Control Laboratory: **a.a. 2015/2016**

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23.1 Hamiltonian and optimal control problem

Recalling the previous lectures, optimal LQ control problem is associated with the solution of Riccati equation:

$$-\dot{P} = A^T P(t) + P(t)A + Q - P(t)BR^{-1}B^T P(t),$$

which has final condition $P(T, T) = Q_T$.
We have also shown the following results:

- $V_o^*(x_0) = x_o^T P(0)x_0$, where $x(0) = x_0$;
- $P(t, T) = g(e^{\Lambda_1(t-T)}, e^{\Lambda_2(t-T)}, W)$, where W comes from decomposition of Hamiltonian matrix H $= \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}^{-1}$;
- $\lim_{T-t\to+\infty} P(t, T) = \lim_{T\to+\infty} P(0, T) = \bar{P} = W_{21}W_{11}^{-1}$, where W_{21} and W_{11} are blocks of W;

We are up to find a proper feedback gain for the system,

$$K(t, T) = R^{-1}B^T P(t, T),$$

with is sued to compute the optimal input $u^*(t) = -K(t, T)x(t)$.

To solve the optimal control problem, we see that if we consider $\bar{P}$, solution of Riccati equation in infinte horizon, then we obtain the time-invariant gain

$$K_{LQ} = R^{-1}B^T \bar{P},$$

and input becomes

$$\lim_{T\to+\infty} u^*(t) = -K_{LQ}x(t).$$

It is now necessary for this procedure to make two hypotheses:

- $(A, Q^{\frac{1}{2}})$ observable;
- (A, B) reachable.

These hypotheses guarantee that:

1. $\bar{P}$ exists on infinite horizon;
2. $\bar{P}$ is unique and does not depend on the particular Q_T we choose;
3. $A - BK_{LQ}$ is asymptotically stable, i.e. the controller will stabilize the closed loop system;
4. the eigenvalues of $A_c = A - BK_{LQ}$ coincide with stable eigenvalues of H.

Hypotheses made on observability and reachability directly imply points 1 and 2.

23.1.1 F asymptotically stable

We give a proof for point 3. Re-writing F as $F = A - BK_{LQ}$, we obtain

$$\begin{aligned} F^T\bar{P} + \bar{P}F^T &= (A - BR^{-1}B^T\bar{P})^T\bar{P} + \bar{P}(A - BR^{-1}B^T\bar{P}) = \\ &= A^T\bar{P} - \bar{P}BR^{-1}B^T\bar{P} + \bar{P}A - \bar{P}BR^{-1}B^T\bar{P} = \\ &= -Q - \bar{P}BR^{-1}B^T\bar{P} \end{aligned}$$

using the equivalence given by the fact that $\lim_{T\to+\infty} \dot{P}(t) = 0$, and so

$$A^T\bar{P} + A\bar{P} + Q - \bar{P}BR^{-1}B^T\bar{P} = 0.$$

Let us now suppose *ab absurdum* that it exists $v \in \mathbb{C}^n$, eigenvector of F for eigenvalue $\lambda \in \mathbb{C}$, such that $v \neq 0$ and $\Re(\lambda) > 0$. We can write the expression above as:

$$\begin{aligned} v^*F^T\bar{P}v + v^*\bar{F}Fv &= -v^*Qv - v^*\bar{P}BR^{-1}B^T\bar{P}v = \\ \bar{\lambda}v^*\bar{P}v + \lambda v^*\bar{P}v &= -\|Q^{\frac{1}{2}}v\|^2 - \|R^{-\frac{1}{2}}B^T\bar{P}v\|^2 \\ (\bar{\lambda} + \lambda)\|\bar{P}^{-\frac{1}{2}}v\|^2 &= -\|Q^{\frac{1}{2}}v\|^2 - \|R^{-\frac{1}{2}}B^T\bar{P}v\|^2 \\ 2\Re(\lambda)\|\bar{P}^{-\frac{1}{2}}v\|^2 &= -\|Q^{\frac{1}{2}}v\|^2 - \|R^{-\frac{1}{2}}B^T\bar{P}v\|^2 \end{aligned}$$

Quantity on the right hand side of the expression is surely ≤ 0, but quantity on the left hand side is ≥ 0 because $\bar{P} \geq 0$, so this implies that to satisfy the equivalence all terms must be equal to zero and in particular:

$$Q^{\frac{1}{2}}v = 0, \quad B^T\bar{P}v = 0$$

We can express F as

$$Fv = (A - BR^{-1}B^T\bar{P})v = Av - BR^{-1}B^T\bar{P}v = Av = \lambda v$$

but since $Q^{\frac{1}{2}}v = 0$ this implies that PBH test is violated since this could imply that $(A, Q^{\frac{1}{2}})$ is not observable. So matrix F must be asymptotically stable.

23.1.2 Coincidence of eigenvalues

We said on point 4 that eigenvalues of matrix $F = A - BK_{LQ}$, obtained with the application of optimal control input $u^* = -R^{-1}B^T\bar{P}x$, coincide with the stable eigenvalues of matrix H. To prove that, we must choose a full-rank matrix T such that it is possible a change of basis for H:

$$T = \begin{bmatrix} I & 0 \\ -\bar{P} & I \end{bmatrix}, T^{-1} = \begin{bmatrix} I & 0 \\ \bar{P} & I \end{bmatrix}$$

Knowing that the eigenvalues are invariant to change of basis, we can compute:

$$THT^{-1} = \begin{bmatrix} I & 0 \\ -\bar{P} & I \end{bmatrix} \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I & 0 \\ \bar{P} & I \end{bmatrix} =$$

$$= \begin{bmatrix} A - BR^{-1}B^T\bar{P} & -BR^{-1}B^T \\ -\bar{P}A - Q + \bar{P}BR^{-1}B\bar{P} - A^T\bar{P} & -A^T + \bar{P}BR^{-1}B^T \end{bmatrix} = \begin{bmatrix} F & -BR^{-1}B^T \\ 0 & -F^T \end{bmatrix}$$

Last matrix is upper triangular, so eigenvalues are all contained in diagonal blocks, in particular F contains all stable eigenvalues, while $-F^T$ has their opposite values (unstable).
In Matlab there is a specifical function that given the weights Q, R and the matrices A, B returns the gain for optimal control, K_{LQ}: `[K_LQ] = lqr(A,B,Q,R)`.

23.2 LQ design: Root locus approach

In a SISO system, the eigenvalues of the closed-loop resulting matrix $F = A - BK_{LQ}$ can be also computed through a suitable root locus approach. Let us consider the LTI system:

$$\begin{cases} \dot{x} = Ax(t) + Bu(t), & x \in \mathbb{R}^n \\ y(t) = Cx(t), & u, y \in \mathbb{R} \end{cases} \tag{23.1}$$

and the infinite-horizon cost:

$$\int_0^{+\infty} y(t)^2 + ru(t)^2 dx,$$

where $r \geq 0$ is the control weight. We could further distinguish two different behaviours of the controller, based on the structure of the cost:

$$r \to 0 \quad \Rightarrow \quad cheap - control$$

$$r \to +\infty \quad \Rightarrow \quad expensive - control.$$

Thanks to the optimal control cost-function $\int_0^{+\infty}(x(t)^TQx(t) + u(t)^TRu(t))dx$, we can relate Q and C through the following formulation:

$$y^2(t) = y^T(t)y(t) = x^T(t)C^TCx(t) \Rightarrow Q = C^TC \quad (rank = 1)$$

For this reason, the only free parameter which can be chosen in the LQ design is $R = r$. The Hamiltonian system associated to 23.1 is defined as follows:

$$\begin{bmatrix} A & -\frac{1}{r}BB^T \\ -C^TC & -A^T \end{bmatrix} \tag{23.2}$$

where the spectrum of $F = A - BK_{LQ}$ coincides with the stable eigenvalues of (23.2); $\lambda(F)$ corresponds to the poles of the closed-loop transfer function.

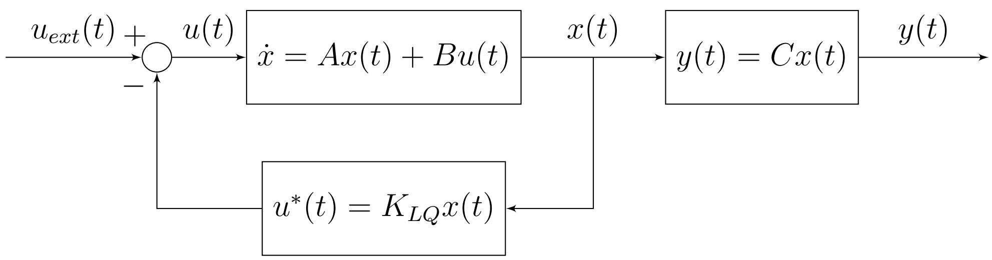

Figure 23.1. Closed-loop system: LQ control

$$Y(s) = C(sI - F)^{-1}BU_{ext}(s) = \frac{n_c(s)}{d_c(s)}U_{ext}(s),$$

where $P_c(s) = C(sI - F)^{-1}B$, and $\lambda(F)$ = poles of $P_c(s)$. Under the formulation $\lambda(H) = \det(sI - H) = 0 = d_c(s)d_c(-s)$, we show these preliminary results.

23.2.1 Preliminary results

1. $\det(AB) = \det(A)\det(B)$;

2. $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det(D - CA^{-1}B)$;

3. $\text{trace}(AB) = \text{trace}(BA)$;

4. $\text{trace}(A) = \sum_i \lambda_i \Rightarrow \lambda_i$ is eigenvalue of A (possibly with repetition according to the Jordan diagonalization)

   $PROOF:$

   if $\exists T$ such that $J = TAT^{-1} \quad \Rightarrow \quad \text{trace}(A) = \text{trace}(T^-1JT) = \text{trace}(JTT^{-1}) = \text{trace}(J)$

5. $x, y \in \mathbb{R}^n, \quad E = xy^T \in \mathbb{R}^{n\times n}$ which is a rank-1 matrix, therefore there are $n - 1$ eigenvalues in 0

$\Lambda(E) = \{\lambda_1 \neq 0, \lambda_2 = ... = \lambda_n = 0\}$

$\Lambda(I + E) = \{\lambda_1 + 1, \lambda_2 = ... = \lambda_n = 1\}$

$\text{trace}(E) = \lambda_1 = \text{trace}(xy^T) = \text{trace}(y^T x) = y^T x$

$\det(I + E) = \prod_i \bar{\lambda}_i = 1 + \lambda_1 = 1 + \text{trace}(E); \qquad \bar{\lambda}_i \in \Lambda(I + E)$

Now we determine the characteristic polynomial of the Hamiltonian about the system

$$\det\left(\begin{bmatrix} sI & 0 \\ 0 & sI \end{bmatrix} - \begin{bmatrix} A & -\frac{1}{r}BB^T \\ -C^TC & -A^T \end{bmatrix}\right) = \det\left(\begin{bmatrix} sI - A & \frac{1}{r}BB^T \\ C^TC & sI + A^T \end{bmatrix}\right) \overset{(2)}{=}$$

$$\overset{(2)}{=} \det(sI - A)\det(sI + A^T - C^TC(sI - A)^{-1}\tfrac{1}{r}BB^T) =$$

remembering that $P(s) = C(sI - A)^{-1}B = \frac{n(s)}{d(s)}$, where are open loop poles and zeros

$$= \det(sI - A)\det(sI + A^T - \tfrac{P(s)}{r}C^TB^T) =$$

$$= \det(sI - A)\det\left((sI + A^T)\left(I - \tfrac{P(s)}{r}(sI + A^T)^{-1}C^TB^T\right)\right) \overset{(1)}{=}$$

$$= \det(sI - A)\det(sI + A^T)\det\left(I - \tfrac{P(s)}{r}\underbrace{(sI + A^T)^{-1}C^TB^T}_{rank-1}\right) \overset{(5)}{=}$$ A and A^T have the same eigenvalues

$$= \det(sI - A)\det(sI + A)\det\left(1 - \tfrac{P(s)}{r}\text{trace}\left((sI + A^T)^{-1}C^TB^T\right)\right) \overset{(3)}{=}$$

$$= \det(sI - A)\det(sI + A)\det\left(1 - \tfrac{P(s)}{r}\text{trace}(\underbrace{B^T(sI + A^T)^{-1}C^T}_{scalar})\right) =$$

$$= \det(sI - A)\det(sI + A)\det\left(1 - \tfrac{P(s)}{r}B^T(sI + A^T)^{-1}C^T\right) =$$

$$= \det(sI - A)\det(sI + A)\det\left(1 - \tfrac{P(s)}{r}\left(B^T(sI + A^T)^{-1}C^T\right)^T\right) =$$

$$= \det(sI - A)\det(sI + A)\det\left(1 - \tfrac{P(s)}{r}C(sI + A)^{-1}B\right) =$$

$= \det(sI - A)\det(-(-sI - A))\det\left(1 + \frac{P(s)}{r}C(-sI - A)^{-1}B\right) =$

$= d(s)(-1)^n d(-s)\left(1 + \frac{1}{r}P(s)P(-s)\right) =$

$= d(s)d(-s)\left(1 + \frac{n(s)}{d(s)r}\frac{n(-s)}{d(-s)}\right) = 0$

$d(s)d(-s) + \frac{1}{r}n(s)n(-s) = 0$ Poles of closed loop system