Control Laboratory:	**a.a. 2015/2016**

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22.1 Properties of the Hamiltonian matrix

As we have seen in the previous lecture, we can solve the Riccati equation in continuos time using the Hamiltonian matrix H given by:

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}.$$

In this lecture we will analyze some properties of this particular matrix.

Property 1: If we define a matrix J such as

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

then we have

$$JHJ = H^T.$$

The proof is simply done by inspection:

$$JHJ = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} = \begin{bmatrix} A^T & -Q \\ -BR^{-1}B^T & -A \end{bmatrix}$$

which is of course the transpose of H.

Property 2: If $\lambda \in \Lambda(H)$, then also $-\lambda \in \Lambda(H)$; in particular we will see that $-\lambda \in \Lambda(H^T)$ since H and H^T have the same spectrum.
Let's take $v \in \mathbb{C}^{2n\times 2n}$ as an eigenvector for the eigenvalue λ:

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

so we have $Hv = \lambda v$.
We can now define a new vector w as:

$$w = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}.$$

We will show now that $H^T w = -\lambda w$. In fact, for the previous property we have:

$$H^T w = JHJw = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} H \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} H \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Then, by definition of eigenvector:

$$H^T w = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_2 \\ -v_1 \end{bmatrix} = -\lambda \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix} = -\lambda w.$$

Property 3: If (A, B) is reachable and $(A, Q^{1/2})$ is observable, if $\lambda \in \Lambda(H)$ then $\mathbb{R}[\lambda] \neq 0$. We will prove it by contradiction: we define a vector $v \neq 0$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{C}^{2n}$$

and $\mu \in \mathbb{R}$ such that $Hv = j\mu v$. We also define a vector w as

$$w = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} \in \mathbb{C}^{2n}.$$

We can write

$$w^* Hv = \begin{bmatrix} v_1^* & v_2^* \end{bmatrix} \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1^* & v_2^* \end{bmatrix} \begin{bmatrix} Av_1 - BR^{-1}B^T v_2 \\ -Qv_1 - A^T v_2 \end{bmatrix}$$

So we get:

$$w^* Hv = v_2^* A v_1 - v_2^* BR^{-1}B^T v_1 - v_1^* Q v_1 - v_1^* A^T v_2$$

Note that the last term is a complex number and its transpose is equal to the first term, so we obtain:

$$w^* Hv = -v_2^* BR^{-1}B^T v_1 - v_1^* Q v_1$$

This is a negative real number because Q and R are positive semidefinite matrices. Decomposing these quadratic forms we obtain:

$$w^* Hv = -\|R^{-1/2}B^T v_2\|^2 - \|Q^{1/2} v_1\|^2 \leq 0$$

Using now the fact that $Hv = j\mu v$, we can write:

$$w^* Hv = w^* j\mu v = j\mu \begin{bmatrix} v_2^* & v_1^* \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = j\mu(v_2^* v_1 + v_1^* v_2) = j\mu(v_2^* v_1 + \overline{v_2^* v_1}) = j\mu \mathbb{R}e[v_2^* v_1]$$

which is a purely imaginary number.

As we are equating an imaginary number with a real number, therefore the identity will be satisfied if only if both terms are equal to 0. From this we can easily derive

$$\begin{cases} R^{-1/2}B^Tv_2 = 0 \\ Q^{1/2}v_1 = 0 \end{cases} \qquad \Rightarrow \qquad \begin{cases} B^Tv_2 = 0 \\ Q^{1/2}v_1 = 0. \end{cases} \tag{22.1}$$

Recalling that $Hv = j\mu v$ we can see

$$Hv = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} Av_1 - BR^{-1}B^Tv_2 \\ -Qv_1 - A^Tv_2 \end{bmatrix} = \begin{bmatrix} Av_1 \\ -A^Tv_2 \end{bmatrix} = j\mu \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Finally we get:

$$\begin{cases} Av_1 = j\mu v_1 \\ A^Tv_2 = -j\mu v_2 \end{cases} \tag{22.2}$$

this means that v_1 and v_2 are a right and left eigenvector of A for the eigenvalues $j\mu$ and $-j\mu$, respectively.

Since we are proving the property by contadiction, we will show now that these two conditions (22.1 − 22.2) will violate the PBH test. In fact, in order for the system to be reachable, it has to be:

$$rank[sI - A|B] = n \qquad \forall s$$

If this is not the case, i.e. the rank is smaller that n, then there exists a vector $x \in \mathbb{C}^n, x \neq 0$ such that

$$x^T[sI - A|B] = 0.$$

If we take $x = v_2$, which is $v_2 \neq 0$, and $s = -j\mu$, then we find that

$$v_2^T[-j\mu I - A|B] = [-j\mu I v_2^T - v_2^TA|v_2^TB] = [0|0] = 0$$

which contradicts the initial hypothesis.

So we have found that the eigenvalues of H will be split into stable eigenvalues ($\mathbb{R}e[\lambda] < 0$) and unstable eigenvalues ($\mathbb{R}e[\lambda] > 0$), but they cannot be purely imaginary; since we showed above that if λ in the spectrum of H and also λ is, this means that the eigenvalues in the complex plane present a mirror symmetry with respect to both the imaginary and the real axes.

22.2 Solution of the Riccati equation using the Hamiltonian

Given the properties of the Hamiltonian matrix derived in the first part of the lecture, we can now find the explicit solution for the Riccati equation:

$$-\dot{P}(t) = P(t)A + A^T P(t) + Q - P(t)BR^{-1}B^T P(t), \quad P(T) = Q_T \tag{22.3}$$

It is standard linear algebra fact that there exists a the basis change matrix $W \in \mathbb{C}^{2n\times 2n}$ such that:

$$H = W \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} W^{-1},$$

where Λ_1 and Λ_2 are in Jordan canonical form and the spectrum of Λ_1 contains all the n "stable" eigenvalues of H while Λ_2 the "unstable" ones. Recall that, under the assumptions of (A, B) reachable and $(A, Q^{1/2})$ observable, we know, from property 3, that the Hamiltonian has no eigenvalues on the imaginary axis. Thus we can say that the eigenvalues of H can be divided in "stable" and "unstable", i.e. with positive and negative real part respectively. Moreover, from property 2, there is an equal number n of "stable" and "unstable" eigenvalues: so Λ_1 and Λ_2 have both dimension equals to n.

We now partition W as follows:

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix},$$

where the four matrices are in $\mathbb{R}^{n\times n}$ and the first n columns are the eigenvectors (possibly generalized) relative to the "stable" eigenvalues and the remaining columns are the eigenvectors relative to the "unstable" eigenvalues.

The natural evolution of the system

$$\begin{bmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = H \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} \tag{22.4}$$

from initial time t to final time T is:

$$\begin{bmatrix} X(T) \\ Y(T) \end{bmatrix} = e^{H(T-t)} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} \tag{22.5}$$

thus, given that the exponential matrix is always invertible, we can write:

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^{-H(T-t)} \begin{bmatrix} X(T) \\ Y(T) \end{bmatrix}. \tag{22.6}$$

Applying the basis change with matrix W to equation 22.6 and given that $X(T) = I$ and $Y(T) = Q_T$, we derive:

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = W e^{\left[\begin{smallmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{smallmatrix}\right](t-T)} W^{-1} \begin{bmatrix} I \\ Q_T \end{bmatrix}; \tag{22.7}$$

instead of computing the inverse of W, we can use the following property that can be verified by direct substitution:

$$W^{-1}\begin{bmatrix} I \\ Q_T \end{bmatrix} = \begin{bmatrix} I \\ S \end{bmatrix} L \tag{22.8}$$

where $L \in \mathbb{C}^{n\times n}$ invertible and $S \in \mathbb{C}^{n\times n}$

$$S = -\left(W_{22} - Q_T W_{12}\right)^{-1}\left(W_{21} - Q_T W_{11}\right). \tag{22.9}$$

Equation 22.7 becomes:

$$\begin{aligned}
\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} &= \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}\begin{bmatrix} e^{\Lambda_1(t-T)} & 0 \\ 0 & e^{\Lambda_2(t-T)} \end{bmatrix}\begin{bmatrix} I \\ S \end{bmatrix} L \\
&= \begin{bmatrix} W_{11}e^{\Lambda_1(t-T)} & W_{12}e^{\Lambda_2(t-T)} \\ W_{21}e^{\Lambda_1(t-T)} & W_{22}e^{\Lambda_2(t-T)} \end{bmatrix}\begin{bmatrix} I \\ S \end{bmatrix} L \\
&= \begin{bmatrix} W_{11}e^{\Lambda_1(t-T)} + W_{12}e^{\Lambda_2(t-T)}S \\ W_{21}e^{\Lambda_1(t-T)} + W_{22}e^{\Lambda_2(t-T)}S \end{bmatrix} L.
\end{aligned} \tag{22.10}$$

Recalling that the solution of the Riccati equation can be expressed as $P(t) = Y(t)X(t)^{-1}$, it is now possible to write:

$$P(t) = \left(W_{21}e^{\Lambda_1(t-T)} + W_{22}e^{\Lambda_2(t-T)}S\right) LL^{-1}\left(W_{11}e^{\Lambda_1(t-T)} + W_{12}e^{\Lambda_2(t-T)}S\right)^{-1}; \tag{22.11}$$

now, given that $e^{\Lambda_1(T-t)}e^{-\Lambda_1(T-t)} = I$ we derive

$$\begin{aligned}
P(t) &= \left(W_{21}e^{-\Lambda_1(T-t)} + W_{22}e^{-\Lambda_2(T-t)}S\right) e^{\Lambda_1(T-t)}e^{-\Lambda_1(T-t)}\left(W_{11}e^{-\Lambda_1(T-t)} + W_{12}e^{-\Lambda_2(T-t)}S\right)^{-1} \\
&= \left(W_{21} + W_{22}e^{-\Lambda_2(T-t)}Se^{\Lambda_1(T-t)}\right)\left(W_{11} + W_{12}e^{-\Lambda_2(T-t)}Se^{\Lambda_1(T-t)}\right)^{-1}.
\end{aligned}$$

We have found the solution to the Riccati equation (which is a non-linear differential equation) by simply computing eigenvalues and relative eigenvectors of the Hamiltonian matrix. Recalling the example for the scalar case we note that there is an analogy between the expressions for $P(t)$: in the scalar case $P(t)$ is the ratio of two terms while here it is a product between a matrix and the inverse of a second one.

Furthermore, if we are interested only in the steady-state value, i.e. for $T \to \infty$, we see that $P(0) = W_{21}W_{11}^{-1}$, so we need to compute only the eigenvectors relative to "stable" eigenvalues. Given that the feedback matrix for the optimal control input is $K(t) = R^{-1}B^T P(t)$, the DC gain is equal to:

$$K(0) = R^{-1}B^T P(0) = R^{-1}B^T W_{21}W_{11}^{-1}.$$