

Lezione 24 — 4 May

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24.1 LQ control and root locus for SISO systems

We have the dynamical SISO system:

$$\begin{cases} \dot{x} = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

whose associated transfer function is:

$$P(s) = C(sI - A)^{-1}B = \frac{n(s)}{d(s)}$$

We are interested in understanding what happens when a state feedback is applied, whose gain is determined by the infinite horizon LQ control K_{LQ} . From:

$$\begin{cases} \dot{x} = Ax(t) + Bu(t) \\ y(t) = Cx(t) \\ u(t) = u_{ext}(t) - K_{LQ}x(t) \end{cases}$$

we get:

$$\begin{cases} \dot{x} = (A - BK_{LQ})x(t) + Bu_{ext}(t) \\ y(t) = Cx(t) \end{cases}$$

whose corresponding transfer function is given by:

$$P_d = C(sI - (A - BK_{LQ}))^{-1}B = \frac{n_c(s)}{d_c(s)}$$

Note that the zeros of $n(s)$ are the same of $n_c(s)$ (under the assumption that no cancellation occurs), in fact the state feedback does not move the zeros. The poles in $d(s)$ and in $d_c(s)$, instead, are different.

Remember that K_{LQ} is the gain that minimizes the cost function:

$$K_{LQ} = \operatorname{argmin}_K \int_0^{+\infty} y^2(t) + ru^2(t) dt$$

We are interested in finding poles of closed loop system as a function of the weight r , which is the only parameter we can choose. We know that the spectrum of the closed loop function $\Lambda(A - BK_{LQ})$ is equal to the stable spectrum of the Hamiltonian H (the eigenvalues of H

with real part < 0). So now we will try to compute $\det(sI - H) = 0$ in order to find stable and unstable eigenvalues of H , that are symmetrical with the respect to imaginary axis. During the last lesson, we found that computing $\det(sI - H) = 0$ is equivalent to:

$$d(s)d(-s) + \frac{1}{r}n(s)n(-s) = 0 \quad (24.1)$$

$$1 + \frac{1}{r} \frac{n(s)}{d(s)} \frac{n(-s)}{d(-s)} = 0 \quad (24.2)$$

$$1 + \frac{1}{r} P(s)P(-s) = 0 \quad (24.3)$$

If we call:

$$k = \frac{1}{r}, \text{ with } k > 0 \text{ and } G(s) = P(s)P(-s)$$

we obtain:

$$1 + kG(s) = 0$$

Solving this expression corresponds to computing the poles of the closed loop system as a function of k . The problem that we want so solve is now a root locus problem. The solution to the last expression will give all the eigenvalues of H , positive and negative, so we will find poles both positive and negative but exactly symmetrical. However, remember that $A - BK_{LQ}$ has only stable poles, so the root locus that represents the poles of the closed loop system lie on the left hand side of the complex plane.

24.1.1 Example: Integrator

$$\begin{cases} \dot{x} = u(t) \\ y(t) = x(t) \end{cases}$$

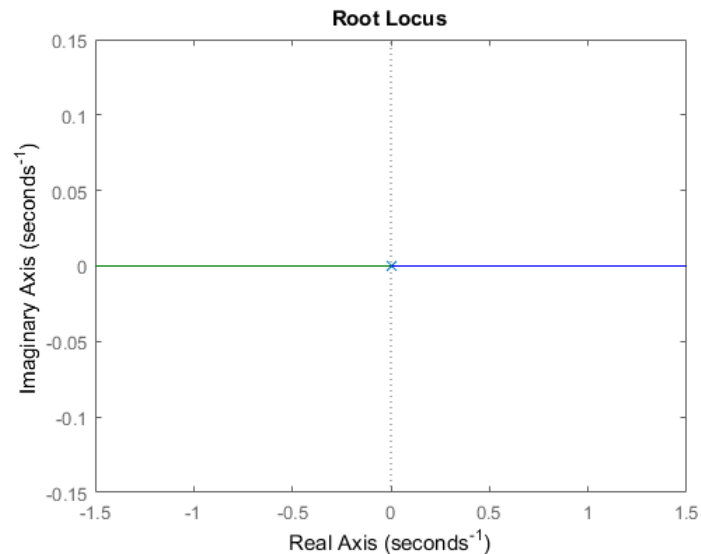
The corresponding transfer function is:

$$P(s) = 1 * s^{-1} * 1 = \frac{1}{s}$$

We are interested in understanding where the closed loop poles will go when optimal LQ control is applied. We have to look at the function

$$G(s) = -\frac{1}{s^2}$$

$G(s)$ mirrors the poles and the zeros of the original transfer function with respect to the imaginary axis. In this case $G(s)$ has 2 poles in 0. Note that there is a minus sing in $G(s)$, so insted of $G(s)$ we can consider $G'(s) = -G(s)$, but we have to compute the negative

Negative root locus for $G(s)$

root locus. For the LQ control it is very important to figure out if we need to compute the negative or the positive root locus, otherwise we would get a completely wrong solution.

An interesting thing is to see what happens for

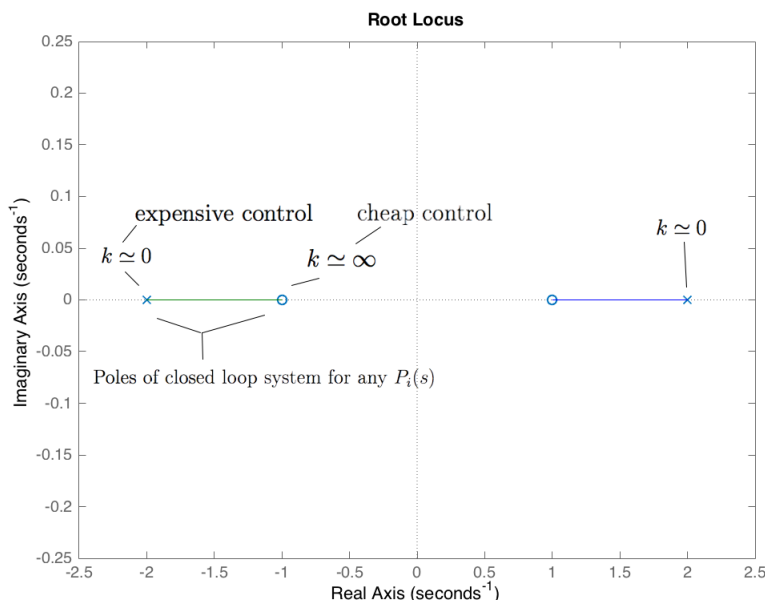
- $r \rightarrow 0 \leftrightarrow k \rightarrow +\infty$ which is the cheap control and pushes poles on the left side
- $r \rightarrow +\infty \leftrightarrow k \rightarrow 0$ which is the expensive control and keeps poles near imaginary axis

Let us now consider 4 different transfer functions relative to the open loop system to be controlled:

- $P_1(s) = \frac{s+1}{s+2} \rightarrow G_1(s) = P_1(s)P_1(-s) = \frac{s+1}{s+2} \frac{1-s}{2-s} = \frac{s^2-1}{s^2-4}$
- $P_2(s) = \frac{s-1}{s+2} \rightarrow G_2(s) = \frac{s-1}{s+2} \frac{-1-s}{2-s} = \frac{s^2-1}{s^2-4}$
- $P_2(s) = \frac{s+1}{s-2} \rightarrow G_3(s) = \frac{s^2-1}{s^2-4}$
- $P_2(s) = \frac{s-1}{s-2} \rightarrow G_4(s) = \frac{s^2-1}{s^2-4}$

Since they have the same $G(s)$, they have the same root locus. We know that the the root locus cannot cross the imaginary axis, so we can easily understand if the positive or negative one is needed. In our case is the positive root locus. So, let us go back to the original problem; we have exactly the same root locus, for the same r we find exactly the same poles in the close loop but we have totally different systems. It is important to remember that we have to consider the original system. For example, we can choose a specific \bar{r} such that our $d_c(s) = (s + \frac{3}{2})$

In this way, we get:

Root locus for $G(s)$

- $P_{1,cl}(s) = \alpha_1 \frac{s+1}{s+\frac{3}{2}}$
- $P_{2,cl}(s) = \alpha_2 \frac{s-1}{s+\frac{3}{2}}$
- $P_{3,cl}(s) = \alpha_3 \frac{s+1}{s+\frac{3}{2}}$
- $P_{4,cl}(s) = \alpha_4 \frac{s-1}{s+\frac{3}{2}}$

where α_i are some suitable constants. Note that the output in the close loop system of $P_{1,cl}(s)$ and $P_{2,cl}(s)$ is not the same.

The output from $P_{1,cl}(s)$ and $P_{3,cl}(s)$ are identical but $K_{LQ,1}$ and $K_{LQ,3}$ are different, in this case $K_{LQ,3} \gg K_{LQ,1}$. This means that we have to push much more in order to make the closed loop system of an unstable system to behave like $P_{1,cl}$ which is already stable. The same will happens for $P_{2,cl}(s)$ and $P_{4,cl}(s)$.

Observation: if we have a system and we want to minimize the cost $\int y^2(t) + ru^2(t)dt$, if $r \approx 0$ we would expect to put the closed loop poles very negative in order to go to zero faster. But looking at the root locus, the optimal control tell us to do the opposite, we should approach the zero in -1 . Placing the poles far away from the zero will give rise to a larger cost, since the open loop zero is helping cancelling the effect of the closed loop pole close to it.

What is probably even more surprising is what happens when you have a transfer function $P_2(s)$ with root locus shown in Fig. 24.2

We cannot move the zero, but the LQ optimal control problem tell us that even if we do

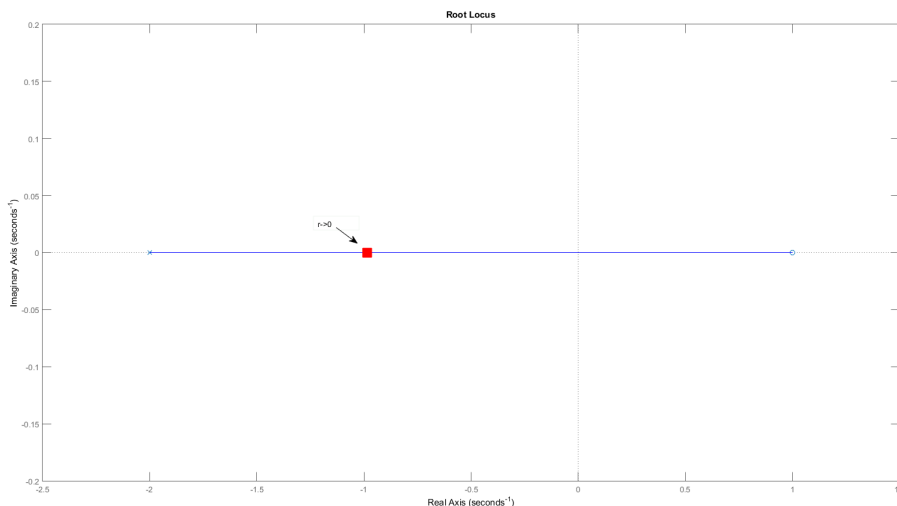


Figura 24.1: Root locus for $P_2(s)$

not have any physical open loop zero, we still have to place one closed loop pole there and if we want to minimize the cost, eventually we have to place the closed loop pole in -1 . Any other choice of the closed loop poles will make the cost larger.

So, the root locus for LQ optimal control tells us how the poles should be placed in the closed loop for a general SISO system. Knowing the poles and the zeros and using the root locus we can infer where they should be approximately located in order to obtain very good performances, and in particular for the expensive or the cheap control scenarios.

24.2 DC motor

As was mentioned in previous lectures, the DC motor model under consideration lead to a transfer function of a second order system which has two poles (one of which is at the origin) and no zeros, namely

$$P(s) = \frac{b}{s(s+p)}$$

and the cost function which has to be minimized is

$$J(x_0, u(t)) = \int (\theta_l^2(t) + ru^2(t)) dt.$$

In order to see where the closed-loop system poles are located as r (free parameter) changes from 0 to ∞ , the "special" root locus of $1 + kG(s)$, where $k = \frac{1}{r}$, should be examined. In

this particular case, $G_1(s)$ is defined as follows

$$G_1(s) = P(s)P(-s) = \frac{b}{s(s+p)} \frac{b}{s(s-p)} = \frac{b^2}{s^2(s+p)(s-p)}$$

so that the positive root locus has to be plotted (see figure 24.1).

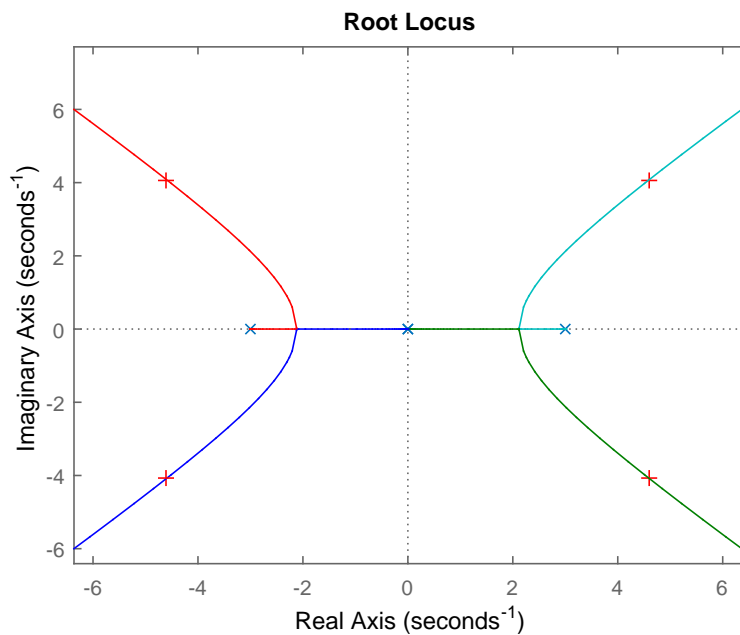


Figure 24.2: Root locus of $G_1(s)$.

Notice that, since $P(-s)$ has no physical meaning, the only relevant part of the root locus is the one on the left half plane (blue and red braces). Starting from the pole at the origin and the pole at $-p$ with $k = 0$ (i.e. $r = \infty$, with abuse of notation) the closed-loop poles are located, as k increases, in the performance region determined by the desired rising time, settling time and overshoot. Therefore LQ control approach automatically performs (once r free parameter has been selected) poles placement according to the dominant pole approximation approach seen in previous lectures.

24.3 DC motor with integral control

If an integral action is required, the DC motor model transfer function can be modified fictitiously (including an additional pole at the origin) as follows

$$P(s) = \frac{b}{s^2(s+p)}.$$

The special root locus which has to be plotted is referred to $G_2(s)$ defined as

$$G_2(s) = \frac{b}{s^2(s+p)} \frac{b}{-s^2(-s+p)} = -\frac{b^2}{s^4(s^2-p^2)}$$

so that the negative root locus need to be examined (see figure 24.3).

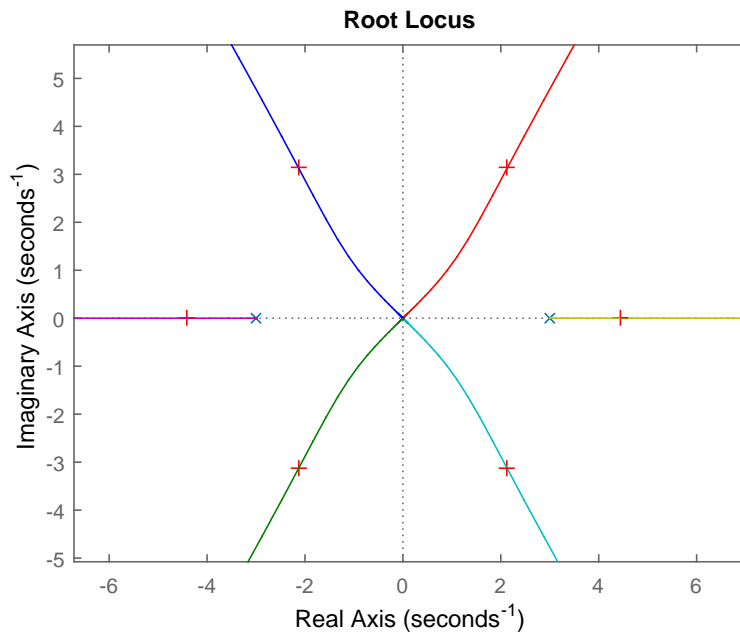


Figura 24.3: Root locus of $G_2(s)$.

Once again, the only relevant part of the root locus is the one on the left half plane (blue, green and violet braces) and can be noticed that poles locations determined by LQ control approach form a so-called Butterworth filter configuration which is similar to configuration that one would propose based on the standard poles placement design approach.

24.4 Asymptotes calculation rules

In order to plot the asymptotes along which root locus branches go to infinity, a few simple rules has to be followed. Let us consider the transfer function $P(s)$ with

- n = number of poles of $P(s)$
- m = number of zeros of $P(s)$

and the problem of plotting the root locus of $G(s) = P(s)P(-s)$. As k goes to $+\infty$:

- $2m$ branches goes to zeros of $P(s)$
- $2(n - m)$ branches goes to infinity along asymptotes (which are straight lines starting at the origin)

The angles between asymptotic straight lines and real positive semi-axes (counterclockwise direction) are:

- if $(n - m)$ is odd $\rightarrow \vartheta = \frac{\pi}{n - m}h$ $h = 0, 1, \dots, 2(n - m) - 1$
- if $(n - m)$ is even $\rightarrow \vartheta = \frac{\pi}{n - m}(h + \frac{1}{2})$ $h = 0, 1, \dots, 2(n - m) - 1$

For example, if the transfer function $P(s) = \frac{1}{s^4}$ is considered, the root locus of $G_3(s) = P(s)P(-s)$ is the one showed off in figure 24.3.

24.5 DC motor with flexible joint

As we will see later, the model of the DC motor with flexible joint leads to the following transfer function

$$P(s) = \frac{K(s + z)(s + \bar{z})}{s(s + p_1)(s + p_2)(s + \bar{p}_2)}.$$

The system has 4 poles and 2 zeros, so that

- $n = 4$
- $m = 2$
- $(n - m) = 2$

The root locus of $G_4(s) = P(s)P(-s)$ is showed in figure 24.4.

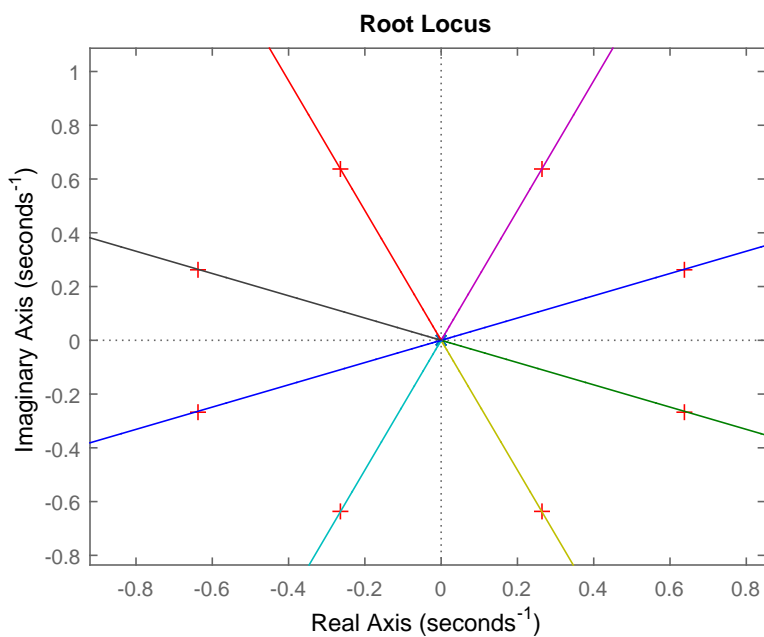


Figura 24.4: Root locus of $G_3(s) = \frac{1}{s^8}$.

24.6 Weights (Q, R) design

If we consider the SISO system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

with $x(t) \in \mathbb{R}^n$ and the cost function

$$J(x_0, u(t)) = \int_0^{+\infty} (x^T(t)Qx(t) + ru^2(t)) dt$$

the simplest choice for the matrix Q is $Q = C^T C$ so that $x^T(t)Qx(t) = y^2(t)$ and the cost function becomes

$$J(x_0, u(t)) = \int_0^{+\infty} (y^2(t) + ru^2(t)) dt$$

which correspond to penalizing the output instead of the state.

An alternative choice could be to define a new "virtual" output $\tilde{y}(t) = \tilde{C}x(t)$, with \tilde{C} such that $Q = \tilde{C}^T \tilde{C}$ and (A, \tilde{C}) observable (which is equivalent of saying that $(A, Q^{\frac{1}{2}})$ observable).

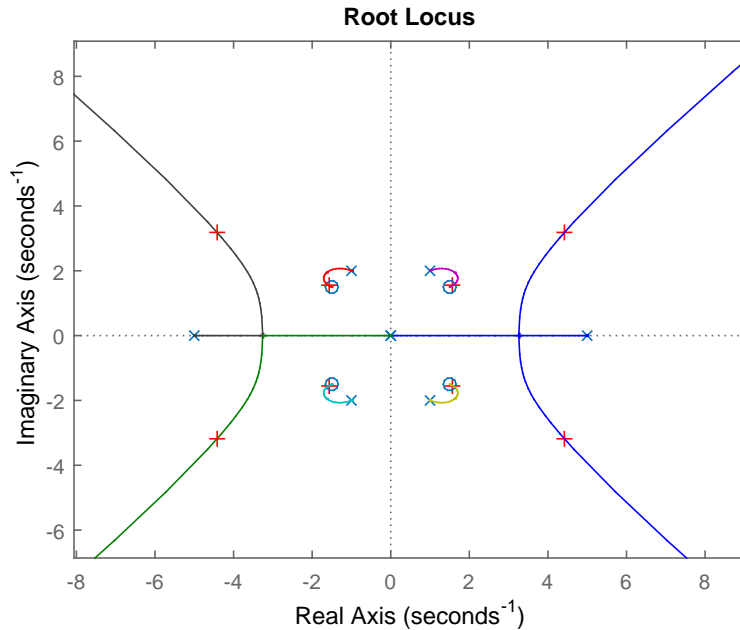


Figure 24.5: Root locus of $G_4(s)$.

In this particular case, we have

$$\int_0^{+\infty} (\tilde{y}^2(t) + ru(t)) dt \xrightarrow{\text{optimal } LQ} u(t) = \tilde{K}_{LQ} x(t) \quad (A - B\tilde{K}_{LQ}) \text{ asymptotically stable}$$

The transfer functions associated to \tilde{C} and C are respectively

$$\tilde{P}(s) = \tilde{C}(sI - A)^{-1}B = \frac{\tilde{n}(s)}{d(s)}$$

and

$$P(s) = C(sI - A)^{-1}B = \frac{n(s)}{d(s)}.$$

Notice that they share the same denominator, but the numerators are different, so playing with \tilde{C} the numerator can be manipulated (i.e. up to $n - 1$ zeros can be moved).