Control Laboratory: **a.a. 2015/2016**

Lezione 15 — 13 April

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15.1 Internal Model Principle (continued)

In the last lecture we have began to study the internal model principle, which is a generalization of the integral control. Our purpose is to track a certain signal ($y(t) \rightarrow r(t)$) and reject another signal ($w(\mathrm{t})$), which satisfy the following property:

$$r^{(m)} + \alpha_{m-1} r^{(m-1)} + ... + \alpha_0 r = 0 \tag{15.1}$$

$$w^{(m)} + \alpha_{m-1} w^{(m-1)} + ... + \alpha_0 w = 0 \tag{15.2}$$

where $\{\alpha_i\}_{i=0}^{m-1}$ are known.

We have introduced a new state: $z \triangleq \left[\begin{array}{c} e \\ \vdots \\ e^{(m-1)} \\ \hline \xi \end{array}\right] \in \mathbb{R}^{(n+m)}$,

where

$$e^{(k)} = Cx^{(k)} - r^{(k)} \qquad ,$$

$$\xi \triangleq x^{(m)} + \alpha_{m-1} x^{(m-1)} + ... + \alpha_0 x, \qquad ,$$

because with this new representation the reference signal and the input disturbance vector disappear. We found:

$$\dot{z} = \left[\begin{array}{c} e^{(1)} \\ \vdots \\ e^{(m)} \\ \hline \dot{\xi} \end{array}\right] = \overbrace{\left[\begin{array}{ccccc|c} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{m-1} & C \\ \hline 0 & 0 & 0 & \cdots & 0 & A \end{array}\right]}^{A_z} \left[\begin{array}{c} e \\ \vdots \\ e^{(m-1)} \\ \hline \xi \end{array}\right] + \overbrace{\left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \hline \mathrm{B} \end{array}\right]}^{B_z} u_\xi \qquad ,$$

with

$$u_\xi \triangleq u^{(m)} + \alpha_{m-1} u^{(m-1)} + ... + \alpha_0 u, \qquad ,$$

so $\dot{z} = A_z z + B_z u_\xi$.

If we introduce a negative state feedback we obtain: $\begin{cases} \dot{z} = A_z z + B_z u_\xi \\ u_\xi = -K_z z \end{cases}$.

If (A_z, B_z) is reachable, then $\exists K_z \in \mathbb{R}^{(m+n)\times p} : (A_z - B_z K_z)$ is asymptotically stable, because it has eigenvalues that can be arbitrarily placed. So $z(t) \to 0$ for each initial condition of r and w that satisfy 15.1 e 15.2. Therefore $e(t) \to 0$, that is $y(t) \to r(t)$.

If we apply the PBH test for the reachability of (A_z, B_z) we find:

$$\left[\begin{array}{ccccc|c|c} s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & s & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ \alpha_0 & \alpha_1 & \alpha_2 & \cdots & s+\alpha_{m-1} & -C & 0 \\ \hline 0 & 0 & 0 & \ldots & 0 & sI-A & B \end{array}\right],$$

that must have a rank equal to n+m.

The necessary and sufficient conditions which guarantee that (A_z, B_z) is reachable are:

- (A, B) reachable;
- zeroes of $(s^m + \alpha_{m-1}s^{m-1} + ... + \alpha_0)$ are not zeroes of the transfer function of the original system (A, B, C).
  Indeed (A, B) is reachable, the first $m-1$ rows and the last n rows of the PBH matrix are always linearly independent for each $s \in \mathbb{C}$. But we have to verify when the m-th row is linearly independent from all others. If $s = \bar{z}$ is a zero of $s^m + \ldots \alpha_0 = 0$, then the first m components of the m-th row are linearly dependent from the first $m-1$ rows. Besides if $s = \bar{z}$ is also a zero of the transfer function of the system (A, B, C), then the last n components of the m-th row are linearly dependent from the last n rows of the PBH matrix, and so the rank of the latter decreases to $m + n - 1$.

Now we want to describe the system with this new type of control as a function of $x(t)$ and of the error signal $e(t)$. Considering that:

$$u_\xi = -\underbrace{\begin{bmatrix} k_0 & \cdots & k_{m-1} \end{bmatrix.}_{\in \Re^m} \underbrace{\left. k_\xi \right]}_{\in \Re^n} \begin{bmatrix} e \\ \vdots \\ e^{(m-1)} \\ \xi \end{bmatrix}$$

If we expand u_ξ:

$$u^{(m)} + \alpha_{m-1}u^{(m-1)} + ... + \alpha_0 u = -k_0 e - ... - k_{m-1}e^{(m-1)} - k_\xi(x^{(m)} + \alpha_{m-1}x^{(m-1)} + ... + \alpha_0 x)$$

We can collect some terms in this way:

$$(u^{(m)} + k_\xi x^{(m)}) + \alpha_{m-1}(u^{(m-1)} + k_\xi x^{(m-1)}) + ... + \alpha_0 \underbrace{(u + k_\xi x)}_{\widetilde{u}} = -k_0 e - ... - k_{m-1}e^{(m-1)}$$

So we obtain:

$$\widetilde{u}^{(m)} + \alpha_{m-1}\widetilde{u}^{(m-1)} + ... + \alpha_0 \widetilde{u} = -k_0 e - ... - k_{m-1}e^{(m-1)}$$

Applying the Laplace Transform:

$$s^m \widetilde{U}(s) + \alpha_{m-1}s^{m-1}\widetilde{U}(s) + ... + \alpha_0 \widetilde{U}(s) = -k_0 E(s) - ... - k_{m-1}s^{m-1}E(s)$$

So

$$\begin{aligned}\widetilde{U}(s) &= -\underbrace{\frac{k_{m-1}s^{m-1} + ... + k_0}{s^m + \alpha_{m-1}s^{m-1} + ... + \alpha_0}}_{P_e(s)} E(s) \\ &= -P_e(s)E(s)\end{aligned}$$

The complete scheme of the overall system controlled by using the internal control principle is reported in Fig.15.1.

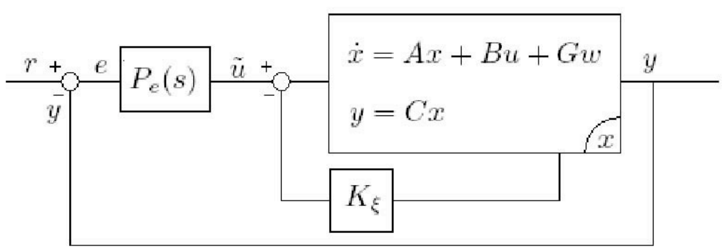


Figura 15.1. Scheme of the internal model control. (Typo: the error $e(t)$ should be replaced with $-e(t)$ in the figure)

We can notice that this is a generalization of the integral control in which we have assumed that the state $x(t)$ is accessible. Indeed in the case of integral control we have:

$$P_e(s) = \frac{k_I}{s} = \frac{k_o}{s}.$$

15.1.1 Summarizing

1. Check if (A, B) is reachable.
2. Check that roots of $(s^m + \alpha_{m-1}s^{m-1} + ... + \alpha_0)$ are *not* zeroes of $P(s) = C(sI - A)^{-1}B$.
3. Build matrices A_z , B_z.
4. Decide where to place eigenvalues: using dominant pole approximation with performance region.
5. $K_z = [k_0...k_{m-1}|k_z]$, so compute $P_e(s)$.

Considering the transfer function of the signal of the closed loop system:

$$Y(s) = P_{ry}(s)R(s)$$

If we are assume to track a sinusoidal signal $r(t) = a\sin(\omega_0 t + \phi)$, at steady state we will have $y(t) = a|P_{ry}(j\omega_0)|\sin(t + \phi + \angle P_{ry}(j\omega_0))$.
Instead if we have the disturbance signal $w(t) = a\sin(t\omega_0 + \phi)$ and $r(t) = 0$, the output will go to zero ($y(t) \to 0$). So the transfer function $Y(s) = P_{wy}(s)W(s)$ must have a zero in ω_0 : $P_{wy}(j\omega_0) = 0$.

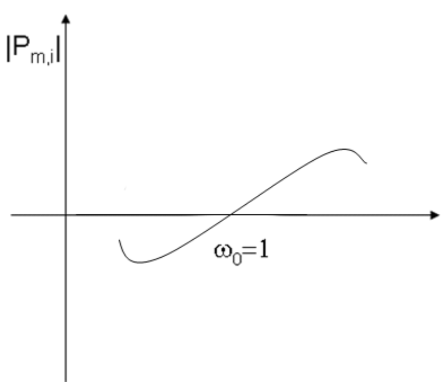


Figura 15.2. *Module of the transfer function $P_{ry}(j\omega)$.*

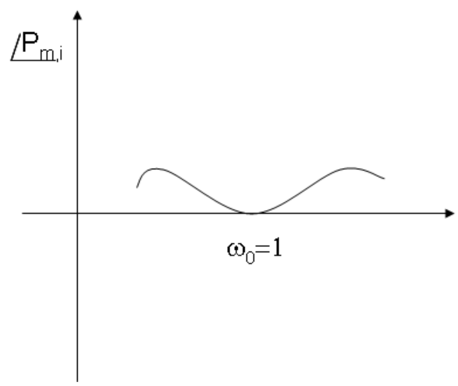


Figura 15.3. *Phase of the transfer function $P_{ry}(j\omega)$.*

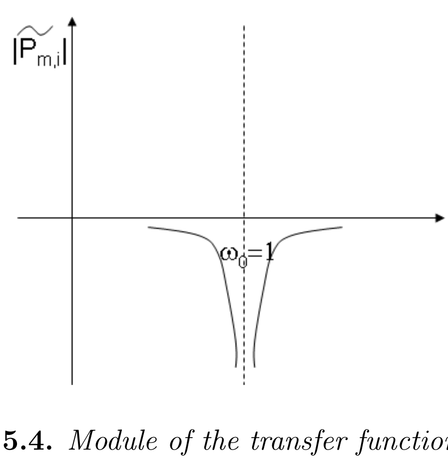


Figura 15.4. *Module of the transfer function $P_{wy}(j\omega)$*

15.2 Observers and Regulators

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu \\ y = C\hat{x} \end{cases}$$

which are the state rappresentation of Plant and Observer.

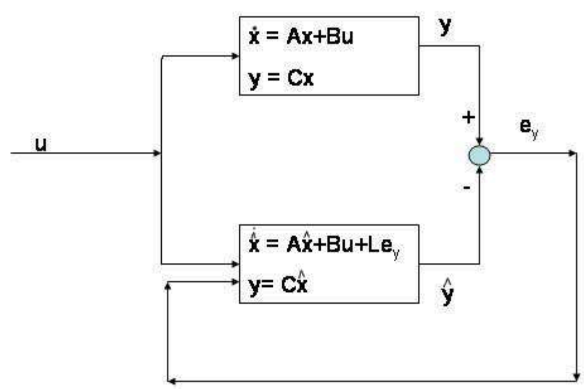
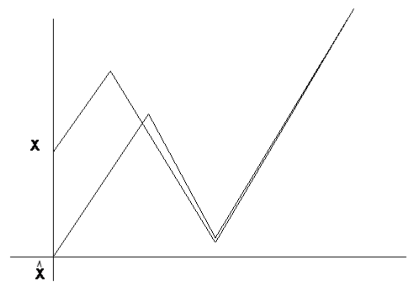


Figura 15.5. *Scheme of Plant and observer.*



State error: $e_x = x - \hat{x}$
measuring error: $e_y = y - \hat{y} = C(x - \hat{x}) = Ce_x$
$\dot{e_x} = Ae_x \Rightarrow \begin{array}{c} e \rightarrow 0 \\ \hat{x} \rightarrow x \end{array} \Leftrightarrow$ A is stable
$\dot{e_x} = \dot{x} - \dot{\hat{x}} = Ax + Bu - (A\hat{x} + Bu + Le_y) = (A - LC)e_x$

If $(A - LC)$ is strictly stable $\Rightarrow \hat{x}(t) \rightarrow x(t)\ \forall u(t), \forall A$

$\Rightarrow$ we can use the state feedback control.

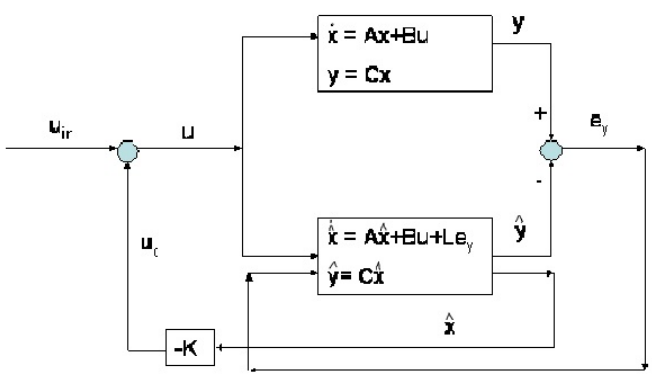
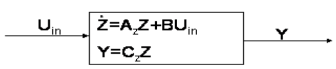


Figura 15.6. *Scheme of Plant, observer and controller.*

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ \dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \qquad \text{equations of the system} \\ \hat{y} = C\hat{x} \\ u = u_{in} + u_c \\ u_c = -k\hat{x} \end{cases}$$



$$z = \begin{bmatrix} x \\ e_x \end{bmatrix} z \in \mathbb{R}^{2n}$$

from the equations we obtain:
$\dot{x} = Ax + Bu_{in} - Bk\hat{x} = Ax + Bu_{in} - Bk(x - e_x)$
$\dot{e_x} = Ax + Bu_{in} - Bk\hat{x} - (A\hat{x} + Bu_{in} - Bk\hat{x} + LCe_x) = (A - LC)e_x$

15.2.1 Dynamical system

The state space representation with the more useful state z is:

$$\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{e_x} \end{bmatrix} = \left[\begin{array}{c|c} A - BK & BK \\ \hline 0 & A - LC \end{array}\right] \left[\begin{array}{c} x \\ \hline e_x \end{array}\right] + \left[\begin{array}{c} B \\ \hline 0 \end{array}\right] u_{in}$$

$$y = \left[\begin{array}{c|c} C & 0 \end{array}\right] \left[\begin{array}{c} x \\ \hline e_x \end{array}\right]$$

The stability of these system depends on the eigenvalues of A_z:

$$\Re[\lambda(A_z)] < 0 \Leftrightarrow Stable$$

The matrix is upper triangular so we can, by using the separation principle, divide the eigenvalues of A_z in the union between the eigenvalues of the controller and the eigenvalues of the observer:

$\lambda(A_z) = \lambda(A - BK) \bigcup \lambda(A - LC)$

$$\begin{cases} \text{(A,B)}, REACHABLE & \Rightarrow \\ \text{(A,C)}, OBSERVABLE & \Rightarrow \end{cases}$$ Exists K, L such that eigenvalues can be placed arbitrarily.

We will choose, in order to stabilize the system, eigenvalues with $Re[\lambda(A_z)] < 0$.

15.2.2 Tracking of a reference signal

How can be the matrices L and K chosen in order to achieve the best performance possible?
How can be $\lambda(A - LC)$ chosen?
In order that $\hat{x} \to x$ as fast as possible, $\lambda(A - LC)$ has to be chosen more negative (faster) than $\lambda(A - BK)$.
It is important to notice that L cannot be chosen too big because it amplifies the measure noise and because in that case the error e_x can have high peaks in the initial phase of the transitory. This peaks can force the controller to have high control signals.

$$\dot{\hat{x}}(t) = (A - LC)x(t) + Bu(t) + L(y(t) + d_y(t))$$

with $d_y(t)$ representing the measurement noise.

A rule of thumb for the design is to choose the eigenvalues of the observer matrix 3-10 times faster than the ones of the controller, in order to have a good compromise between the velocity of the tracking and the rejection of the output disturbances:

$$| \lambda(A - LC) | \simeq \beta | \lambda(A - BK) |$$

with $3 \leq \beta \leq 10$.