Control Laboratory:	**a.a. 2015/2016**

Lezione 16 — 14 April

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16.1 Observers and regulators

The dynamics of the overall systems composed by the original plant and the regulator with its observer and state-feedback controller:

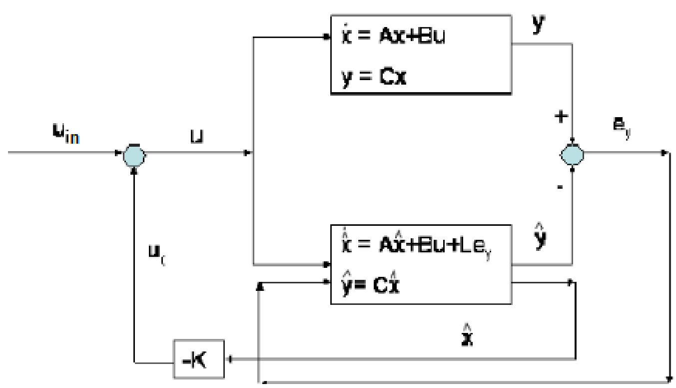


Figura 16.1. Observer/Regulator

Suppose that $u_{in}(t) = 0$:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \\ \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(e_y(t)) \leftarrow \\ \hat{y}(t) = C\hat{x}(t) \\ u(t) = u_c(t) \\ u_c(t) = -K\hat{x}(t) \end{cases}$$

where :
first and second equation concerning the physical model (Plant),
third and fourth equation concerning estimator (Observer),
the last two equations concerning the Controller and feedback.

16.2 Transfer function of regulators

Since $u_{in} = 0$ we have the situation shown below:

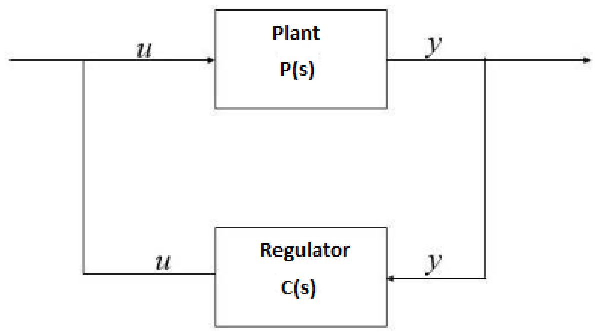


Figura 16.2. Regulator/Plant

where the Regulator (Observer+Controller), in state space, is modelized by the follow equations:

$$\begin{cases} \dot{\hat{x}}(t) = (A - BK - LC)\hat{x}(t) + Ly(t) = F\hat{x}(t) + Ly(t) \\ u_c(t) = -K\hat{x}(t) = u(t) \\ \hat{y}(t) = C\hat{x}(t) \end{cases}$$

So, the representation of the regulator by its transfer function is:

$$C(s) = -K(sI - (A - BK - LC))^{-1}L = -K(sI - F)^{-1}L$$

paying attention about these warnings:

$$\begin{matrix} (A, B) \text{ reachable} \\ (A, C) \text{ observable} \end{matrix} \Rightarrow \begin{matrix} \exists \text{ regulator that stabilizes the closed loop system and} \\ \text{that can place the closed loop eigenvalues arbitrarily.} \end{matrix}$$

$$\left.\begin{matrix} A - BK \\ A - LC \end{matrix}\right\} \text{asymptotically stable} \not\Rightarrow F \text{ asymptotically stable}$$

16.3 Tracking of signals reference

How should we choose L and K values to obtain the best performances of the system? How should we choose $\lambda(A - LC)$? Requiring $\hat{x}(t) \to x(t)$ faster implies that we must to choose

$\lambda(A - LC)$ more negative (faster) with respect to $\lambda(A - BK)$. However, it is important is to know that L should not be too "big" because this amplifies measurement noise, as shown by the following expression:

$$\hat{x}(t) = (A - LC)x(t) + Bu(t) + L(y(t) + d_y(t))$$

where d_y measurement noise.
Moreover a large "L" would give rise to large error signal $e_x(t)$ which in turns would give rise to high control signals during the transients.
One approximative rule to choose eigenvalues of the observer matrix $(A-LC)$ around $3-10$ times faster with respect to the eigenvalues of the control matrix $(A - BK)$. Therefore, in order to have a good compromise between tracking speed and rejection noise at the output a possible rule-of-thumb is the following

$$|\lambda(A - LC)| \simeq 3 - 4|\lambda(A - BK)|$$

16.4 Tracking with observers

Let us consider this plant-regulator block scheme shown below:

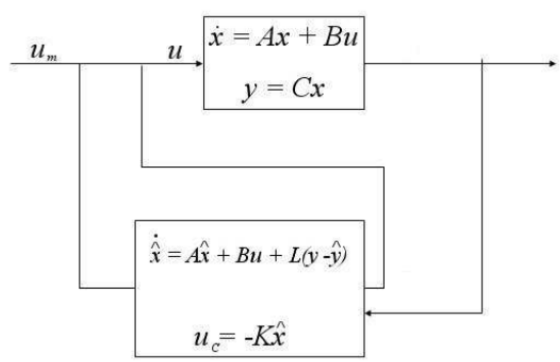


Figura 16.3. Plant/Observer/Controller+feedback

The state equations can be resume as follows:

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{e}_y \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e_x \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_m \\ y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ e_x \end{bmatrix} \end{cases}$$

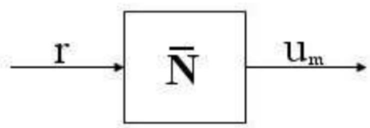


Figura 16.4. Blocks system scheme

As we already shown in pure state-feedback control, tracking of a step input $r(t) = 1(t) \Rightarrow y(t) \to r(t) = 1(t)$ can be achieved as follows:

1) nominal tracking (feedforward control).
2) robust tracking (integral control).

1) **Nominal tracking:** in this case the problem is how to pick $\bar{N}$:
First of all, N must be chosen such that:

- $y_{DC} = r_{DC}$
- $u_m = \bar{N} r_{DC}$

If $x \to x_{DC}$ and $e_x \to e_x^{DC} \Rightarrow \dot{x} \to 0$ and $\dot{e}_x \to 0$

So:

$(A - LC)e_x^{DC} = 0 \Rightarrow e_x^{DC} = 0 \Leftrightarrow \hat{x}_{DC} = x_{DC}$
$(A - BK)x_{DC} + B\bar{N}r_{DC} = 0$ since that $\dot{x} \to 0$ and $e_x \to 0$
$x_{DC} = -(A - BK)^{-1}B\bar{N}r_{DC}$

where we used the fact that $(A - LC)$ and $(A - KB)$ are invertible since they are assumed to be asymptotically stable. As a consequence, the output y_{DC} is:

$y_{DC} = Cx_{DC} = -C(A - BK)^{-1}B\bar{N}r_{DC} = r_{DC}$

with:

$$\boxed{\bar{N} = -\frac{1}{C(A - BK)^{-1}B}}$$

So the feedforward control scheme in dynamic state-feedback is equal to the static state-feedback and does not depend on the observer gain L. Similar considerations also hold for the integral control scheme, only paying attention to use in the observer the same input signal that enters the process, including possible saturations or non-linear functions.

16.5 Reduced order observers

We are now interested in deriving observers whose dynamics can be represented by fewer states in the scenario that some of the state variables are directly observable and the corresponding measurement noise is negligible. In other works, the objective is to estimate only the component of the whole state x that are not directly observable. Let as assume that

$$x \in \mathbb{R}^n, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x_1 \in \mathbb{R}^m,\ x_2 \in \mathbb{R}^{n-m}, y = x_1$$

Our objective is to estimate only x_2 since our global observer we be given by

$$\widehat{x} = \begin{bmatrix} y \\ \widehat{x}_2 \end{bmatrix}$$

The state x_2 will be indirectly estimated via an additional state variable $z \in \mathbb{R}^{n-m}$ defined as follows

$$z(t) = x_2(t) - Lx_1(t) \in R^{n-m}$$

where $L \in \mathbb{R}^{(n-m)\times m}$ is a matrix that will be designed later. Even this variable z is not directly observable, but if we can find an observer $\widehat{z}$ such that $\widehat{z}(t) \to z(t)$, then we can design an observer for x_2 as follows:

$$\widehat{x}_2(t) = \widehat{z}(t) + Ly(t)$$

since

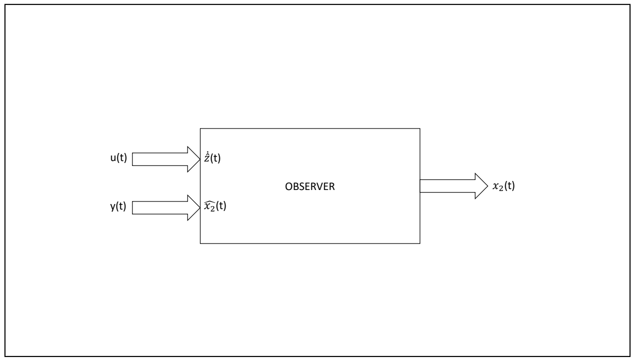
$$\widehat{z}(t) \to z(t) \Longrightarrow \widehat{x}_2(t) = \widehat{z}(t) + Ly(t) = \widehat{z}(t) + Lx_1(t) \to z(t) + Lx_1(t) = x_2(t) - Lx_1(t) + Lx_1(t) = x_2(t)$$

Note that we can partition the matrices A, B as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

Now we can write:

$$\begin{aligned} z(t) &= \dot{x}_2(t) - L\dot{x}_1 = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) - L[A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t)] \\ &= (A_{22} - LA_{12})x_2(t) + (A_{21} - LA_{11})x_1(t) + (B_2 - LB_1)u(t) \\ &= (A_{22} - LA_{12})(z(t) + Lx_1(t)) + (A_{21} - LA_{11})y(t) + (B_2 - LB_1)u(t) \\ &= \underbrace{(A_{22} - LA_{12})}_{A_r} z(t) + \underbrace{(A_{21} - LA_{11} - LA_{12}L + A_{22}L)y(t) + (B_2 - LB_1)u(t)}_{u'(t)} \end{aligned} \tag{16.1}$$



which can be summarized by

$$\dot{z}(t) = A_r z(t) + u'(t)$$

where the "virtual input" $u'(t)$ is a known signal for the observer while $z(t)$ is not measurable directly. Recall that the standard approach to derive an observer for the variable $z(t)$ is given by:

$$\dot{\hat{z}}(t) = A_r \hat{z}(t) + u'(t) + H(y - \hat{y})$$

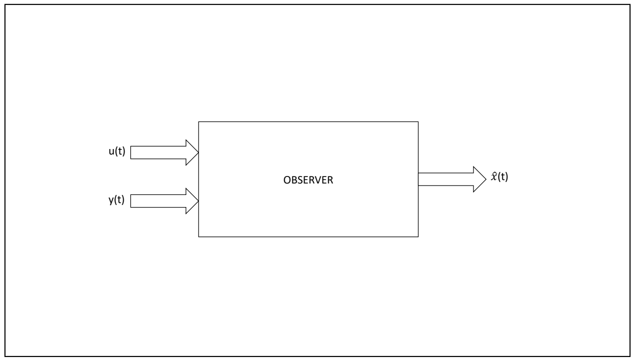
However, we have also seen that if A_r is asymptotically stable, the correcting term $H(y - \hat{y})$ is not needed. As so the problem that we haev to solve is to verify if there exists an L such that $A_r = A_{22} - LA_{12}$ is A.S. This is equivalent to determine whether the pair (A_{22}, A_{12}) is observable. Let us assume that this is the case, then by setting $H = 0$, the overall reduced observer can be written as the output of an LTI system of dimension $(n - m)$ that requires as input the input to the plant $u(t)$ and the measurement vector y:

$$\begin{cases} \dot{\hat{z}}(t) = (A_{22} - LA_{12})\hat{z}(t) + [B_2 - LB_1 \mid A_{21} - LA_{11} - LA_{12}L + A_{22}L] \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \\ \hat{x}_2(t) = [\ I\ ]\hat{z}(t) + [\ 0 \mid L\ ] \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \end{cases} \tag{16.2}$$

We now want to show that the pair (A_{22}, A_{12}) is observable is and only if the pair (A, C) is observable, i.e. if the original system is observable. In this scenario, since $y = x_1$, then the observability matrix can be written as $C = [\ I_m \mid 0\ ]$. We now prove the equivalence between observability of (A_{22}, A_{12}) and (A, C) via the PBH test. The PBH test for the pair (A, C) is given by:
rank $\left[\frac{C}{sI-A}\right] = n \quad \forall s$ which can be expanded as follows;

$$\text{rank} \begin{bmatrix} I_m & 0 \\ sI - A_{11} & -A_{12} \\ -A_{21} & sI - A_{22} \end{bmatrix} = n \Leftrightarrow \text{rank} \begin{bmatrix} 0 \\ -A_{12} \\ sI - A_{22} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{12} \\ sI - A_{22} \end{bmatrix} = n - m$$

that is exactly the PHB for observability of the pair (A_{22}, A_{12}).



The previous scenario can be generalized to $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$
where D is different from 0 and $\operatorname{rank}(C) = m$.

16.6 Example

I could try to apply the *reduce order* observer to our motor:

$$\begin{bmatrix} \dot{\theta}_c \\ \ddot{\theta}_c \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} \begin{bmatrix} \theta_c \\ \dot{\theta}_c \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_c \\ \dot{\theta}_c \end{bmatrix}$$

where $a, b > 0$.
In this case, the reduced observer matrix $L \in \mathbb{R}$ since $n - m = 1$, i.e. it is just a scalar.

$$\begin{cases} \dot{\hat{z}} = (-a - L)\hat{z} + (-aL - L^2)y(t) + bu(t) = -(a+L)\hat{z} + \left[\, b \mid -L(a+L) \right] \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \\ \hat{x}_2 = \dot{\hat{\theta}}_c = \hat{z} + Ly(t) = \hat{z} + \left[\, 0 \mid L \,\right] \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \end{cases} \tag{16.3}$$

which can be written also in terms of the transfer functions:

$$\begin{aligned} \widehat{X}_2(s) &= P_u(s)U(s) + P_y(s)Y(s) = \frac{b}{s+a+L}U(s) + \left(-\frac{L(a+L)}{s+a+L} + L \right) Y(s) \\ &= \frac{b}{s+a+L}U(s) + \frac{sL}{s+a+L}Y(s) \end{aligned}$$