**Control Laboratory:** **a.a. 2015/2016**

# Lezione 25 — $5^{th}$ May

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## 25.1 Weight Design for Optimal Control: R, Q

At this point in the study of the optimal control the question we want to answer is: how to design the weights represented by the matrices R and Q?

As introduced in the last lecture, we consider an alternative measurement $\widetilde{y}(t)$, which can be seen as a virtual sensor, in order to design the feedback of the state x(t).

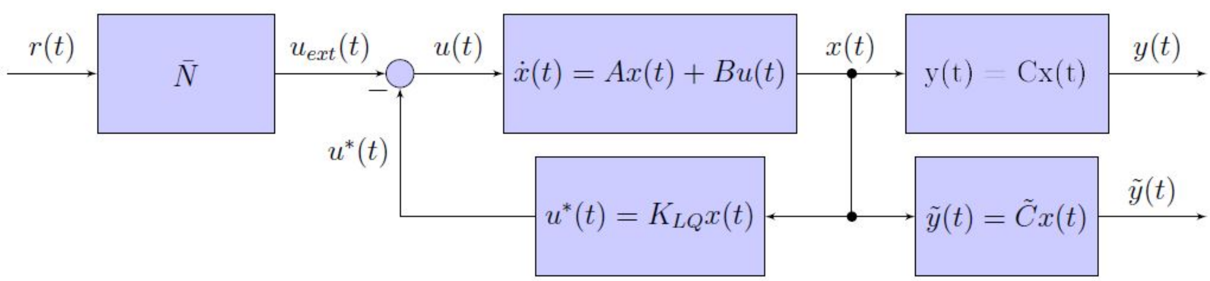


**Figura 25.1.** Block representation with optimal control system and virtual oputput

The control cost $\int_0^\infty \widetilde{y}^2 + ru^2(t)dt$ will be based on $\widetilde{y}(t)$, although what we are really interested in is controlling $y(t)$. Considering different $\widetilde{y}(t)$ is not an issue as far as stability is concerned, because, as seen in the more general problem of tracking, first we want to make sure that the closed-loop system is stable and then you track the signal, either using feed-forward control (nominal tracking) or extending the dynamics of the system by adding an integrator and applying integral control (robust tracking).

Of course the conditions $(A, B)$ reachable and $(A, \widetilde{C})$ observable are necessary in order for the closed-loop system to be asymptotically stable.

Adding the virtual sensor is useful because with this strategy is possible to force n-1 poles of the closed-loop system to be close to n-1 specific locations, for $r \to 0$.

Let us clarify this statement with an example. $P(s) = \frac{n(s)}{d(s)}$ represents the original system and $\widetilde{P}(s) = \frac{\widetilde{n}(s)}{d(s)}$ the transfer function associated to the the virtual sensor.

While is obvious that the two share the same denominator having both the same dynamics, we will see that the location of the zeros of $\widetilde{P}(s)$ can be determined by $\widetilde{C}$.

Let's assume that the original system P(s) has got three poles and a zero. We can apply optimal control and the result is illustrated in the following figures.

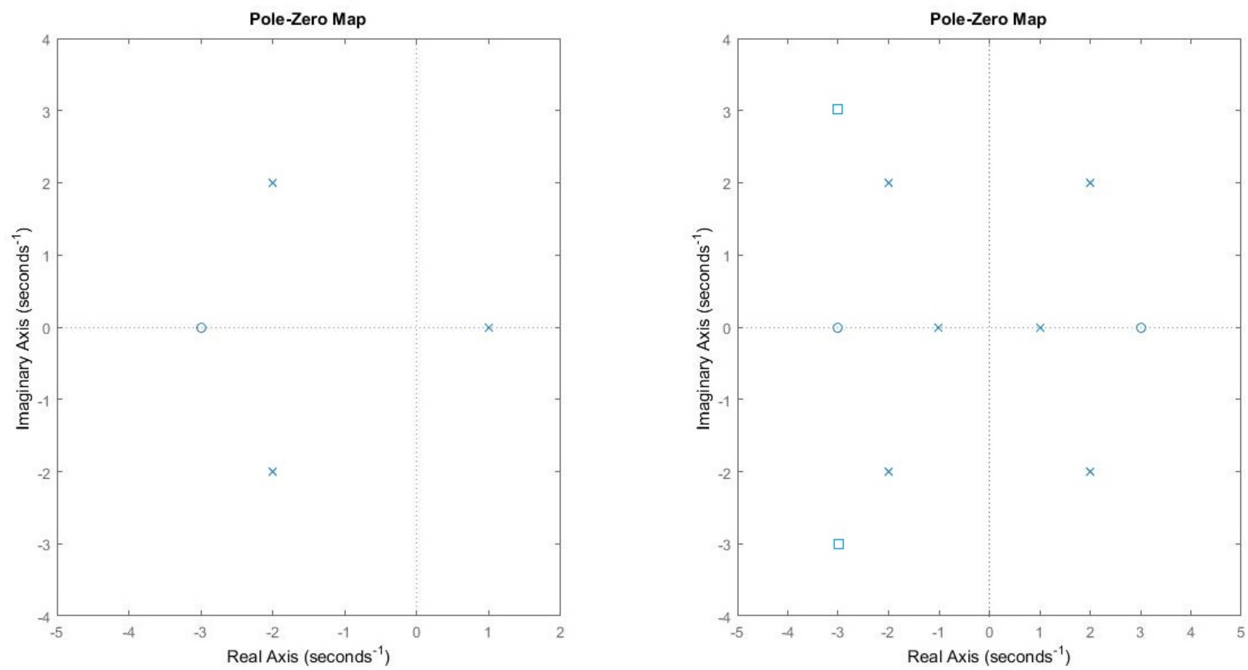


**Figura 25.2.** (On the left) Zeros and poles of the original system. (On the right) Zeros and poles of the system with optimal control

The squares in the figure represent desired locations where for some reason you would like the three poles to be. So the natural question is: can we move the poles in closed-loop, if not on the exact desired locations, at least near them?

In this case we cannot move all three poles where we want to because we can place only n-1 poles. So we design $\widetilde{C}$ such that the zeros of $\widetilde{P}(s)$ will correspond exactly to two of the desired locations.

These zeros can be seen as virtual zeros that do not exist in the real system, but we pretend they do. Now if I use the LQ control with $\widetilde{C}$ the result will be as represented in figure 25.3.

The LQ control will not place the poles exactly on the desired location, but the smaller r is, the closer the closed-loop poles are to the "virtual" zeros.

So we have seen an alternative way to proceed and try to get a better performance, according to some other specifications.

Using optimal control the poles will not be exactly as desired and the shape of the root locus will depend on the specific system.

So one could think about using the pole placement approach to place the closed-loop poles exactly in the desired locations, but this method will not take into account the possibility

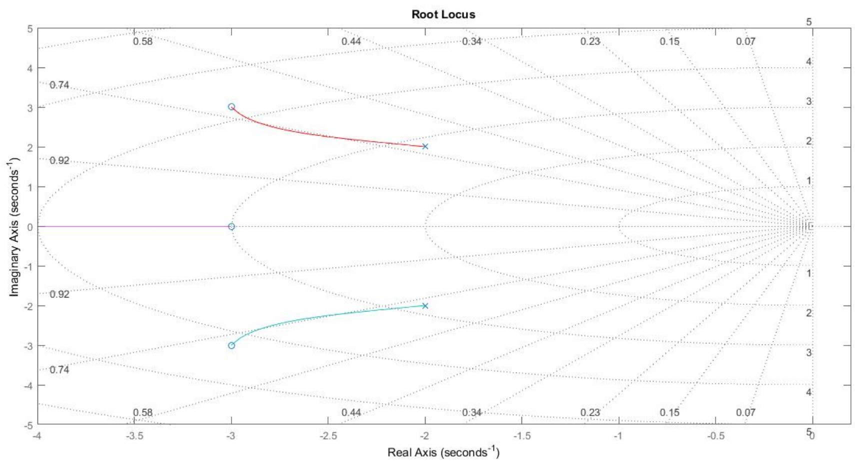


**Figura 25.3.** Root locus of the optimal control system with zeros of $\widetilde{P}(s)$ placed in desired locations

that some locations can be difficult to reach or will generate some difficulty in the control. Instead the optimal control is more robust and will take care of that while trying to satisfy the location requests.

Now we want to find out how to fix the zeros with $\widetilde{C}$. To do this let's consider a strictly rational transfer function

$$\widetilde{P}(s) = \frac{b_{n-1}s^{n-1} + ... + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_0},$$

and try to obtain $\widetilde{C}$ from $\{b_i\}_{i=1..n}$.

We can rewrite the numerator of the transfer function as follows:

$$b_{n-1}s^{n-1} + ... + b_0 = b_{n-1}(s^{n-1} + \frac{b_{n-2}}{b_{n-1}}s^{n-2} + .. + \frac{b_0}{b_{n-1}}) = b_{n-1}\prod_{i=1}^{n-1}(s - z_i),$$

where $z_i$ are known.

Then, being $\widetilde{P}(s) = \widetilde{C}(sI - A)^{-1}B = \frac{\widetilde{n}(s)}{d(s)}$, we get $\widetilde{n}(s)$ as a linear polynomial in $\{\widetilde{C}_i\}_{i=1..n}$.

Finally, with the notation:

$$b = \begin{bmatrix} b_{n-1} & .. & b_0 \end{bmatrix}^T \in \mathbb{R}^n,$$
$$\widetilde{C} = \begin{bmatrix} \widetilde{C}_1 & .. & \widetilde{C}_n \end{bmatrix} \in \mathbb{R}^{1\times n};$$

the final result is a linear system of equations:

$$b = F\widetilde{C}^T \Longrightarrow \widetilde{C} = (F^{-1}b)^T,$$

where F is a suitable invertible matrix which is a function of the system parameters.

## 25.2 Closed loop poles inside a specific half plane

We would like to force the contoller to have certain properties.
For example we want a desired rising time $t_r$, obtained by forcing the poles of the closed loop system to be in a certein half plane which is on the left of the imaginary axis.

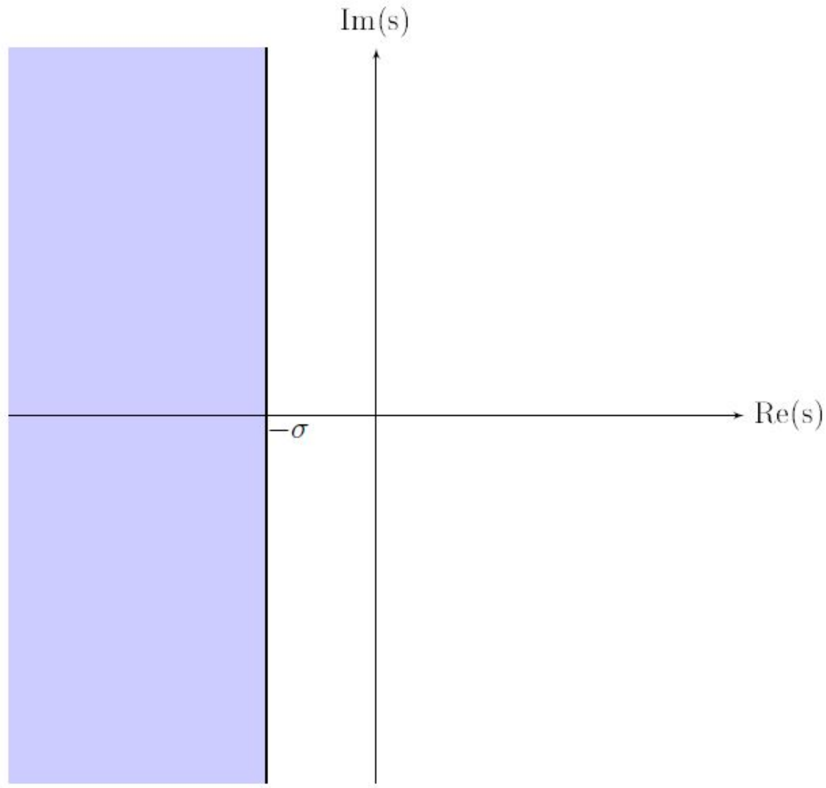


**Figura 25.4.** Dominant poles approximation region of desired $tr$

Can we modify our control problem such that the poles of the close loop are in the half plane in figure? We want an LQ optimal control where the eigenvalues of the closed loop system have the additional constrain:

$$\mathrm{Re}[\lambda(A - BK_{LQ})] < -\alpha, \tag{25.1}$$

A possible solution to enforce this additional property is by solving the optimal LQ control on the following modified dynamical systems:

$$\begin{aligned} \dot{x}(t) &= (A + \sigma I)x(t) + Bu(t) \\ &= A'x(t) + Bu(t) \end{aligned} \tag{25.2}$$

This fictitious system $(A', B)$ and our original weights $Q$, $R$ still preserve the required reachability/observability properties:

$$\begin{cases} (A', B) & \textit{reachable} \\ (A', Q^{1/2}) & \textit{observable} \end{cases} \tag{25.3}$$

With this two hypothesis we can apply optimal control and obtain the guarantee that the closed loop system poles respect

$$\lambda(A' - BK_{LQ})] < -\sigma, \tag{25.4}$$

By replacing $A' = (A + \sigma I)$ in (25.4) we obtain

$$\begin{aligned} \lambda(A' - BK_{LQ}) &= \lambda(A' + \sigma I - BK_{LQ}) \\ &= \sigma + \lambda(A - BK_{LQ}) \end{aligned}, \tag{25.5}$$

so basically the eigenvalues of $(A' - BK_{LQ})$ are exactly the same of $(A - BK_{LQ})$, but shifted to the left by an amount $-\alpha$, while the eigenvectors remain the same.

If we want a desired overshoot $M_p$, we need to guarantee that the poles of the close loop are are in a certain region of the plane, for example by forcing them to be in a circle centered in $-\alpha$ and with desired radius $\rho$.

With this choose of the region of the plane in figure 25.5 we will guarantee a desired rising time $t_r$ given by $\alpha$ and a desired overshoot $M_p$ gived by $\rho$.

How do we force the system to have this additional property? If we want to solve that problem we have to talk about discrete time optimal LQ control.

## 25.3 Discrete time optimal control

We have the discrete time linear system

$$\begin{cases} x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_k \end{cases} \tag{25.6}$$

and the cost

$$J_T(x_0, u) = \frac{1}{T} \sum_{k=0}^{T-1} (x_k^T Q x_k + u_k^T R u_k) + x_T^T Q_T x_T \tag{25.7}$$

where $u = (u_0, \ldots, u_{T-1})$. The optimal control problem consists in optimizing the signal $u$ to minimize this cost, so we want to

$$u^* = \underset{u}{\text{argmin}} J_T(x_0, u) \tag{25.8}$$

where $u^* = (u_0^*, \ldots, u_{T-1})$. Even in this case we can define optimal cost-to-go function

$$V^*(x_k, k) = \underset{u_k, \ldots, u_{T-1}}{\text{argmin}} \sum_{h=k}^{T-1} (x_h^T Q x_h + u_h^T R u_h) + x_T^T Q_T x_T \tag{25.9}$$

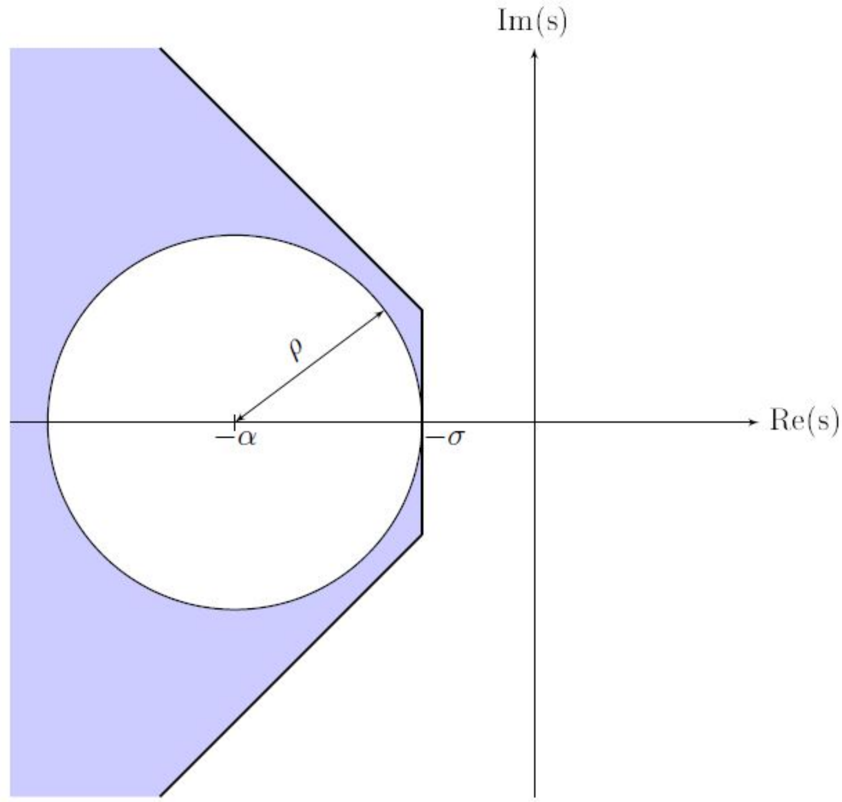


**Figura 25.5.** Dominant poles approximation region of desired $M_p$ and $tr$

which can be seen to have the following quadratic expression:

$$V^*(x_k, k) = x_k^T P_k x_k \tag{25.10}$$

where

$$P_k = AP_{k+1}A^T + Q - A^T P_{k+1} B(B^T P_{k+1} B + R)^{-1} B^T P_{k+1} A \tag{25.11}$$

It is also possible to show that the optimal control is given by:

$$u_k^* = -K_k x_k \tag{25.12}$$

where

$$K_k = (R + B^T P_k B)^{-1} B^T P_k \tag{25.13}$$

If

$$\begin{cases} T \longrightarrow +\infty \\ (A, B) \qquad \textit{reachable} \\ (A, Q^{1/2}) \qquad \textit{observable} \end{cases} \tag{25.14}$$

we can show that

- $P_0 \longrightarrow \overline{P}$ (unique) $\quad\forall\ Q_T$ (initial condition)

- $K_k \longrightarrow K_{LQ} = (R + B^T\overline{P}B)^{-1}B^T\overline{P}$ where $\mid \lambda(A - BK_{LQ}) \mid < 1$

Now, we want to use this property to solve the countinous time problem where we have the system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \\ u(t) = -K_{LQ}x(t) \end{cases} \tag{25.15}$$

with the addition property such that the eigenvalues $\lambda(A - BK_{LQ})$ belong to a circle with center in $-\alpha$ and radius $\rho$.

Consider, now, this discrete time system

$$z_{k+1} = \frac{1}{\rho}Az_k + \frac{1}{\rho}Bu_k = A_d z_k + B_d u_k \tag{25.16}$$

It is easy to verify that we still have

$$\begin{cases} (A_d, B_d) \qquad \textit{reachable} \\ (A_d, Q^{1/2}) \qquad \textit{observable} \end{cases} \tag{25.17}$$

So, if now we apply discrete time LQ control, we find $K_{LQ}$ such that

$$\begin{aligned} &\mid \lambda(A_d - B_d K_{LQ}) \mid < 1 \Longrightarrow \mid \lambda(\frac{A}{\rho} - \frac{1}{\rho}BK_{LQ}) \mid < 1 \\ &\Longrightarrow \mid \lambda(\frac{1}{\rho}(A - BK_{LQ})) \mid < 1 \Longrightarrow \frac{1}{\rho} \mid \lambda(A - BK_{LQ}) \mid < 1 \\ &\Longrightarrow \mid \lambda(A - BK_{LQ}) \mid < \rho \end{aligned} \tag{25.18}$$

In this case, the eigenvalues of this matrix are inside the circle of radius $\rho$ centered in the origin, however we want to center the circle in $-\alpha$. We can do so by considering the following fictitious system:

$$z_{k+1} = \underbrace{\frac{1}{\rho}(A + \alpha I)}_{A'_d} z_k + \underbrace{\frac{1}{\rho}B}_{B_d} u_k = A'_d z_k + B_d u_k \tag{25.19}$$

and, once again,

$$\begin{cases} (A'_d, B_d) \qquad \textit{reachable} \\ (A'_d, Q^{1/2}) \qquad \textit{observable} \end{cases} \tag{25.20}$$

Then, we apply the **discrete time** optimal control to this dynamical system with the original Q and R system and we obtain that $\mid \lambda(A_d - B_d K_{LQ}) \mid$ will be inside the circle with center in $-\alpha$ and radius $\rho$, as seen before in figure 25.5.
In principle we can place the circle also taking into consideration the desired raising time and overshoot for the system.

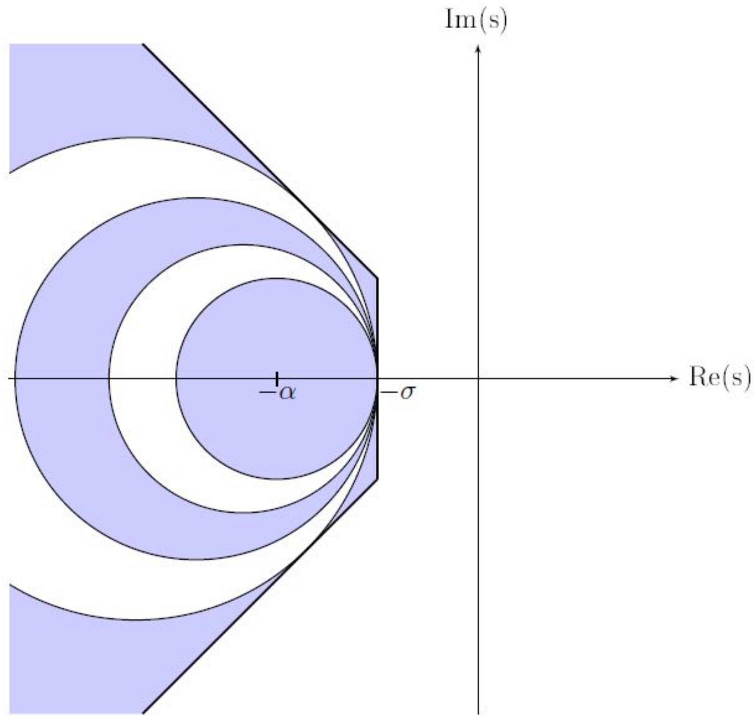


**Figura 25.6.** Possible choises of circles centered in *alpha* with radius $\rho$ where to place the poles of the closed loop system

So, we must choose $\rho$ and $\alpha$ such that the circle is inside the specification region and it is as large as possible, as we can see in the figure 25.6.

## 25.4 Selection of Q and R

Next step is to determine $R = r$ and $Q \in \mathbb{R}^{n \times n}$. We have to consider physically meaningful state and control variables to select Q and R. In particular, there are different choices for Q:

1. $Q = C^T C$. The latter expression allows us to write:

$$x(t)^T Q x(t) = x(t)^T C^T C x(t) = y^T(t) y(t) = y^2(t).$$

Therefore, the quadratic cost function to minimize is the following:

$$J(u, x_0) = \int_0^\infty y^2(t) + r u^2(t) dt$$

The choice $Q = C^T C$ is useful when there is a specific output $y(t) = Cx(t)$ that needs to be kept small.

2. $Q = \widetilde{C}^T\widetilde{C}$. In this way we can force $n-1$ poles of the closed-loop system to be close to some locations (the zeros of $\widetilde{P}(s) = \widetilde{C}(sI-A)^{-1}B$).

3. $Q = I$. Similarly to the first case, we have that the related quadratic cost function is given by:
$$J(u, x_0) = \int_0^{\infty} \sum_i x_i^2(t) + ru^2(t)dt = \int_0^{\infty} \|x\|^2 + ru^2(t)dt$$

4. Choose Q (and R) to be diagonal in the absence of information about coupling. For instance, let us define:
$$Q = \begin{bmatrix} q_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & q_n \end{bmatrix} \qquad q_i > 0$$
This choice leads us to have:
$$x(t)^T Q x(t) = \sum_i q_i\, x_i^2(t)$$
We want to obtain acceptable excursions among the components of the signal:
$$|x_i(t)| < x_{i,max} \quad \forall i \quad \Rightarrow \quad \left|\frac{x_i(t)}{x_{i,max}}\right| < 1$$
where we are normalizing the state.
A good idea is to use the following choice for the weights $q_i$:
$$q_i = \frac{1}{x_{i,max}^2}$$
The purpose is to chose the $q_is$ so that all entries equally contribute to the cost to be minimized.
For example, if we want to bring the position of the load connected to a motor from $x_1(0) = \theta_l(0) = 0°$ to $x_1(0.1) = \theta_l(0.1) = 100°$ in 0.1 seconds (we denote the mean position as $\bar{x}_1 = 50°$), we have that the (mean) velocity of the load must be:
$$x_2 = \dot{\theta}_l = \frac{100}{0.1} = 1000°/sec$$
Thus, a possible Q is given by:
$$Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \quad \text{with} \quad q_1 = \frac{1}{\bar{x}_1^2} = \frac{1}{50^2} \quad \text{and} \quad q_2 = \frac{1}{x_2^2} = \frac{1}{1000^2}$$

We do not know which is the best option among the 4 choices of Q and this, normally, requires some trial-and-error procedures.

## 25.5 Frequency shaping for LQ optimal control

Let us consider the system shown in Figure 25.7

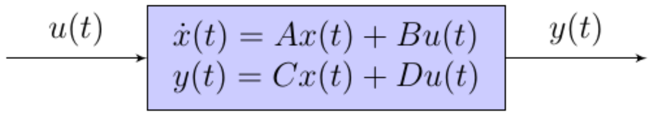


**Figura 25.7.** A generic linear system.

The LQ optimal control is bounded to minimize the following cost function in the time domain:

$$J(u, x_0) = \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)dt.$$

In many situations, it is more advantageous to specify the criteria in frequency domain. For example, it might be useful to penalize control inputs that excites some resonances of the plant at a specific frequency or to reduce some high-frequency noise at the output. (figure 25.8 ) .

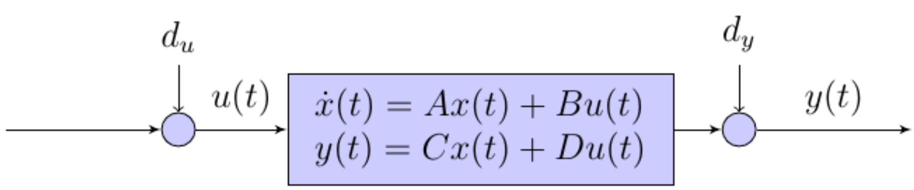


**Figura 25.8.** A generic linear system with additional disturbances.

If we want to reduce noises with high frequencies we can introduce a low-pass filter and consider $\widetilde{y}$ as output (figure 25.9 ) .

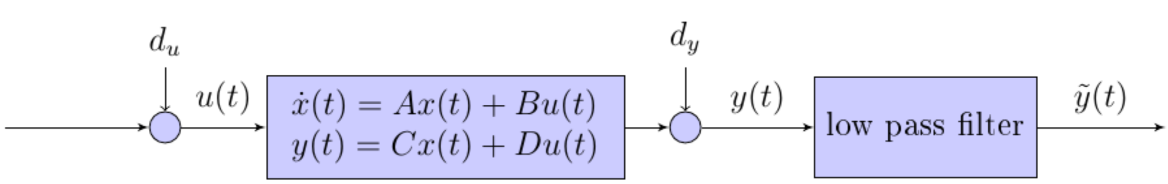


**Figura 25.9.** A linear system influenced by disturbances with low-pass filter

However, the insertion of the low-pass filter involves the slowing down of the overall system. If the system is characterized by some natural resonances, we would like to have an input which does not amplify the inout noises around the resonance frequency. To do that, we can introduce another dynamical systems with transfer functions $P_x(s)$, $P_u(s)$ and $P_y(s)$. The outputs of the latter systems are filtered versions of x(t), u(t) and y(t) respectively (figure 25.10 ).

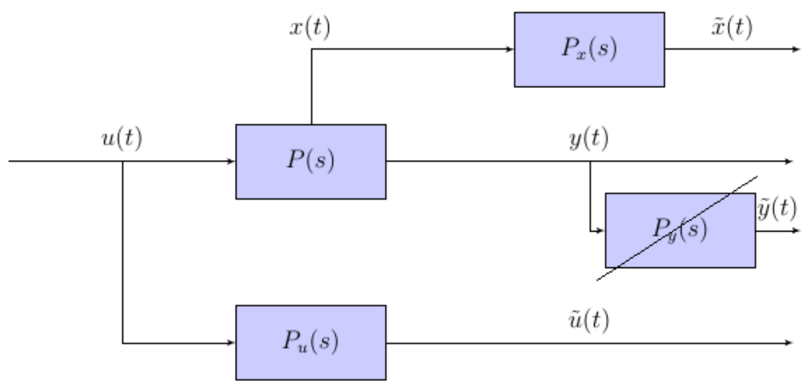


**Figura 25.10.** A linear system with additional dynamical systems $P_u(s)$, $P_y(s)$ and $P_x(s)$

We wonder what happens if we consider $\widetilde{u}(t) = P_u(s)u(t)$. The new LQ optimal control that penalizes $\widetilde{u}$ is given by:

$$J(u, x_0) = \int_0^\infty x^T(t)Qx(t) + \widetilde{u}^T(t)R\widetilde{u}(t)dt.$$

The controller will be forced by the system to have high frequencies at natural frequencies. This is clear by using the Parseval's theorem.

## 25.6 Parseval's theorem

Parseval's theorem has the following form:

$$\int_{-\infty}^\infty \widetilde{u}^T(t)\widetilde{u}(t)dt = \frac{1}{2\pi}\int_{-\infty}^\infty \widetilde{U}^*(j\omega)\widetilde{U}(j\omega)d\omega = \int_{-\infty}^\infty U^*(j\omega)P_u^*(j\omega)P_u(j\omega)U(j\omega)d\omega$$

If the signal is causal, i.e. $u(t) = 0, t < 0$, we have:

$$\int_0^\infty \widetilde{u}^T(t)\widetilde{u}(t)dt = \int_{-\infty}^\infty U^*(j\omega)P_u^*(j\omega)P_u(j\omega)U(j\omega)d\omega$$

This is stating that the integral of the signal in the time domain is exactly equal to the integral of the square of the signal in the frequency domain shaped by the transfer function $P_u(j\omega)$.

In this way, we are going to penalize the components of the original input $u$ at some specific frequencies. This is a way to enforce frequency domain requirements into time domain specifications. This can be done also for the output, using $P_y(s)$. So that, we want to penalize $\widetilde{u}$ and $\widetilde{y}$ and what we do is to take our original system, with some input and some output and shape them differently. Actually, we not only can shape the output, but we can shape the all state itself. We assume that we can access the state and we can pass through some linear system and obtain $\widetilde{x}(t)$. This is what we optimize, i.e. we try to minimize the cost not on the original state and the original input, but there will be a frequency shaped version of both state and input.