**Control Laboratory:** **a.a. 2015/2016**

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## 21.1 Riccati equation in continuous time

Let us solve the HJB equation considering the following LTI[1] system:

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0 \tag{21.1}$$

with:

$$\ell(x(t), u(t), t) = x^T(t)Qx(t) + u^T(t)Ru(t),\ Q, R \geq 0 \tag{21.2}$$

$$m(x(t)) = x^T(t)Q_Tx(t),\ Q_T \geq 0. \tag{21.3}$$

We guess the cost function to be a quadratic cost function:

$$V^*(x(t), t) = x^T(t)P(t)x(t) \tag{21.4}$$

where $P(t) \in \mathbb{R}^{n\times n}$ such that $P(t) \geq 0$ and $P(t) = P^T(t)$ without any loss of generality. Now we try to compute the optimal input $u^*(t)$ and the matrix $P(t)$. To simplify the notation, computations will be done using matrixes constant in the time, however they can be generalized to time-varying matrices $A(t), B(t), Q(t), R(t)$. Starting from:

$$\frac{\partial V^*}{\partial t} = x^T\dot{P}(t)x = -\min_u \left\{x^TQx + u^TRu + 2x^TP(Ax + Bu)\right\} \tag{21.5}$$

using the fact that $\frac{\partial}{\partial x}(x^TAx) = 2x^TA$, we obtain

$$x^T\dot{P}(t)x = -\min_u \left\{x^T(Q + 2PA)x + u^TRu + 2x^TPBu\right\}. \tag{21.6}$$

To find the minimum we can derive with respect to $u$ and set it equal to zero:

$$\frac{\partial}{\partial u}\left(x^T(Q + 2PA)x + u^TRu + 2x^TPBu\right) = 0$$

recalling that $\frac{\partial}{\partial x}(b^Tx) = b^T$, we obtain

$$2u^TR + 2x^TPB = 0$$

[1]the following equations are the same in the case of linear time variant systems using the substitutions $A = A(t), B = B(t), R = R(t)$.

now doing the transpose and pre-multiply $R^{-1}$ we get

$$R^{-1}(u^T R + x^T PB)^T = 0 \qquad \Rightarrow \qquad R^{-1}(Ru + B^T Px) = 0$$

so the optimal input $u^*(t)$ is given by a linear function of the state

$$u^*(t) = -R^{-1}B^T P(t)x(t)$$

Replacing it in the equation (21.5):

$$\begin{aligned} x^T \dot{P}(t)x &= -\{x^T(Q + 2PA)x + x^T\big(PBR^{-1}RR^{-1}B^T P\big)x - 2xPBR^{-1}BPx\} \\ &= -xQx - 2x^T PAx + x^T PBR^{-1}B^T Px. \end{aligned} \tag{21.7}$$

As mentioned before, we are looking for a symmetric matrix P(t), so we observe that

$$2x^T PAx = x^T PAx + (x^T PAx)^T = x^T PAx + x^T PA^T x$$

and therefore we get that P(t) must satisfy the following differential equation:

$$-\dot{P}(t) = Q + P(t)A + A^T P(t) - P(t)BR^{-1}B^T P(t)$$

this equation is called **Riccati differential equation**. It is possible to get P(t) from the final condition P(T), integrating backward.

## 21.2 Riccati equation solution in scalar case

$$\begin{aligned} \dot{x(t)} &= ax(t) + bu(t) \qquad x(0) = x_0 \\ \ell(x, u, t) &= qx^2(t) + ru^2(t) \\ p(T) &= q_T \end{aligned} \tag{21.8}$$

with $x, u \in \mathbb{R}, p \in \mathbb{R}, n = 1$. If we call $\tilde{u}(t) = bu(t)$ then we obtain from 21.8:

$$\ell(x, u, t) = qx^2(t) + \frac{r}{b^2}\tilde{u}^2(t)$$

In the Riccati equation we consider $b = 1, q = 1$ without loss in generality:

$$-\dot{p} = 2ap(t) + q - \frac{b^2}{r}p^2(t) = 2ap(t) + 1 - \frac{1}{r}p^2(t) = 2ap(t) + 1 - \rho p^2(t) \tag{21.9}$$

where $\rho = \frac{1}{r}$. Solving in the explicit form

$$\frac{dp}{dt} = -2ap(t) - 1 + \rho p^2(t) = \rho(p^2(t) - \frac{2a}{\rho}p(t) - \frac{1}{\rho}) \tag{21.10}$$

that gives a second order equation with two roots, $\lambda_1$ and $\lambda_2$

$$\frac{dp}{dt} = \rho(p(t) - \lambda_1)(p(t) - \lambda_2) \qquad \lambda_{1,2} = \frac{a}{\rho} \pm \sqrt{\frac{a^2}{\rho} + \frac{1}{\rho}} \tag{21.11}$$

Since $-\frac{1}{\rho} = \lambda_1\lambda_2$, the roots must be real, one positive and the other negative, $\lambda_1 < 0, \lambda_2 > 0$. We order the differential form to semplify the integration,

$$\frac{dp}{(p - \lambda_1)(p - \lambda_2)} = \rho dt \tag{21.12}$$

and calculating the integral

$$\int_{p(t)}^{p(T)} \frac{dp}{(p - \lambda_1)(p - \lambda_2)} = \int_t^T \rho d\tau \tag{21.13}$$

The rigth part of the equation is instantly resolvable

$$\int_t^T \rho d\tau = \rho(T - t) \tag{21.14}$$

For the left part we have to use partial fraction decomposition:

$$\begin{aligned} \frac{1}{(p - \lambda_1)(p - \lambda_2)} &= \frac{\alpha}{(p - \lambda_1)} + \frac{\beta}{(p - \lambda_2)} \\ &= \frac{\alpha p - \alpha\lambda_2 + \beta p - \beta\lambda_1}{(p - \lambda_1)(p - \lambda_2)} \end{aligned}$$

$$\begin{cases} (\alpha + \beta) = 0 \\ \lambda_1\beta + \lambda_2\alpha = -1 \end{cases} \qquad \rightarrow \qquad \begin{cases} \alpha = \dfrac{-1}{\lambda_2 - \lambda_1} \\ \beta = \dfrac{1}{\lambda_2 - \lambda_1} \end{cases}$$

And now we can put this result in the integral 21.13

$$\begin{aligned} \rho(T - t) &= \int_{p(t)}^{p(T)} \left( \frac{-1}{\lambda_2 - \lambda_1} \frac{1}{p - \lambda_1} + \frac{1}{\lambda_2 - \lambda_1} \frac{1}{p - \lambda_2} \right) dp \\ &= \frac{1}{\lambda_1 - \lambda_2} \int_{p(t)}^{p(T)} \left( \frac{1}{p - \lambda_1} - \frac{1}{p - \lambda_2} \right) dp \end{aligned} \tag{21.15}$$

and solving the integral

$$\begin{aligned} \rho(T - t) &= \frac{1}{\lambda_1 - \lambda_2} \left( \ln(p - \lambda_1)\Big|_{p=p(t)}^{p(T)} - \ln(p - \lambda_2)\Big|_{p=p(t)}^{p(T)} \right) \\ &= \frac{1}{\lambda_1 - \lambda_2} \ln \frac{p - \lambda_1}{p - \lambda_2}\Big|_{p=p(t)}^{p(T)} = \frac{1}{\lambda_1 - \lambda_2} \ln \frac{(p(T) - \lambda_1)(p(t) - \lambda_2)}{(p(T) - \lambda_2)(p(t) - \lambda_1)} \end{aligned} \tag{21.16}$$

and calling $q_T = p(T)$

$$\rho(T-t) = \frac{1}{\lambda_1 - \lambda_2} \ln \frac{(q_T - \lambda_1)(p(t) - \lambda_2)}{(q_T - \lambda_2)(p(t) - \lambda_1)} \tag{21.17}$$

$$e^{(\lambda_1 - \lambda_2)\rho(T-t)} = \frac{(q_T - \lambda_1)(p(t) - \lambda_2)}{(q_T - \lambda_2)(p(t) - \lambda_1)} \tag{21.18}$$

Now let $g(t) = e^{(\lambda_1 - \lambda_2)\rho(T-t)}$

$$(q_T - \lambda_1)p(t) - \lambda_2(q_T - \lambda_1) = g(t)((q_T - \lambda_2)p(t) - \lambda_1(q_T - \lambda_2))$$
$$p(t)[g(t)(q_T - \lambda_2) - (q_T - \lambda_1)] = g(t)\lambda_1(q_T - \lambda_2) - \lambda_2(q_T - \lambda_1)$$

and finally

$$p(t) = \frac{e^{(\lambda_1 - \lambda_2)\rho(T-t)}\lambda_1(q_T - \lambda_2) - (q_T - \lambda_1)\lambda_2}{e^{(\lambda_1 - \lambda_2)\rho(T-t)}(q_T - \lambda_2) - (q_T - \lambda_1)} \tag{21.19}$$

We can verify that $p(T) = q_T$

$$p(T) = \frac{\lambda_1(q_T - \lambda_2) - \lambda_2(q_T - \lambda_1)}{(q_T - \lambda_2) - (q_T - \lambda_1)} = q_T \tag{21.20}$$

Now we compute $p(0)$

$$p(0) = \frac{e^{(\lambda_1 - \lambda_2)\rho T}\lambda_1(q_T - \lambda_2) - (q_T - \lambda_1)\lambda_2}{e^{(\lambda_1 - \lambda_2)\rho T}(q_T - \lambda_2) - (q_T - \lambda_1)} \tag{21.21}$$

and we take te limit for $T \longrightarrow +\infty$

$$\lim_{x \to \infty} p(0) = \frac{-(q_T - \lambda_1)\lambda_2}{-(q_T - \lambda_1)} = \lambda_2 > 0 \tag{21.22}$$

So we see that this limit does not depend on $q_T$.

In general we observe that we get the solution $p(t)$ solving two linear systems associated to the roots of the Riccati equation. The fact that it is possible to compute the evolution of $P(t)$ from the solution of a linear system of dimension twice that of $P(t)$ is a structural feature of the Riccati equation, true also in multivariable case.

## 21.3 Riccati equation solution for MIMO systems

Now we consider the following MIMO system and the associated Riccati equation:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{21.23}$$

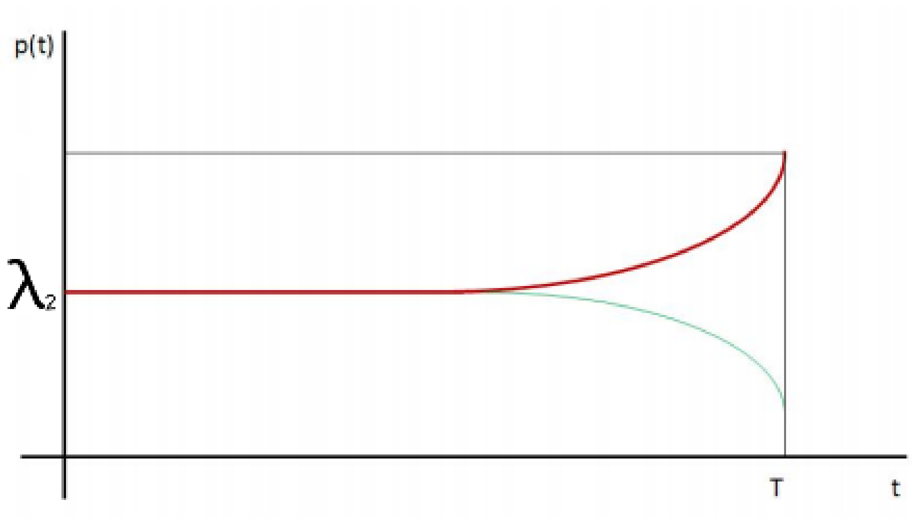


**Figura 21.1.** Riccati equation solution $p(t)$ for two different values of $q_T$.

$$-\dot{P(t)} = P(t)A + A^T P(t) + Q - P(t)BR^{-1}B^T P(T) \tag{21.24}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$, $R > 0$ and $P(T) = Q_T$.

Let us now consider the following system:

$$\begin{pmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{pmatrix} = \left(\begin{array}{c|c} A & -BR^{-1}B^T \\ \hline -Q & -A^T \end{array}\right) \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$$

$X(t), X(t) \in \mathbb{R}^{n \times n}$, $X(T) = I$ and $X(T) = Q_T$.

The matrix is called *Hamiltonian* of $(A, B, Q, R), H \in \mathbb{R}^{2n \times 2n}$.
This system can be solved in closed form, for example with the Jordan decomposition:

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^{H(t-T)} \begin{bmatrix} X(Y) \\ Y(T) \end{bmatrix}$$