Control Laboratory:	a.a. 2015/2016

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17.1 Discrete time systems

Figura 17.1. Continuous control model

We start clarifying the concept of discrete time by giving some definitions:

- Quantized systems: systems whose output and/or input are signals that take values on a finite set:
  $$y(t) \in \{y_1, \cdots, y_n\} \qquad u(t) \in \{u_1, \cdots, u_n\}$$
  The definition holds for both continuous or discrete time systems.

- Discrete systems (DT): basically where t takes values on a finite set
  $$t \in kT \qquad k \in N$$
  with T the sampling period.

- Digital system: the combination of a quantized and a discrete time system

Speaking of a discrete time systems, or more properly of digital systems, we can see that the controller is composed by three different blocks: a ADC (Analog-Digital converter) which returns a discrete time signal $e(k)$ by the sampling of $e(t)$, a discrete time controller $C(z)$ and a DAC (Digital-Analog converter) which returns a continuous time output. They all operates at the same sampling time T.

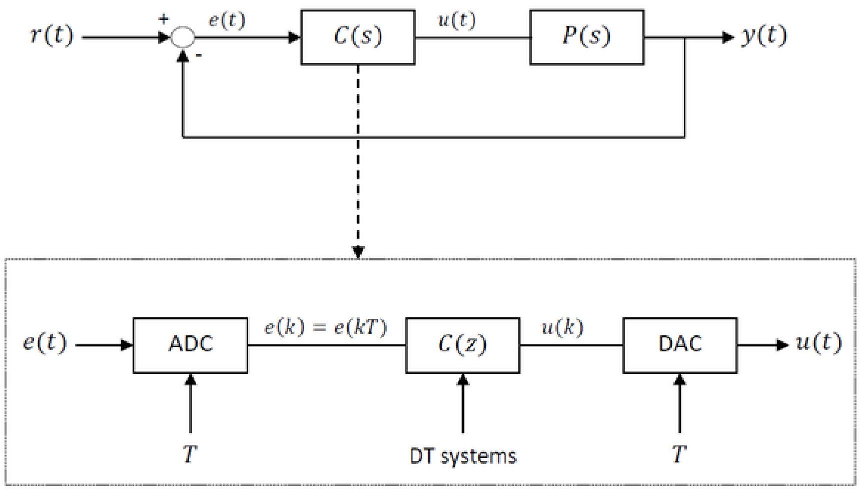

Figura 17.2.

17.1.1 Representations

We can represent the same discrete time system in several ways:

1. Difference equations: (the equivalent of ODE's ordinary differential equation in continuous time)

$$y(k) = -a_n y(k-n) - \cdots - a_1 y(k-1) + b_m u(k-m) + \cdots + b_0 u(k)$$

   with initial conditions: $\{y(h)\}_{h=-1}^{-n}$ $\{u(h)\}_{n=0}^{-m}$
   Tipically $\{u(h)\}_{n=0}^{-m}$ are assumed to be 0.

2. Transfer functions (obtained by $\mathcal{Z}$-transform) expressed by a ratio of polynomials:

$$F(z) = \frac{N(z)}{D(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

$$F(z^{-1}) = \frac{N(z^{-1})}{D(z^{-1})} = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

3. Space State (MIMO/SISO Systems):

$$\begin{cases} x(k+1) = & Ax(k) + Bu(k) \\ y(k) = & Cx(k) + Du(k) \end{cases}$$

   with initial condition $x_0 \in \mathrm{R}$.

4. Impulse response:

$$g(k) = \begin{cases} 0 & k < 0 \\ g(k) & k \geq 0 \end{cases}$$

   a casual function defined in the discrete-time domain.

The main tool for changing rappresentation is the $\mathcal{Z}$-transform.
The $\mathcal{Z}$-transform is the equivalent of the Laplace trasform in the discrete time; given the function $f(k):\ \mathbb{N} \to \mathbb{R}$, its trasform is the function $F(k):\ \mathbb{C} \longrightarrow \mathbb{C}$ that result from:

$$F(z) = \sum_{k=0}^{+\infty} f(k) z^{-k}$$

For example:

$$f(k) = a^k \quad a \in \mathbb{R}$$

$$F(z) = \sum_{k=0}^{+\infty} a^k z^{-k} = \sum_{k=0}^{+\infty} (az^{-1})^k = \frac{1}{1 - az^{-1}}$$

17.2 Properties of the $\mathcal{Z}$-transform

17.2.1 Delay operator

The equivalent in $\mathcal{Z}$-domain of the Laplace transform of the derivative is the $\mathcal{Z}$-transform of the one step delay signal $f(k-1)$: using the definition and adding and subtracting 1 in the esponential, we obtain

$$\mathcal{Z}[f(k-1)] = \sum_{k=0}^{+\infty} f(k-1) z^{-k+1-1} = \sum_{k=0}^{+\infty} f(k-1) z^{-(k-1)} z^{-1} =$$

$$= z^{-1} \sum_{k=0}^{+\infty} f(k-1) z^{-(k-1)} = z^{-1}(f(-1) + \sum_{k=1}^{+\infty} f(k-1) z^{-(k-1)}) =$$

$$= z^{-1}(f(-1)z + \sum_{k'=0}^{+\infty} f(k'-1) z^{-k'}) = f(-1) + z^{-1} F(z)$$

We can see that there is a term due to the initial condition, $f(-1)$, and that the $\mathcal{Z}$-transform of the delay operator is multiplied for z^{-1}.
Substituting this in the equation shown in the beginning

$$y(k) = -a_1 y(k-1) - \cdots - a_n y(k-n) + b_0 u(k) + b_1 u(k-1) + \cdots + b_m u(k-m)$$

using the $\mathcal{Z}$-transform, we obtain

$$\begin{aligned} Y(z) \;&=\; -\, a_1 z^{-1} Y(z) \;-\; \cdots \;-\; a_n z^{-n} Y(z) \;+ \\ &\quad +\, b_0 U(z) \;+\; \cdots \;+\; b_m z^{-m} U(z) \;+\; N_0(z^{-1}) \end{aligned}$$

where $N_0(z^{-1})$ is a polinomial of degree $n-1$, depending on the initial conditions. Now, collecting $Y(z)$ and $U(z)$, we get:

$$\begin{aligned} Y(z) &= \tfrac{b_0+\cdots+b_m z^{-m}}{1+a_1 z^{-1}+\cdots+a_n z^{-n}} U(z) + \tfrac{N_0(z^{-1})}{1+a_1 z^{-1}+\cdots+a_n z^{-n}} \\ &= \underbrace{\frac{N(z^{-1})}{D(z^{-1})} U(z)}_{Y_f(z)} + \underbrace{\frac{N_0(z^{-1})}{D(z^{-1})}}_{Y_0(z)} \end{aligned}$$

where $Y_f(z)$ is the $\mathcal{Z}$-transform of the forced response and $Y_0(z)$ the natural response one; both polinomials with the same denominator.
This operator is useful to pass from a differential representation to the transfer function of a discrete time system.

17.2.2 Convolution

Given the following output

$$y(k) = g(k) * u(k)$$

by the $\mathcal{Z}$-transform, we get

$$Y(z) = G(z)U(z)$$

where $G(z) = \mathcal{Z}[g(k)]$.
So if we consider the input $u(k)$ equal to the Kronecker delta ($\delta(k)$, equal to 1 in zero and to 0 everywhere else), whose $\mathcal{Z}$-transform is unitary, then the output is equal to $Y(z) = G(z) \xrightarrow{\mathcal{Z}^{-1}} g(k)$, the impulse response of the system.

17.2.3 Shift operator

About the shift operator $f(k+1)$, we may proceed as for the delay operator, defining $\mathcal{Z}[f(k+1)]$ as the $\mathcal{Z}$-transform of the operator. Again, adding and subtracting 1 to the esponential, we have:

$$\mathcal{Z}[f(k+1)] = \sum_{k=0}^{+\infty} f(k+1) z^{-k-1+1} = z(\sum_{k=0}^{+\infty} f(k+1) z^{-(k+1)}) =$$

$$= z(\sum_{k=-1}^{+\infty} f(k+1) z^{-(k+1)}) = z(\sum_{k=0}^{+\infty} f(k+1) z^{-(k+1)} - f(0)) =$$

$$= z(F(z) - f(0))$$

Starting from a system described by the state space form

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + D_k \end{cases}$$

then using the zeta transform

$$z(IX(z) - x_0) = AX(z) + BU(z)$$

we get the transfer function of the system, where $X(z)$ could be a complex number.

$$\Rightarrow (zI - A)X(z) = zx_0 + BU(z) \Rightarrow X(z) = z(zI - A)^{-1}x(0) + (zI - A)^{-1}BU(z)$$

$$\begin{aligned} Y(z) &= C(zI - A)^{-1}BU(z) + DU(z) + zC(zI - A)^{-1}x_0 \\ \Rightarrow \quad &= \underbrace{[C(zI - A)^{-1}B + D]}_{F(z)=\frac{N(z)}{D(z)}} U(z) + zC(zI - A)^{-1}x_0 \end{aligned}$$

So, the shift operator is useful to pass from a state space representation to a transfer function representation.
If we do the opposit, from transfer function to the state space form, we have to consider that there are infinite state space representations, so we have to use the canonical forms: observability, controllability, modal (Jordan) and balance realizations.

17.3 Discrete time behaviour of a dynamical system

We recall now the signal in the z domain:

$$Y(z) = \frac{N(z^{-1})}{D(z^{-1})}\,U(z) + \frac{N_0(z^{-1})}{D(z^{-1})}.$$

Similarly to the continuos time system, the stability will be associated to the roots of the denominator

$$D(z^{-1}) = 1 + \alpha_1 z^{-1} + ... + \alpha_n z^{-n} = \prod_{i=1}^{n}(1 - p_i\, z^{-1})$$

which can also be written as a product of monomials (possibly with repetions), with the roots $\{p_i\}_{i=1}^{n}$ such that

$$D(z = p_i) = 0.$$

17.3.1 Natural response

If we look at the natural response

$$Y_o = \frac{N_o(z^{-1})}{D(z^{-1})}$$

we see that the numerator has has maximum degree $n-1$.
We assume for simplicity that all the $\{p_i\}$ are distinct; our Y_o can be decomposed in simple fraction

$$Y_o(z) = \sum_{i=1}^{n} \frac{\alpha_i}{(1 - p_i\, z^{-1})}$$

where α_i is possibly a complex number which depends on the initial conditions.
If now we apply the inverse of the z-transform, we get that the natural response, due to the initial conditions, is the linear combination of the modes of the system

$$y_o(k) = \sum_{i=1}^{n} \alpha_i\, p_i^k.$$

This will converge to zero for all the values of α_i, namely all the initial conditions, only if $|p_i| < 1$. This correspond to the condition of taking the poles within the unitary circle. Without the conditions of p_i distinct $y_o(k)$ will be the linear combination of the modes $p_i^k,\ kp_i^{k-1},\ \frac{k(k-1)p_i^{k-2}}{2!}, \ldots$

17.3.2 Forced response

The forced response, instead, depends on the choice of the input.
Let us consider two typical choice of input:

The Kronecker's delta $\quad u(k) = \delta(k) \to U(z) = \mathcal{Z}[\delta(k)] = 1$

The step function $\quad u(k) = 1(k) \to U(z) = \mathcal{Z}[1(k)] = \frac{1}{1-z^{-1}}$

Assuming the poles distinct and asymptotically stable, if we look at the forced response with δ as an input we get:

$$Y_f(z) = \sum_{i=1}^{n} \frac{\beta_i}{(1 - p_i\, z^{-1})} \quad \xrightarrow{\mathcal{Z}^{-1}} \quad y_f(k) = \sum_{i=1}^{n} \beta_i\, p_i^k \to 0$$

and we know that the output is the linear combination of the modes of the system, so directly related to the roots of the denominator.

With a step input $U(z) = \frac{1}{1-z^{-1}}$ under the previous assumption, we have

$$Y_f(z) = \underbrace{\frac{N(z^{-1})}{D(z^{-1})}}_{roots<1} \underbrace{\frac{1}{1-z^{-1}}}_{roots=1} = \sum_{i=1}^{n} \frac{\gamma_i}{(1 - p_i\, z^{-1})} + \frac{\gamma_o}{(1 - z^{-1})}$$

that leads to

$$y_f(k) = \underbrace{\sum_{i=1}^{n} \gamma_i\, p_i^k}_{\to 0} + \gamma_o 1(k) \to \gamma_o \qquad for\ k \to \infty.$$

To compute γ_o we multiply each term by $1 - z^{-1}$, such that we can apply the final value theorem

$$y_f(k) = \sum_{i=1}^{n} \gamma_i \frac{1 - z^{-1}}{1 - p_i z^{-1}} + \gamma_o \frac{1 - z^{-1}}{1 - z^{-1}} \longrightarrow \gamma_o = F(1)$$

17.4 Discrete time controller design

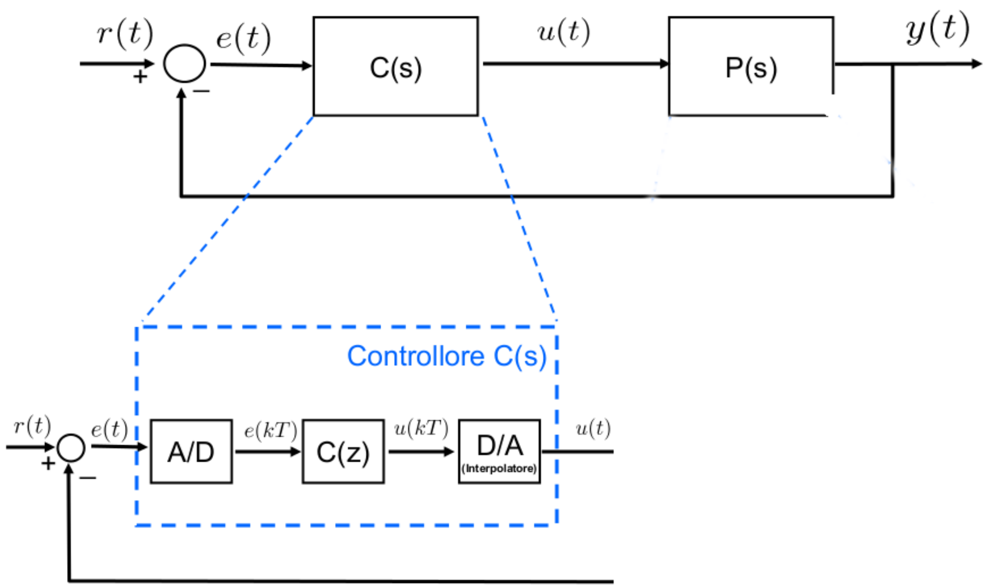

Figura 17.3. Scomposition of C(s) in discrete time

How do we design C(z)? We have two ways:

1) Design C(s) and then discretize it into C(z).

2) Discretize P(s) and then design C(z).

17.4.1 Discretization of C(s)

The new controller is defined by three block syncronized by the same sampling time T. Typically the A/D ,the analog-to-digital converter, is a sampler;

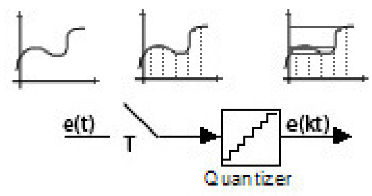

Figura 17.4.

while the D/A ,the digital-to-analog controller, is a *Zero Order Holder*.

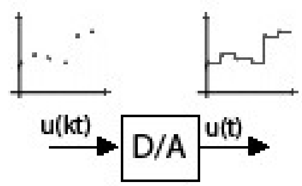

Figura 17.5.

What we want is the discrete time output to be almost equal to the continuos time output.

Of course the two signals will never be the same, since the sampler and the holder introduce an approximation.