Control Laboratory:	**a.a. 2015/2016**
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26.1 Frequency shaping for LQ control: general case

We recall some notions about LQ optimal control. Given the dynamic system described by the equations

$$\begin{cases} \dot{x} = Ax + Bu & \text{con } x(0) = x_0 \\ y = Cx \end{cases}$$

we want to control this system such that we minimize the quadratic cost index

$$J(x_0, u) = \int_0^{+\infty} x^T(t)Qx^T(t) + u^T(t)Ru(t)dt$$

Now the question is if it is possible to define a cost function that can include some useful considerations about the design in frequency domain. The fondamental result that allows us to obtain that is the Parseval theorem. It says that the time domain signal and its Fourier transform have the same energy. We have:

$$\int_0^{+\infty} ||y(t)||^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} ||Y(j\omega)||^2 d\omega$$

Under the assumption that $x(t) = 0$ for $t < 0$, so, in the MIMO case we have:

$$\int_0^{+\infty} x^T(t)Qx(t)dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega)QX(j\omega)d\omega$$

It is possible to penalize some frequencies choosing a different Q. If we have a resonance at a determined frequency at ω_r we will have a great amplification of the input components at the output if we do nothing. The solution is to shape $u(t)$ or the $y(t)$, or even both of them, in order to allow the LQ control to penalize these frequencies. We will try to design K that doesn't excite the resonance frequencies through the shaping of the input or the output of the system using LQ control. We will minimize a cost function as:

$$\int_0^{+\infty} ||\tilde{y}(t)||^2 + r||\tilde{u}(t)||^2 dt$$

The system we are going to consider will be the following, where $R_1(s)$ and $Q_1(s)$ are called *shaping filters*:

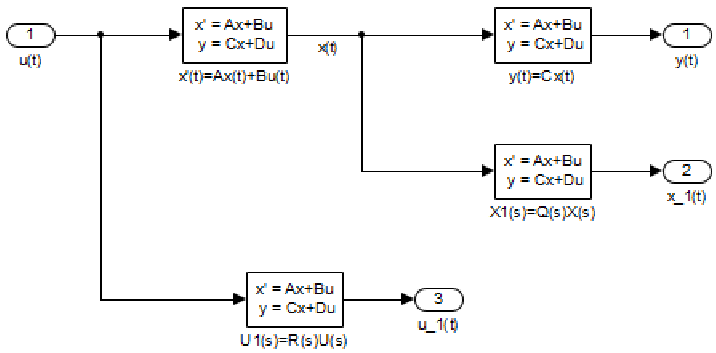

Figura 26.1. Block scheme

For SISO system the classical LQ control design is equivalent of minimizing

$$\int_0^{+\infty} ||y(t)||^2 + r||u(t)||^2 dt$$

which lead to the constant, i.e. independent of the frequency, weights:

$$Q(j\omega) = Q_1^*(j\omega)Q_1(j\omega) = C^T C, \tag{26.1}$$
$$R(j\omega) = R_1^*(j\omega)R_1(j\omega) = r \tag{26.2}$$

So, suppose that we want to generalize this criteria by shaping the weights in the frequency domain as:

$$\begin{aligned} \frac{1}{2\pi} \quad & \int_{-\infty}^{+\infty} \quad X^*(j\omega)Q_1^*(j\omega)Q_1(j\omega)X(j\omega) + U^*(j\omega)R_1^*(j\omega)R_1(j\omega)U^*(j\omega)dw = \\ & \int_0^{+\infty} \quad x_1^T(t)x_1(t) + u_1^T(t)u_1(t)dt = \end{aligned}$$

where

$$X_1(s) = Q_1(s)X(s),$$
$$U_1(s) = R_1(s)U(s)$$

If we know the dynamics of x_1 and u_1 is the control input, then we can solve using the standard LQ technique. Now remember the relations (26.1) and (26.2). We begin by describing the shaping filters in state space introducing the states z_1 for $Q(s)$ and z_2 for $R(s)$. The dynamics of the two systems will be the following:

$$\begin{bmatrix} \dot{z_1}(t) \\ \dot{x_1}(t) \end{bmatrix} = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right] \begin{bmatrix} z_1(t) \\ x_1(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{z_2}(t) \\ \dot{u_1}(t) \end{bmatrix} = \left[\begin{array}{c|c} A_R & B_R \\ \hline C_R & D_R \end{array}\right] \begin{bmatrix} z_2(t) \\ u_1(t) \end{bmatrix}$$

We remark that it is possible to penalize some frequencies thanks to the choice of $Q(s)$ and $R(s)$.

Now that we have these 3 dynamical systems, we would like to use a single system to express the dynamics in order to achieve the global LQ control. We will now create an extended system $\bar{\Sigma}$ with variable extended state $\bar{x}$:

$$\bar{\Sigma}: \qquad \begin{cases} \dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) \\ u_1(t) = \bar{C}\bar{x}(t) + \bar{D}_u u(t) \\ x_1(t) = \bar{C}_x\bar{x}(t) + \bar{D}_x u(t) \end{cases} \qquad \bar{x} = \begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x(t)} \\ \dot{z_1(t)} \\ \dot{z_2(t)} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ B_Q & A_Q & 0 \\ 0 & 0 & A_R \end{bmatrix} \begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ B_R \end{bmatrix} u(t)$$

$$u_1(t) = \begin{bmatrix} 0 & 0 & C_R \end{bmatrix} \bar{x} + D_r u(t)$$

$$x_1(t) = \begin{bmatrix} D_Q & C_Q & 0 \end{bmatrix} \bar{x}$$

considering $u_1(t)$ and $x_1(t)$ as the output of this system (for evaluating the cost). The cost expressed in this new state space becomes:

$$\int_0^{+\infty} \begin{bmatrix} \bar{x}^T(t) & u^T(t) \end{bmatrix} \begin{bmatrix} Q_{eq} & N^T \\ N & R_{eq} \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ u(t) \end{bmatrix} dt$$

with:

$$Q_{eq} = \begin{bmatrix} D_Q^T D_Q & D_Q C_Q & 0 \\ C_Q^T D_Q & C_Q^T C_Q & 0 \\ 0 & 0 & C_R^T C_R \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ 0 \\ C_R^T D_R \end{bmatrix}, \quad R_{eq} = D_R^T D_R$$

This form is extremely inconvenient for computation purposes, so I rewrite the cost as

$$\int_0^{+\infty} \begin{bmatrix} \bar{x}^T(t) & v^T(t) \end{bmatrix} \begin{bmatrix} \bar{Q} & 0 \\ 0 & \bar{R} \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ v(t) \end{bmatrix} dt = \int_0^{+\infty} \bar{x}^T(t)\bar{Q}\bar{x}(t) + v^T(t)\bar{R}v(t)dt$$

by introducing the new control input for the global system:

$$v(t) = u(t) + R_{eq}^{-1} N \bar{x}(t)$$

The new Q and R matrices becomes:

$$\bar{Q} = Q_{eq} - N^T R_{eq} N, \quad \bar{R} = R_{eq}$$

We have thus arrived to standard LQ control where the frequency dependence of the weights $\bar{Q}$ and $\bar{R}$ has disappeared, since it has been included in the extended dynamics.
The equation for the state-space needs to be adjusted, since I want to write it in terms of the virtual input $v(t)$, but now the input is still $u(t)$. If we substitute $u(t)$ with the expression

$$u(t) = -R_{eq}^{-1} N \bar{x}(t) + v(t)$$

the equation for $\dot{\bar{x}}(t)$ becomes

$$\begin{aligned}\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}u(t) = \\ &= \bar{A}\bar{x}(t) + \bar{B}v(t) - \bar{B}R_{eq}^{-1}N\bar{x}(t) = \\ &= \underbrace{(\bar{A} - \bar{B}R_{eq}^{-1}N)}_{F}\,\bar{x}(t) + \bar{B}v(t)\end{aligned}$$

Now, we want to apply LQ control to the system with matrices F and $\bar{B}$ and weights $\bar{Q}$ and $\bar{R}$. These matrices will be bigger, since we extended the state with the state of the filter.
Once we are done, we get that the control is

$$v(t) = -\bar{K}_{LQ}\bar{x}(t)$$

The real input we work with is actually $u(t)$, which we can write as

$$u(t) = -R_{eq}^{-1}N\bar{x}(t) - \bar{K}_{LQ}\bar{x}(t) = -(R_{eq}^{-1}N + \bar{K}_{LQ})\bar{x}(t)$$

In order to implement the controller, we need the state $\bar{x}(t)$, so we need to implement in software the two filters $Q(s)$ and $R(s)$ in state space to get $z_1(t)$ and $z_2(t)$.