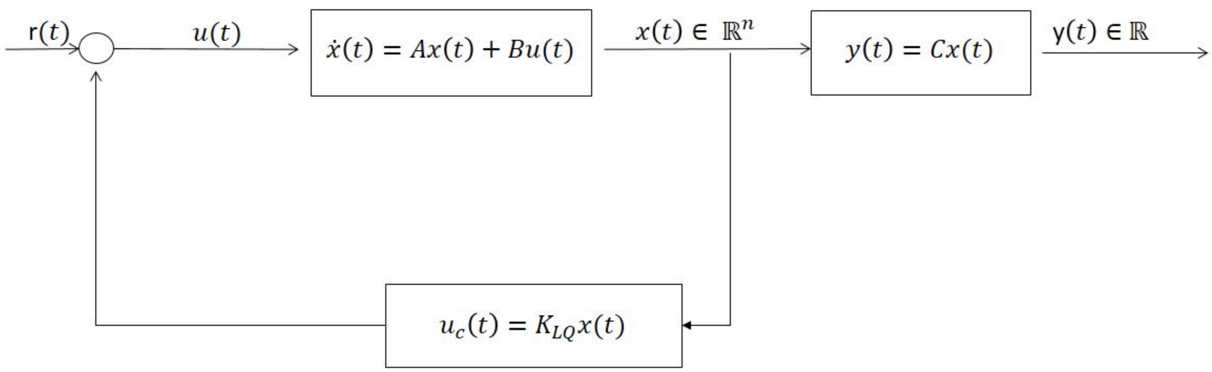
Control Laboratory:	**a.a. 2015/2016**
Lezione 28 — 17 May	
Instructor: Luca Schenato	*Scribes: grups: 17,18,19*

28.1 Structure property of LQ control

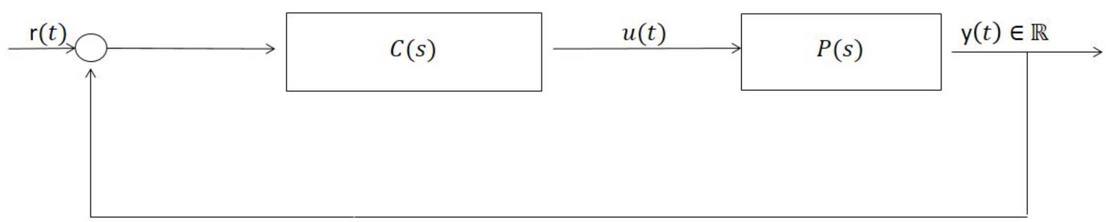
In this section we examine, with the frequency-domain analysis, the phase margin in the LQ control. The LQ control scheme is display in Figure 28.1.



For the classical control we have the open-loop transfer function given by:

$$G(s) = C(s)P(s) \qquad G(jw) : \mathbb{R} \to \mathbb{C}$$

the close-loop transfer function is:



$$W(s) = \frac{G(s)}{1 + G(s)}$$

For a LQ-control these function are:

$$\begin{cases} \dot{x}(t) & = Ax(t) + Bu(t) \\ u_c(t) & = K_{LQ}x(t) \end{cases}$$

If we consider the open-loop transfer function from $u(t) \in \mathbb{R}$ to $u_c(t) \in \mathbb{R}$ we get:

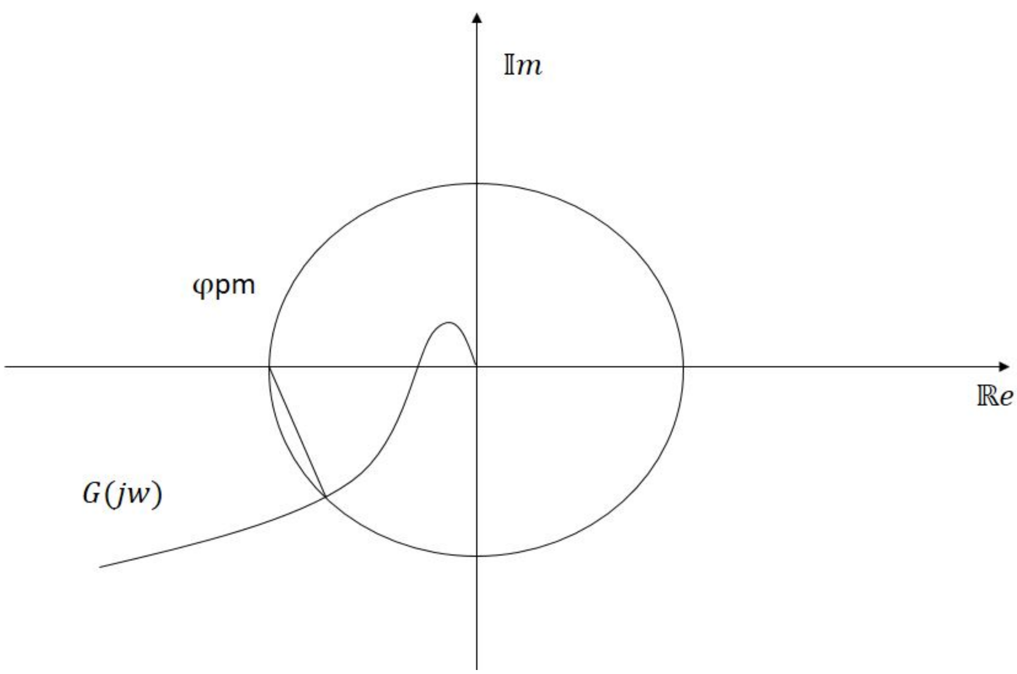
$$G_{LQ}(s) = K_{LQ}(sI - A)^{-1}B \in \mathbb{C}$$

and the close-loop transfer function became:

$$W_{LQ}(s) = \frac{G_{LQ}(s)}{1 + G_{LQ}}$$

28.1.1 Nyquist diagram of $G_{LQ}(jw)$

Property of the optimal LQ control:



$$\begin{cases} u^* = -u_c = -K_{LQ}x(t) \\ K_{LQ} = R^{-1}B^TP \\ \overline{P}A + A^T\overline{P} + Q - \overline{P}BR^{-1}B^T\overline{P} = 0 \end{cases} \tag{28.1}$$

where:

1. $B^T\overline{P} = RK_{LQ}$

2. $K_{LQ}^T = \overline{P}BR^{-1}$

3. $\overline{P}B = K_{LQ}^T R$

We add and subtract $s\overline{P}$ in 28.1 obtaining:

$$\overline{P}A + A^T\overline{P} + Q - \overline{P}BR^{-1}B^T\overline{P} + s\overline{P} - s\overline{P} = 0 \Leftrightarrow$$
$$\overline{P}A + A^T\overline{P} + Q - K_{LQ}^T RR^{-1}B^T\overline{P} + s\overline{P}I + s\overline{P}I = 0 \Leftrightarrow$$
$$Q = K_{LQ}^T RK_{LQ} + s\overline{P}I - \overline{P}A - A^T\overline{P} - s\overline{P}I \Leftrightarrow$$
$$Q = K_{LQ}^T RK_{LQ} + \overline{P}\left(sI - A\right) + \left(-A^T - sI\right)\overline{P} \Leftrightarrow$$

$$B^T\left(-sI - A^T\right)^{-1} Q\left(sI - A\right)^{-1} B = B^T\left(-sI - A^T\right)^{-1}\left[K_{LQ}^T RK_{LQ} + \overline{P}\left(sI - A\right) + \right.$$
$$\left. + \left(-sI - A^T\right)\overline{P}\right]\left(sI - A\right)^{-1} B$$

and after some steps we obtain:

$$R + B^T\left(-sI - A^T\right)^{-1} Q\left(sI - A\right)^{-1} B = \left(I + B^T\left(-sI - A^T\right)K_{LQ}^T\right) R\left(I + K_{LQ}\left(sI - A\right)^{-1} B\right)$$

the right part of the equation is semi-definite positive.

For a general MIMO system we have:

$$R + B^T(-sI - A^T)^{-1}Q(sI - A)^{-1}B = (I + B^T(-sI - A^T)^{-1}K_{LQ}^T)R(I + K_{LQ}(sI - A)^{-1}B).$$

Let's now consider the case of a SISO system, so that $R = r$, is a scalar.
Evaluating the previous equation for $s = jw$ we have:

$$r + B^T(-jw - A^T)^{-1}Q(jw - A)^{-1}B = (1 + B^T(-jw - A^T)^{-1}K_{LQ}^T)r(1 + K_{LQ}(jw - A)^{-1}B).$$

We can see the term in the parenthesis are transpose complex conjugate of each other, so we can rewrite the equation as :

$$r + \underbrace{||(Q^{1/2}(jw - A)^{-1}B||^2}_{square\ norm\ \geq 0} = \underbrace{|1 + K_{LQ}(jw - A)^{-1}B|^2}_{real\ number\ \geq 0} r.$$

This imply $|1 + K_{LQ}(jw - A)^{-1}B|^2 r \geq r$, and taking the square root, we have a simple and important result:

$$|1 + G_{LQ}(jw)| \geq 1$$

which is a property of the open loop transfer function of the optimal LQ control.

In the Nyquist diagram, this property means that the function has to be outside the circle of radius one centered in -1, so $G_{LQ}(jw)$ has, in the worst case scenario, a phase margin equal or greater than $60°$ and a minimum gain margin of $\frac{1}{2}$.
In reality $\phi_{PM}^{LQ} \geq 60°$ in LQ control is not always guaranteed, because the state is not accessible. With the introduction of an observer, a dynamical system, the phase margin typically decreases. So it has to be taken in consideration to design carefully the observer to keep a good phase margin.

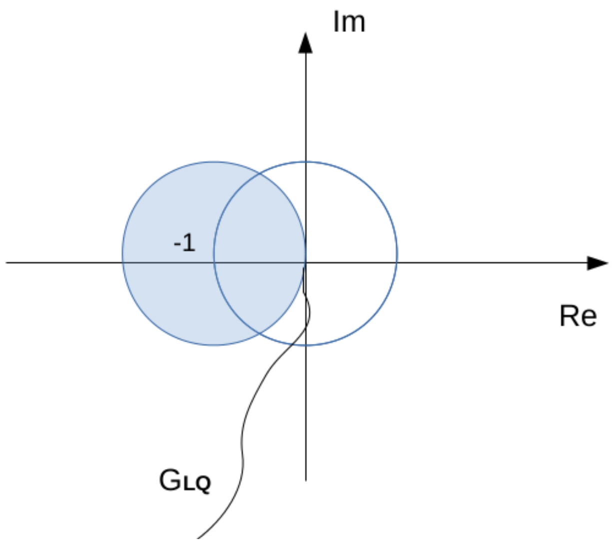


Figura 28.1. Nyquist diagram of G_{LQ}

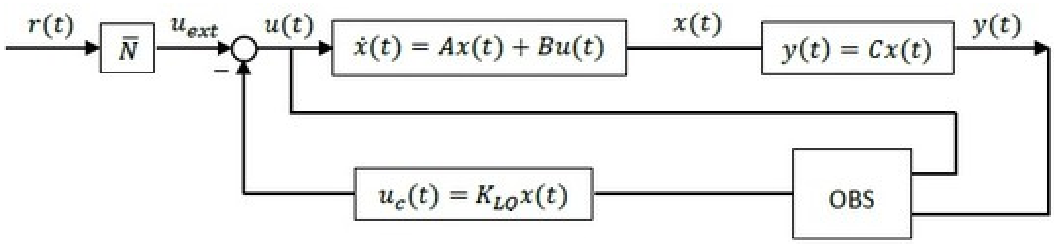


Figura 28.2. The system with the introduction of the observer

28.2 Sensitivity function

Recalling that $G(s) = C(s)P(s)$ and $W(s) = \frac{1}{1+G(s)}$ are the open-loop and the closed-loop transfer function, the sensitivity function is defined as:

$$S(s) = \frac{1}{1 + G(s)}.$$

So the sensitivity for the LQ control is:

$$S_{LQ}(s) = \frac{1}{1 + G_{LQ}(s)}$$

with

$$G_{LQ}(s) = K_{LQ}(sI - A)^{-1}B \qquad W_{LQ}(s) = \frac{G_{LQ}(s)}{1 + G_{LQ}(s)}$$

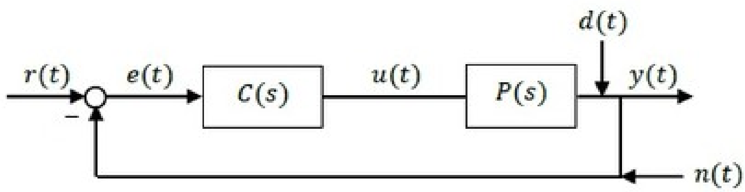


Figura 28.3. The system with two added disturbances

28.3 Importance of the sensitivity function

In this section we examine the two main reasons for which it is important the study of the sensitivity function. The first one is that it is a term that multiplies the external disturbances, as we can see analysing the unitary feedback closed loop control scheme of figure 28.4.

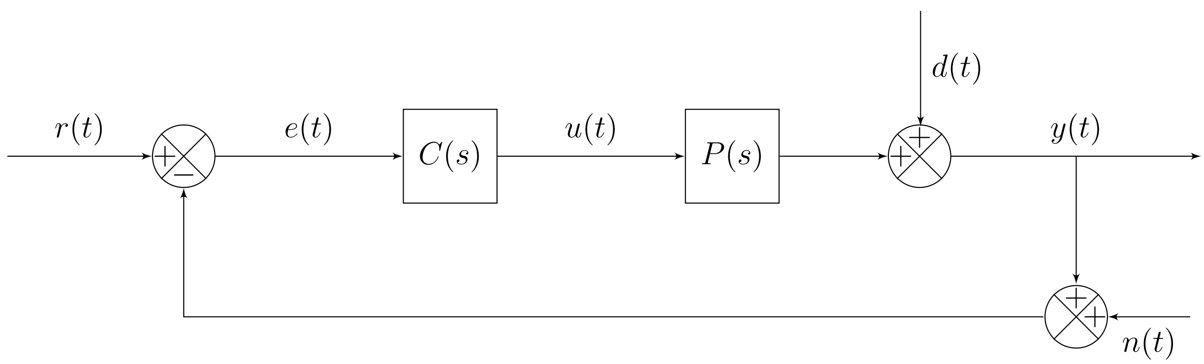


Figura 28.4. Plant with external disturbances

From the block diagram we can rather straightforwardly derive the I/O relashionship from $r(t)$ to $y(t)$ in presence of an external noise $d(t)$ acting at the output of the plant of t.f. $P(s)$ and an additive noise $n(t)$ entering in the feedback loop. For such a derivation we can use the superposition principle. Assuming first $r(t) \neq 0$, $n(t) \neq 0$ and $d(t) = 0$ we can write, in the Laplace domain:

$$\begin{cases} Y_n(s) = C(s)P(s)E(s) \\ E(s) = R(s) - (Y_n(s) + N(s)) \end{cases}$$

which can be solved in $Y_n(s)$ obtaining

$$Y_n(s) = \frac{C(s)P(s)}{1 + C(s)P(s)}[R(s) - N(s)]$$

Then assuming $d(t) \neq 0$ with $n(t) = 0$ and $r(t) = 0$ we have

$$\begin{cases} Y(s) = D(s) + C(s)P(s)E(s) \\ E(s) = -Y(s) \end{cases}$$

that, once solved, gives

$$Y_d(s) = \frac{D(s)}{1 + C(s)P(s)}$$

Then in the general case ($r(t) \neq 0$, $d(t) \neq 0$ and $n(t) \neq 0$) we have that

$$\begin{aligned} Y(s) = Y_n(s) + Y_d(s) &= \frac{C(s)P(s)}{1 + C(s)P(s)}[R(s) - N(s)] + \frac{D(s)}{1 + C(s)P(s)} \\ &= W(s)[R(s) - N(s)] + S(s)D(s) \end{aligned} \tag{28.2}$$

where $S(s) = \frac{1}{1+C(s)P(s)}$ is the sensitivity function and $W(s) = \frac{C(s)P(s)}{1+C(s)P(s)}$ is the closed loop t.f. In a similar way, solving the algebraic system

$$\begin{cases} Y(s) = D(s) + C(s)P(s)E(s) \\ E(s) = R(s) - (Y(s) + N(s)) \end{cases}$$

in $E(s)$, we can also prove that, in the Laplace domain, the error $e(t)$ is related to the control system reference signal $r(t)$ and to the noises $d(t)$ and $n(t)$ by the following relation:

$$E(s) = \frac{1}{1 + C(s)P(s)}[R(s) - D(s) - N(s)] = S(s)[R(s) - D(s) - N(s)] \tag{28.3}$$

Finally, using the fact that $U(s) = C(s)E(s)$, we also immediately find that, in the Laplace domain, the control signal $u(t)$ is related to the signals $r(t)$, $d(t)$ and $n(t)$ by the equation:

$$\begin{aligned} U(s) = C(s)E(s) &= \frac{C(s)}{1 + C(s)P(s)}[R(s) - D(s) - N(s)] = \\ &= C(s)S(s)[R(s) - D(s) - N(s)] \end{aligned} \tag{28.4}$$

From equation (28.2), (28.3) and (28.4) we see the importance of having a small sensitivity function $S(s)$ in the range of frequencies of the signal to be tracked if we want the output not to be too much sensitive to the disturbance $d(t)$ and if we want the error and the control input signal to be small.

If we design the controller of t.f. $C(s)$ in a proper way we should get a closed loop t.f. whose Bode plot should look something like the one of figure 28.5 in which it has been highlighted the angular bandwidth ω_B (dashed vertical line) related to the bandwidth frequency by the relation $\omega_B = 2\pi f_B$. Notice that in the range of angular frequencies $[0, \omega_B]$ we can perfectly track the reference signal $r(t)$ since $|W(\jmath\omega)|_{dB} \approx 1$. In general we will try to design $C(s)$ so as to have a slope of the Bode plot of the module of $W(\jmath\omega)$ which is sufficiently steep for $\omega > \omega_B$ since in the real system there might be resonance effects, not modelled by $P(s)$, which make the Bode diagram of the module of the real closed loop t.f. more similar to the dashed one reported in figure 28.5.

Since for $\omega < \omega_B$ we want, for tracking porposes, $W(\jmath\omega) \approx 1$ then we see that we have to design $C(s)$ in order to have, for $\omega \in [0, \omega_B]$, that $|C(\jmath\omega)P(\jmath\omega)| \gg 1$. In this way we also

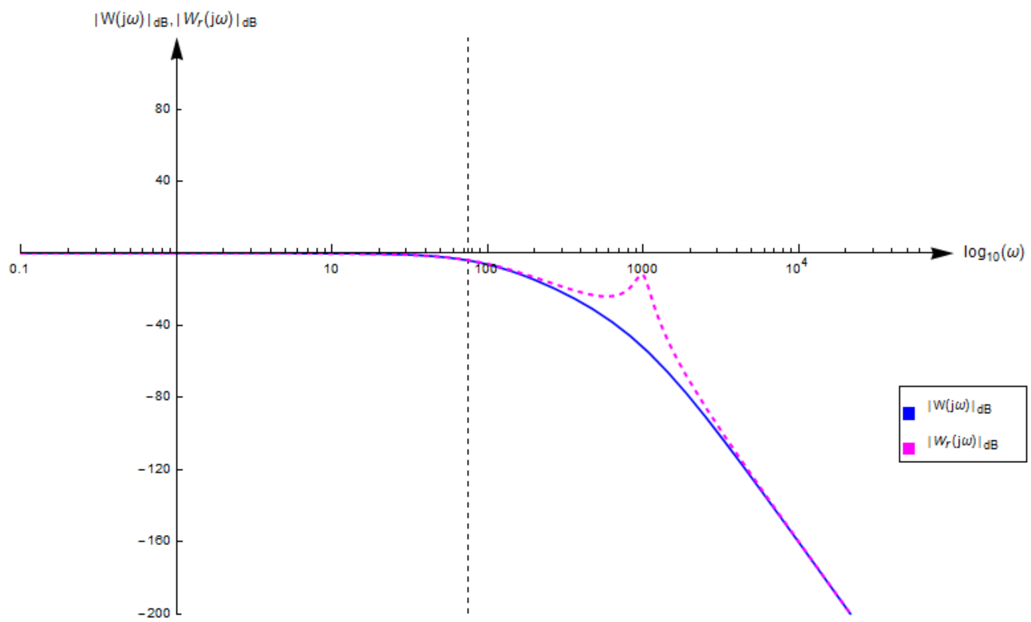


Figura 28.5. Bode plot of the module of the closed loop transfer function $W(\jmath\omega)$

see that in this range of frequencies $S(s) = \frac{1}{1+C(s)P(s)}$ will be very small and approximable as $S(s) \approx \frac{1}{C(s)P(s)} \ll 1$ so that equation (28.4) can be rewritten as

$$\begin{aligned} U(s) &= C(s)S(s)[R(s) - D(s) - N(s)] \approx \\ &\approx C(s)(C(s)P(s))^{-1}[R(s) - D(s) - N(s)] = (P(s))^{-1}[R(s) - D(s) - N(s)] \end{aligned} \tag{28.5}$$

A typical Bode plot of the module of the t.f $P(\jmath\omega)$ and its inverse $(P(\jmath\omega))^{-1}$ is that of figure 28.6. From figure 28.6 and equation (28.5) we see that if the angular bandwidth of the closed loop t.f. ω_B is made too large by the designed $C(s)$ then we will amplify possible noises with frequency components in the range of interest $[0, \omega_B]$. Moreover also some reference signal armonics will be substantially amplified obtaining a very large control input $u(t)$.

The ideal behaviour of the module of the Bode plot of the t.f $S(s)$ is that of a monotonically increasing function of $\log_{10}(\omega)$ which is as smaller as possible in the range of angular frequenciees $[0, \omega_B]$. An acceptable behaviour for the module of the Bode plot of the t.f. $S(s)$ is anyway that of a function of $\log_{10}(\omega)$ which is sufficiently smaller than one in the above mentioned range of angular frequencies. Two examples are reported in figure 28.7

The second, but not less important, reason for which the sensitivity function study is fundamental is the presence of parameter uncertainties in almost every plant model we can do. In general we have in fact a plant t.f. $P(s)$ which depends on a certain number of parameters which we can't know exactly. Therefore in the design of $C(s)$ we will use a nominal plant t.f.

$$P_{nom}(s) = P(s, \theta_{nom}) \tag{28.6}$$

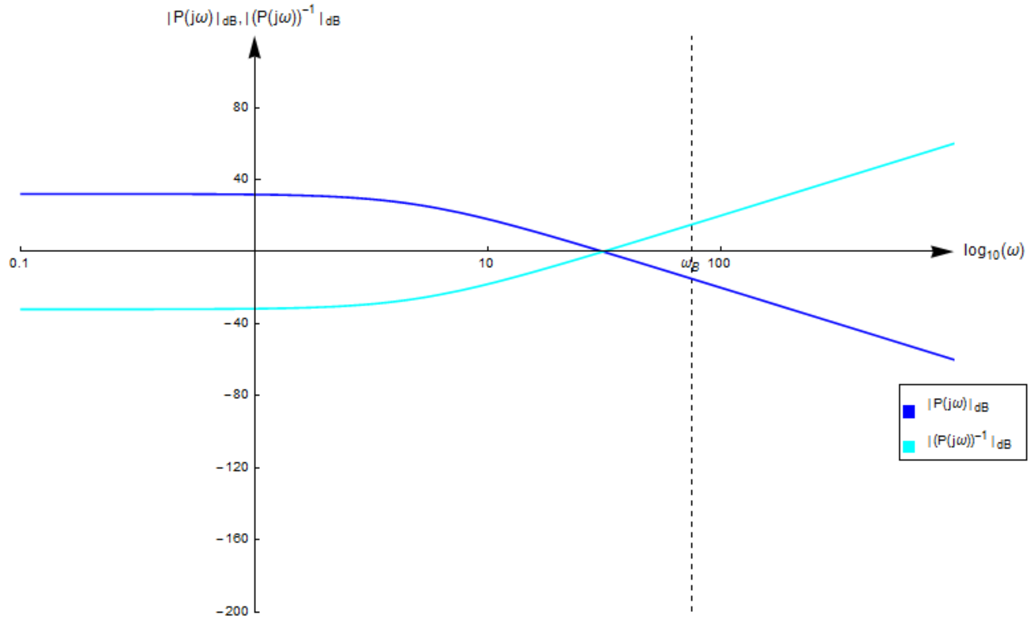


Figura 28.6. Typical bode plot of the module of a plant t.f. $P(\jmath\omega)$

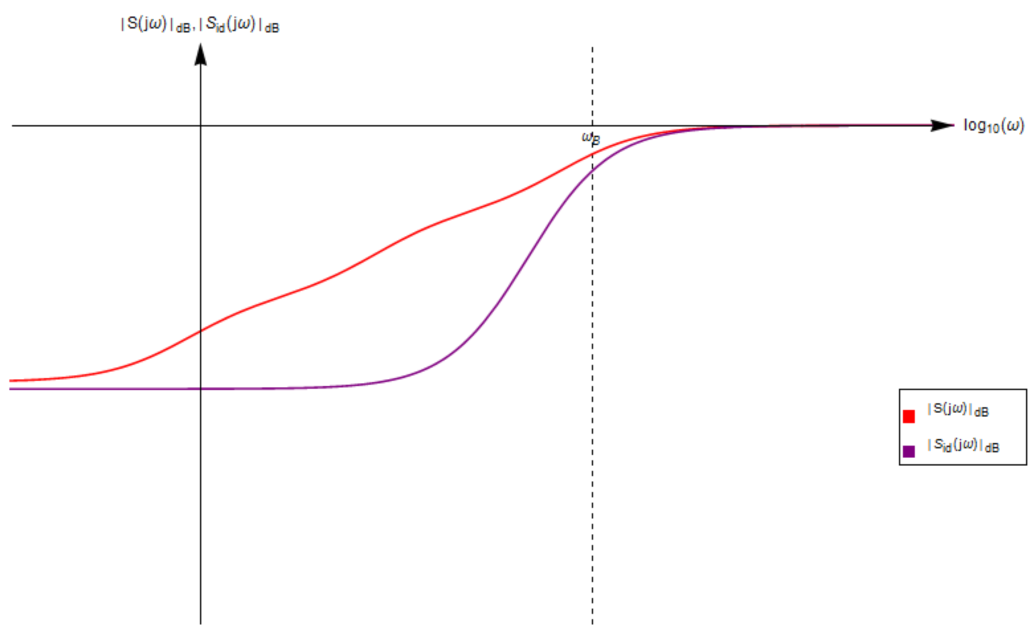


Figura 28.7. Sensitivity t.f. $S(\jmath\omega)$

where θ_{nom} is the vector of nominal parameters. However the actual plant t.f. will be

$$P(s) = P(s, \theta_0) \tag{28.7}$$

where θ_0 is the vector of true parameters. The important thing to notice is that it will be this last t.f. that will determine the effective performances of each controller designed based on $P_{nom}(s)$. We are therefore interested in evaluating the impact on the control system performances of the very likely difference between θ_{nom} and θ_0. Defined $\delta\theta := \theta_0 - \theta_{nom}$ we can compute $\frac{\delta[y(t)]}{\delta\theta}$ and it turns out that

$$\frac{\delta[y_{c.l.t.f}(t)]}{\delta\theta} = S(s)\frac{\delta[y_{o.l.t.f}(t)]}{\delta\theta} \tag{28.8}$$

Again we see the importance of having a small sensitivity function in order to have a closed loop control system that is unsensible to parameter uncertainties. In the block schemes of figures 28.8 and 28.9 are clarified the meanings of the quantities $\delta[y_{o.l.t.f}](t) = y(t) - y_{o.l.nom}(t)$ and $\delta[y_{c.l.t.f}](t) = y(t) - y_{c.l.nom}(t)$.

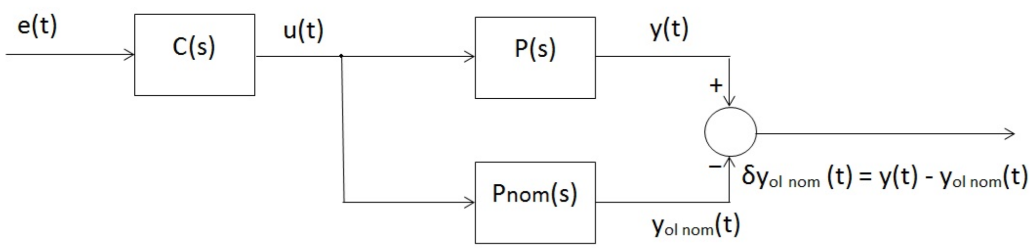


Figura 28.8. $y(t)$ versus $y_{o.l.,nom}(t)$

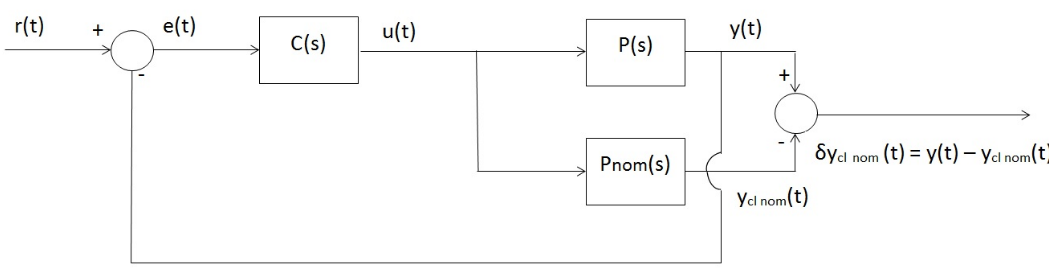


Figura 28.9. $y(t)$ versus $y_{c.l.,nom}(t)$