Control Laboratory: **a.a. 2015/2016**

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20.1 Optimal linear quadratic control (LQ control)

The LQ control is a state space control design technique which aims to solve the problem of pole placement. Given the system (A, B, C, D), we assume it to be observable and reachable and we want to design the feedback matrices $K \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{n\times l}$ so that the eigenvalues $\Lambda(A + BK)$, $\Lambda(A - LC)$ can be placed in a specific region of the complex plain, in order to have a desidered system dynamics. Up to now we have determined this performance region using the dominant pole approximation which

- is based on second order systems,
- does not take into account the zeros of the transfer function that we have to approximate.

Except special cases, the dynamics of the system (A, B, C, D) is not similar to that of a second order system so the dominant pole approximation does not ensure to meet the project specifics. Another way to solve the problem is to us the optimal control which turn the designing problem into an optimization task.

20.1.1 Optimization problem (finite horizon senario)

Consider a linear continuous time MIMO system

$$\begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t) \end{cases}$$

Given an horizon $T > 0$ and a control input $\mathbf{u}(t)$, $t \in [0, T]$, we define the following quadratic cost function

$$J(\mathbf{u}, \mathbf{x_0}) = \int_0^T \mathbf{x}^T(t)Q\mathbf{x}(\mathbf{t}) + \mathbf{u}^T(t)R\mathbf{u}(t)\, dt + \mathbf{x}^T(T)Q_T\mathbf{x}(T), \tag{20.1}$$

where Q, $Q_T \in \mathbb{R}^{n\times n}$ are positive semidefinite matrices and $R \in \mathbb{R}^{m\times m}$ is a positive definite marix[1].

[1] A matrix $F \in \mathbb{R}^{k\times k}$ is said to be *positive semidefinite* if it is symmetric and the scalar $\mathbf{v}^T F\mathbf{v} \geq 0$ for all $\mathbf{v} \in \mathbb{R}^k$. F is said to be *positive definite* if it is positive definite and $\mathbf{v}^T F\mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

Example 1. Let us consider a SISO system ($m = l = 1$). Given the marices

$$Q = C^T C, \qquad Q_T = 0, \qquad R = rI,$$

for some real number $r > 0$, and matrix C (positive semidefinite) of suitable dimensions, the quadratic cost function (20.1) becomes

$$J(\mathbf{u}, \mathbf{x_0}) = \int_0^T ||\mathbf{y}(t)||^2 + r||\mathbf{u}(t)||^2 \, dt. \tag{20.2}$$

The optimization problem will be

$$\mathbf{u}^*(t) = \underset{\mathbf{u}(t)}{\operatorname{argmin}} \, J(\mathbf{u}, \mathbf{x_0}), \qquad t \in [0, T].$$

Supposing we want just to stabilize the system ($\mathbf{y}(t) \to \mathbf{0}$, $t \to \infty$), from (20.2), it is easy to see that very likely it will be

$$\begin{aligned} r \to 0 \implies \mathbf{y}(t) \to \mathbf{0} \quad &\text{fast, but we need a large } \mathbf{u}(t), \\ r \to \infty \implies \mathbf{y}(t) \to \mathbf{0} \quad &\text{slowly, but we need a small } \mathbf{u}(t). \end{aligned}$$

The problem is shifted to the design of the weights $Q, R \geq 0$ and we will see that the solution of the optimization problem will provide us the feedback matrices K. Moreover we will see that the optimal control law will be a static state feedback. Once solved this problem, we will check where the closed loop system poles are placed by the LQ control.

Example 2. Consider the following scalar system

$$\begin{cases} \dot{x}(t) = ax(t) + u(t) \\ y(t) = x(t) \end{cases}$$

and we assume that the optimal control is linear with respect to the system state, i.e. $u^*(t) = -Kx(t)$. We consider the infinite horizon T, $T \to \infty$ and we want to determine the cost function J under the assumption $Q_T = 0$, $Q = 1$ and $R = r$, therefore we obtain:

$$J(K, x_0) = \int_0^{+\infty} y^2(t) + ru^2(t) \, dt. \tag{20.3}$$

With the control input, the dynamics of the system becomes:

$$\begin{cases} \dot{x}(t) = a_c \, x(t) \\ y(t) = x(t) \end{cases}$$

where we define $a_c = a - K$. We can also obtain the expression of $x(t)$ as a function of the initial state, x_0, and the system matrices, such that:

$$x(t) = e^{(a-K)t} x_0 \tag{20.4}$$

Using this expression in (20.3) we get:

$$\begin{aligned}
J(K, x_0) &= \int_0^{+\infty} y^2(t) + ru^2(t)\, dt \\
&= \int_0^{+\infty} x^2(t) + rK^2 x^2(t)\, dt \\
&= (1 + rK^2) \int_0^{+\infty} x^2(t)\, dt \\
&= x_0^2\,(1 + rK^2) \int_0^{+\infty} e^{2(a-K)t}\, dt \\
&= \begin{cases} +\infty, & (a-K) > 0 \\ \\ x_0^2(1 + rK^2)\left[\frac{1}{2(a-K)} e^{2(a-K)t}\right]_0^{+\infty}, & (a-K) < 0 \end{cases} \\
\\
&= \begin{cases} +\infty, & (a-K) > 0 \\ \\ \frac{x_0^2(1+rK^2)}{2(K-a)}, & (a-K) < 0 \end{cases}
\end{aligned}$$

Relying on this result we have to find K^* in order to minimize the cost function, such that:

$$K^* = \underset{K}{\operatorname{argmin}}\, J(K, x_0) = \underset{K}{\operatorname{argmin}} \left[\frac{x_0^2(1 + rK^2)}{2(K-a)}\right] \tag{20.5}$$

If we study the asymptotes of $J(K, x_0)$ we find that it has a vertical one, $K = a$, and an oblique one $J(K, x_0) = \frac{rx_0^2}{2} K$. With this information we can represent easily the cost function as in figure 20.1.
Now, if we consider the same figure we see that $J(K, x_0)$ admits a minimum value; in order to find an expression for K^* we have to consider $\frac{\partial J(K,x_0)}{\partial K} = 0$. We get:

$$\begin{aligned}
\frac{2rK(K-a) - (1 + rK^2)}{(K-a)^2} &= 0 \\
2rK^2 - 2rKa - 1 - rK^2 &= 0 \\
rK^2 - 2rKa - 1 &= 0.
\end{aligned}$$

Finally, we can extract the expression of K^*, i.e.:

$$K^* = \frac{ra + \sqrt{r^2a^2 + r}}{r} = a + \sqrt{a^2 + \frac{1}{r}} \tag{20.6}$$

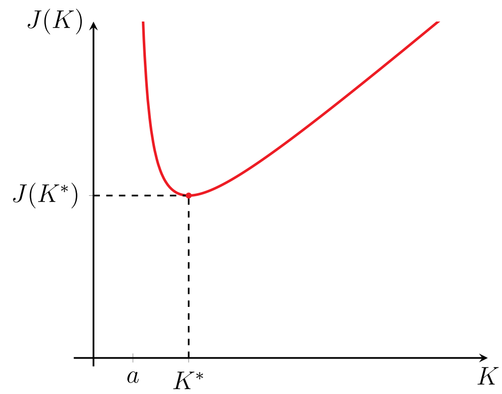

Figure 20.1. Cost function

where we did not take account of the unfeasible solution. If we remember that $a_c = a - K$, where in this case $K = K^*$, we obtain

$$a_c = a - a - \sqrt{a^2 + \frac{1}{r}} = -\sqrt{a^2 + \frac{1}{r}}$$

We have found that the closed loop pole a_c, is a function of the matrix (in this case it is a scalar value) a. In particular, as we can see in figure 20.2, if we consider the case when $r \to +\infty$ the closed loop pole will be placed in $-|a|$. Moreover, we can see that in both cases $a_c \to -\infty$ when $r \to 0$.

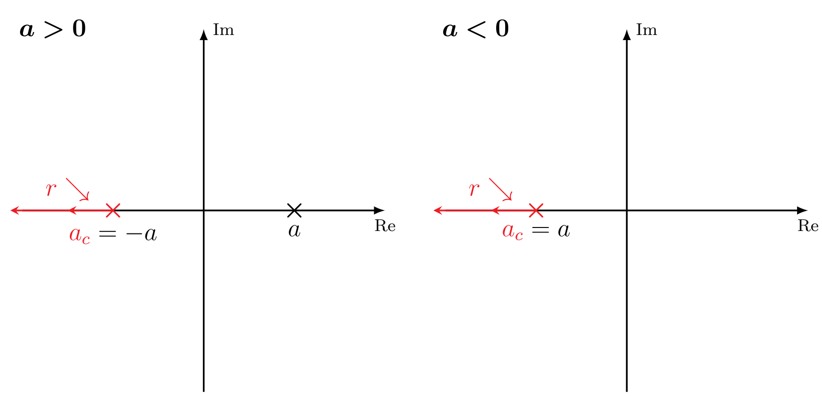

Figure 20.2.

Therefore, if $a < 0$, and we want to use the smallest effort to stabilize the system, then the solution is not to apply any feedback ($K^* = 0$), since the system is already stable. In general, when $r \to \infty$ we are dealing with *expensive control scenario*, while $r \to 0$ is called *cheap control scenario*. It also interesting to notice that if the system is unstable and we would like to minimize the cost using as little input energy as possible, we would expect that placing the closed loop poles just on the left of the real axis might be the best solution, in order to require very little energy. Instead, this is not a good position for the closed loop poles and the reason is given by the cost function:

$$\int_0^{+\infty} y^2(t) + ru^2(t)dt = x_0^2(1 + rK^2) \int_0^{+\infty} x^2(t)dt \tag{20.7}$$

we see that, due to the presence of the quadratic terms $x^2(t)$ and x_0^2 (the initial condition), the result of this integral would be very large if $x(t)$ converges to zero too slowly. Optimal control tells us that the minimum energy location for the closed loop poles is exactly the symmetrical of the original open loop.

20.1.2 Optimal control problem: Hamilton-Jacobi-Bellman equation

Let us consider the generical continuos time system:

$$\dot{x}(t) = f(x(t), u(t), t) \tag{20.8}$$

with $x(0) = x_0$ and $u(\tau)$, $\tau \in [t, T]$.
We define the *cost-to-go function* at time t:

$$V(x(t), t) = \int_t^T \ell(x(\tau), u(\tau), \tau)d\tau + m(x(T)) \tag{20.9}$$

where the term $\ell(x(\tau), u(\tau), \tau)$ and the term $m(x(\tau))$ are called, respectively, *instantaneous cost* and *terminal cost* and they are such that $\ell : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^+$ and $m : \mathbb{R}^n \to \mathbb{R}^+$. Therefore, with this definition, we see that:

$$J(x_0, u) = V(x(0), 0) \tag{20.10}$$

with $u \in [0, T]$. Once defined the *cost-to-go function* we can determine the optimal one:

$$V^*(x(t,)t) = \min_{u[t,T]} V(x(t), t) \tag{20.11}$$

We are looking at all possible inputs which can minimize the cost function: V^* is the optimal cost function and does not depend on the input u. If we assume to start from an initial condition $x(0)$ and to end up at $x(T)$ and if we assume it to be the optimal trajectory, we have, basically, minimized the cost function by choosing u^* among all possible inputs and we can find the trajectory $\dot{x}(t) = f(x(t), u^*(t), t)$. Now, it is interesting to notice that for the *Optimality principle*, if the trajectory from $x(0)$ to $x(T)$ is optimal, also the one from $x(t_1)$ to $x(T)$ is optimal, with $t < t_1 < T$. Therefore, in general, if the input is optimal from t to T, it would also be optimal from t_1 to T, with $t_1 > t$.

Thanks to these observations we can split the optimal cost function in this way:

$$V^*(x(t),t) = \min_{u(\tau),\tau\in[t,T]} V(x(t),t) \tag{20.12}$$
$$= \min_{u(\tau),\tau\in[t,t_1]} \left[\min_{u(\tau),\tau\in[t_1,T]} \int_t^{t_1} \ell(x(\tau),u(\tau),\tau)d\tau + \int_{t_1}^{T} \ell(x(\tau),u(\tau),\tau)d\tau + m(x(T)) \right] \tag{20.13}$$
$$= \min_{u(\tau),\tau\in[t,t_1]} \int_t^{t_1} \ell(x(\tau),u(\tau),\tau)d\tau + \min_{u(\tau),\tau\in[t_1,T]} \int_{t_1}^{T} \ell(x(\tau),u(\tau),\tau)d\tau + m(x(T)) \tag{20.14}$$

where:

$$V^*(x(t_1),t_1) = \min_{u(\tau),\tau\in[t_1,T]} \int_{t_1}^{T} \ell(x(\tau),u(\tau),\tau)d\tau + m(x(T)) \tag{20.15}$$

We would like to understand what happens if we take t_1 very small, in the sense that t_1 is just a little bit larger than t. What we get is:

$$V^*(x(t),t) = \min_{u(\tau),\tau\in[t,t+\epsilon]} \left[\int_t^{t+\epsilon} \ell(x(\tau),u(\tau),\tau)d\tau + V^*(x(t+\epsilon),t+\epsilon) \right] \tag{20.16}$$

Let focus now on what happens when $\epsilon \to 0$. In order to do that we apply the *Taylor expansion* with respect to ϵ:

$$V^*(x(t+\epsilon),t+\epsilon) = V^*(x(t),t) + \epsilon \cdot \left[\frac{\partial V^*}{\partial x}\frac{\partial x}{\partial \epsilon} \right]_{\epsilon=0} + \epsilon \cdot \left[\frac{\partial V^*}{\partial t} \right]_{\epsilon=0} + o(\epsilon) \tag{20.17}$$

and

$$\int_t^{t+\epsilon} \ell(x(\tau),u(\tau),\tau)d\tau = \underbrace{\int_t^{t} \ell(x(\tau),u(\tau),\tau)d\tau}_{=0} + \epsilon \cdot \underbrace{\left[\frac{\partial x}{\partial \epsilon} \int_t^{t+\epsilon} \ell(x(\tau),u(\tau),\tau)d\tau \right]_{\epsilon=0}}_{=\ell(x(t),u(t),t)} + o(\epsilon) \tag{20.18}$$

and the expression becomes:

$$V^*(x(t),t) = \min_{u(\tau),\tau\in[t,t+\epsilon]} \left[\epsilon\, \ell(x(t),u(t),t) + V^*(x(t),t) + \frac{\partial V^*}{\partial x}\frac{\partial x}{\partial \epsilon}\epsilon + \frac{\partial V^*}{\partial t}\epsilon + o(\epsilon) \right]. \tag{20.19}$$

Simplifying:

$$0 = \min_{u(\tau),\tau\in[t,t+\epsilon]} \left[\ell(x(t),u(t),t)\epsilon + \frac{\partial V^*}{\partial x}\frac{\partial x}{\partial t}\epsilon + \frac{\partial V}{\partial \epsilon}\epsilon + + \frac{\partial V^*}{\partial t}\epsilon + o(\epsilon) \right]. \tag{20.20}$$

If we take the limit for $\epsilon \to 0$, the expression becomes:

$$\min_{u(t)} \left[\ell(x(t), u(t), t) + \left[\frac{\partial V^*}{\partial x} \right]_t f(x(t), u(t), t) + \frac{\partial V^*}{\partial t} \right] = 0 \tag{20.21}$$

finally, we get the *Hamilton Jacobi Bellman equation*:

$$\frac{\partial V^*}{\partial t}(x(t), t) = -\min_{u(t)} \left[\ell(x(t), u(t), t) + \frac{\partial V^*}{\partial x}(x(t), t) \cdot f(x(t), u(t), t) \right] \tag{20.22}$$

In general, this is a formidable partial non-linear differential equation which is extremely hard to solve, even simply numerically. However, in the very special case where we pick $\ell(\cdot)$ as a quadratic function, *i.e.* $\ell(x(t), u(t), t) = x^T(t)Qx(t) + u^T(t)Ru(t)$ and a linear dynamical system

$$\begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) = C\mathbf{x}(t) \end{cases}$$

we are able to compute $V^*(x(t), t)$ and $u^*(t)$ in closed form.