

## Lezione 27 — 12 May

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## 27.1 Design of $R(j\omega)$ and $Q(j\omega)$

Given the usual expression of the quadratic cost function:

$$\int_0^{+\infty} \bar{x}(t)^T \bar{Q} \bar{x}(t) + \bar{u}(t)^T \bar{R} \bar{u}(t) dt$$

where  $\dot{\bar{x}}(t) = (\bar{A} - \bar{B}R_{eq}N)\bar{x}(t) + \bar{B}v(t)$ , we have the following hypothesis:

1.  $(F, \bar{B})$  is reachable;
2.  $(F, \bar{Q}^{\frac{1}{2}})$  is observable;
3.  $\bar{R}$  is strictly positive:  $\bar{R} > 0$ ;

Typically the first and the second assumptions are satisfied unless there are some zero/pole cancellations between  $P(s)$ ,  $Q(s)$  and between  $P(s)$  and  $R(s)$ . For the third hypothesis, if it cannot be satisfied, then we can set  $\bar{R} = \varepsilon I$  with  $\varepsilon$  small, so that the solution of this problem is very similar to the solution of the problem with  $\bar{R} = 0$ .

Applying the Parseval Theorem it can be written:

$$\frac{1}{2\pi} \int_0^{+\infty} X(j\omega)^* Q(j\omega) X(j\omega) + U(j\omega)^* R(j\omega) U(j\omega) d\omega$$

So, how to design  $Q(s)$  and  $R(s)$  in order to force the optimal control to have some characterizations of  $u(t)$  and  $x(t)$  in frequency domain?

As an example, consider Figure 27.1 which describes a mechanical system with a resonance. Any small input with frequency near  $\omega_r$  will be amplified. Besides the input is used also to estimate the state, so resonances can generate some problems in the estimation of the state of the system.

There can be specific oscillations around  $\omega_r$ , so they have to be predicted: in order to do that the output has to be passed through a filter  $P_y(s)$ , as it is showed in the figure below.

Considering these equations:

- $\tilde{y}(t) = P_y(s)y(t)$  where the usual abuse of notation stands;
- $y(t) = Cx(t)$ ;

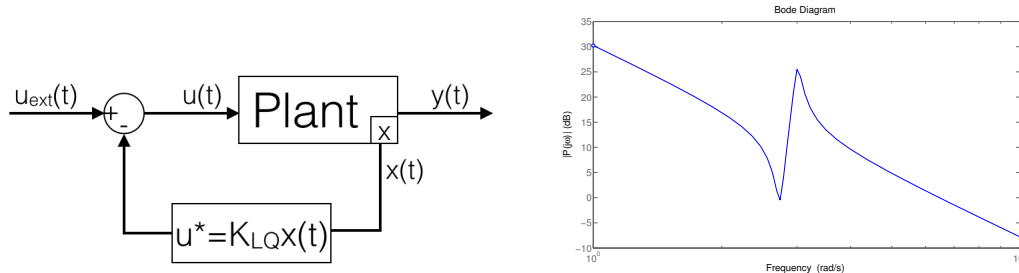


Figure 27.1. Scheme of the  $K_{LQ}$  controller and Bode plot of the collocated system.

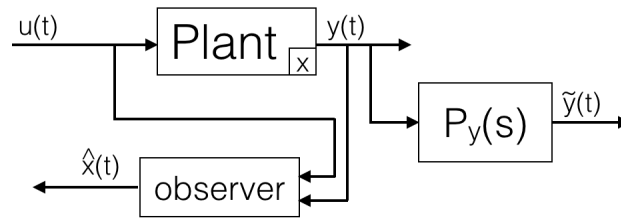


Figure 27.2. First way to reduce the effect of resonances.

$$\Rightarrow \tilde{y}(t) = P_y(s)Cx(t).$$

Looking back at the form of integral 27.1, it follows:

$$\begin{aligned} \int_0^{+\infty} \tilde{y}(t)^2 + ru(t)^2 dt &= \int_0^{+\infty} x(t)^T C^T P_y(s)^T P_y(s) C x(t) + ru(t)^2 dt \\ \xrightarrow{\text{Parseval}} \frac{1}{2\pi} \int_0^{+\infty} X(j\omega)^* C^T P_y(j\omega)^* P_y(j\omega) C X(j\omega) + rU(j\omega)^* U(j\omega) d\omega &= \\ = \frac{1}{2\pi} \int_0^{+\infty} X(j\omega)^* \underbrace{C^T P_y(-j\omega) P_y(j\omega) C}_{Q(j\omega)} X(j\omega) + rU(j\omega)^* U(j\omega) d\omega \end{aligned}$$

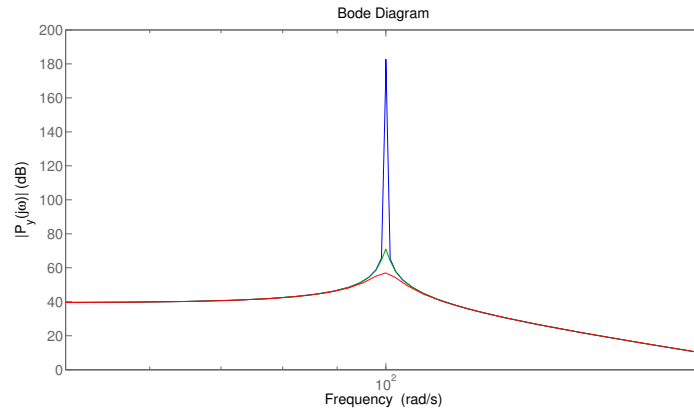
$$\text{If } Q(j\omega) = \begin{bmatrix} |P_y(j\omega)|^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \geq 0,$$

$P_y(s)$  has to be chosen in order to penalize the frequency peak of the output at the resonance frequency  $\omega_r$ ; for example the magnitude of the frequency response,  $|P_y(j\omega)|$ , can be increased in a neighborhood of  $\omega_r$ .

Giving some examples:

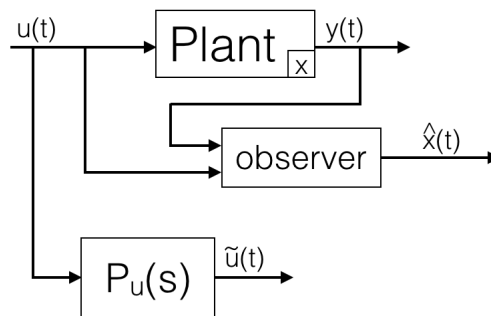
1.  $P_y(s) = \frac{a^2}{s^2 + \omega_r^2}$ , in this case a large frequency range around  $\omega_r$  is penalized

2.  $P_y(s) = \frac{a^2}{s^2 + 2\xi\omega_r s + \omega_r^2}$ , in this case there is a little bit of dumping and the range of frequencies that are penalized is larger;



**Figure 27.3.** Effect of a variation of the damping factor on  $P_y(s)$ .

In order to penalize the frequencies at input that are in a neighborhood of  $\omega_r$ , we can filter the input rather than the output:

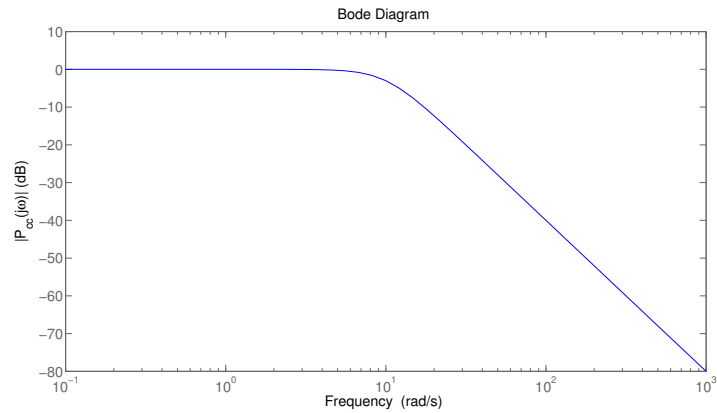


**Figure 27.4.** Second way to reduce the effect of resonances.

So there are two ways to reduce the effects of the resonances in the plant  $P(s)$ .

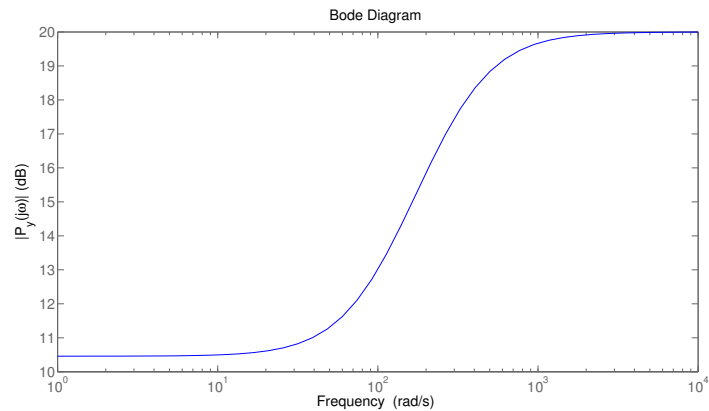
If instead there is high frequency noise produced by the sensor and  $t_r$ ,  $t_s$  are fixed by specifications, also  $\omega_C \simeq \omega_B$  can be derived, where  $\omega_B$  is the angular bandwidth of the closed loop system.

The purpose is to trace every signal with bandwidth  $\omega_B$ , and to penalize any output frequency  $\omega_O > \omega_B$ , with the choice of  $\omega_C$ , in order to obtain the closed loop transfer function in figure (27.5).



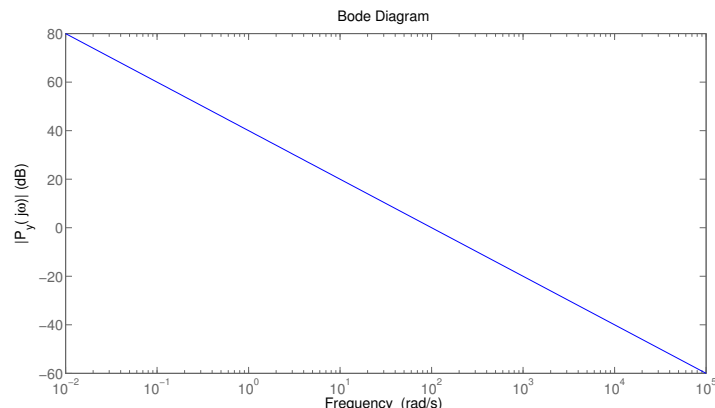
**Figura 27.5.** Bode plot of the desired closed loop transfer function.

In order to achieve that, a suitable  $P_y(s)$  could be  $P_y(s) = \frac{1+\alpha s}{1+\beta s}$  where  $\frac{1}{\alpha} \simeq \omega_B$  and  $\frac{1}{\beta} \simeq (3 \div 10)\omega_B$ .



**Figura 27.6.** Bode plot of a suitable  $P_y(s)$ .

Eventually also the steady-state error can be modified, penalizing the frequency that are very small. A suitable  $P_y(s)$  for this kind of problems is the integrator (27.7):  $P_y(s) = \frac{1}{s}$ , that filters constant disturbances.



**Figure 27.7.** Bode plot of an integrator.

So how to design  $P_y(s)$  and  $P_u(s)$ ? It depends on the situation, in particular on the type of disturbances or the transfer functions of sensors one has to deal with. Suitable weights have to be chosen, for example the well known  $R(s)$ ,  $Q(s)$  are squares of the magnitude of some filters. Reminding the factorization,  $Q(j\omega) = Q_1^*(j\omega)Q_1(j\omega)$  where  $Q_1(j\omega) \in \mathbb{C}^n$  is the Bode representation of the weighting filter and  $Q(j\omega)$  is the matrix power spectrum of the filter.

Considering this special case:  $Q(j\omega)$  is a block matrix, so:

$$Q(j\omega) \in \mathbb{C}^{n \times n}, \quad Q(j\omega) = \left[ \begin{array}{c|c} Q'(j\omega) & 0 \\ \hline 0 & Q''(j\omega) \end{array} \right], \quad Q'(j\omega) \in \mathbb{C}^{n_1 \times n_1}, \quad Q''(j\omega) \in \mathbb{R}^{(n-n_1) \times (n-n_1)}$$

The inputs can be penalized separately!

It's also assumed that  $Q(j\omega)$  is obtained through a filter  $H_Q(j\omega)$ :  $Q(j\omega) = H_Q^*(j\omega)H_Q(j\omega)$ , where  $H_Q(j\omega)$  is the filter associated to  $Q(j\omega)$  (i.e. the factorization of  $Q(j\omega)$ ).

$$H_Q(j\omega) = \left[ \begin{array}{c|c} H'_Q(j\omega) & 0 \\ \hline 0 & H''_Q \end{array} \right], \quad H'_Q(j\omega) \in \mathbb{C}^{n_1 \times n_1}, \quad H''_Q \in \mathbb{R}^{(n-n_1) \times (n-n_1)}$$

where  $H''_Q$  does not depend on frequency.

Being  $H_Q(s)$  a filter, it can be represented in state-space, with input  $x(t)$ :

$$\begin{cases} \dot{x}_Q(t) = A_Q x_Q(t) + B_Q x(t) \\ y_Q(t) = C_Q x_Q(t) + D_Q x(t) \end{cases}$$

where  $x_Q(t) = H(s)x(t)$ .

It follows:

$$B_Q = \left[ B'_Q \mid 0 \right], \quad C_Q = \left[ \begin{array}{c} C'_Q \\ 0 \end{array} \right], \quad D_Q = \left[ \begin{array}{c|c} D'_Q & 0 \\ \hline 0 & H''_Q \end{array} \right].$$

The structure is such that  $(A_Q, B'_Q, C'_Q, D'_Q)$  are associated to the dynamical part  $H'_Q(j\omega)$ , while the other blocks are associated to the static part.

For  $R(j\omega)$  there is the same structure:

$$R(j\omega) \in \mathbb{C}, \quad R(j\omega) = \left[ \begin{array}{c|c} R'(j\omega) & 0 \\ \hline 0 & R''(j\omega) \end{array} \right], \quad R'(j\omega) \in \mathbb{C}^{m_1 \times m_1}, \quad R''(j\omega) \in \mathbb{R}^{(m-m_1) \times (m-m_1)}$$

and  $R(j\omega) = H_R^*(j\omega)H_R(j\omega)$ , where  $H_R(j\omega)$  is the filter associated to  $R(j\omega)$  (i.e. the factorization of  $R(j\omega)$ ). Being also  $H_R(s)$  a filter, it can be represented in state-space, with input  $x(t)$ :

$$\begin{cases} \dot{x}_R(t) = A_R x_R(t) + B_R u(t) \\ y_R(t) = C_R x_R(t) + D_R u(t) \end{cases}$$

It follows:

$$B_R = [ B'_R \mid 0 ], \quad C_R = \left[ \begin{array}{c} C'_R \\ 0 \end{array} \right], \quad D_R = \left[ \begin{array}{c|c} D'_R & 0 \\ \hline 0 & H''_R \end{array} \right].$$

## 27.2 Examples of weighting filters

Design of  $Q(j\omega)$ :

$$Q(j\omega) = \left[ \begin{array}{c|c} Q'(j\omega) & 0 \\ \hline 0 & Q''(j\omega) \end{array} \right], \quad \Rightarrow Q'(j\omega) = \begin{bmatrix} \frac{a_1}{(\omega^2 - \omega_1^2)^2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{a_{n'}}{(\omega^2 - \omega_{n'}^2)^2} \end{bmatrix}$$

where  $\omega_1, \dots, \omega_{n'}$  are the natural resonances of the original system that have to be penalized and  $a_i$  is the specific weight of for the state entry  $x_i(t)$ . The associated filter  $H_Q(j\omega)$ , has the diagonal elements in the form  $h_i(s) = \frac{1}{s^2 + \omega_i^2}$ , whose state-space representation is:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} a_i & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u(t) \end{cases}$$

So there are  $n'$  state-spaces, one for every element of the diagonal of the matrix:

$$H'_Q(j\omega) = \begin{bmatrix} \frac{a_1}{((j\omega)^2 + \omega_1^2)^2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{a_{n'}}{((j\omega)^2 + \omega_{n'}^2)^2} \end{bmatrix}$$

Having the state-space  $(A_Q, B_Q, C_Q, D_Q)$ , it follows:

$$A_Q = \left[ \begin{array}{ccc|c} 0 & & & I \\ \hline -\omega_1^2 & & & \\ & \ddots & & \\ & & -\omega_n^2 & \\ \hline & & & 0 \end{array} \right], \quad B_Q = \left[ \begin{array}{c|c} 0 & 0 \\ \hline I & 0 \end{array} \right], \quad C_Q = \left[ \begin{array}{ccc|c} -a_1 & & & 0 \\ & \ddots & & \\ & & -a_n & \\ \hline & & & 0 \\ \hline & & 0 & \\ \hline & & & 0 \end{array} \right], \quad D_Q = \left[ \begin{array}{c|c} 0 & 0 \\ \hline 0 & H'' \end{array} \right].$$