Control Laboratory:	**a.a. 2015/2016**

Lecture 14 — 12 April

Instructor: Luca Schenato *Scribes: A. Scampicchio, S. Romagnolo, T. Donchev*

14.1 Integral Control, Robust Tracking (continued)

The system we are analyzing is described by the following scheme:

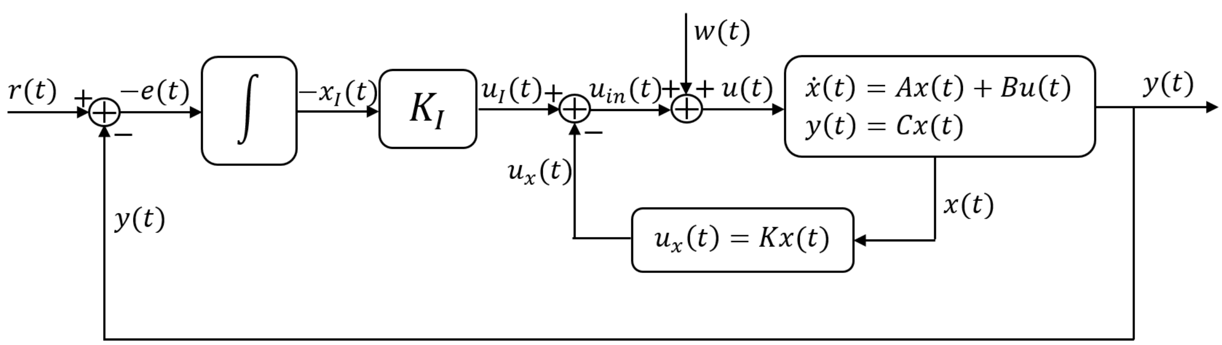

Figura 14.1.

Below some relations involved in the system are recalled:

$$e(t) = y(t) - r(t) \tag{14.1}$$

$$x_I(t) = \int e(t)dt \Leftrightarrow \dot{x}_I(t) = e(t) \tag{14.2}$$

$$u_{in}(t) = -\left[\ K_I \mid K\ \right] \underbrace{\begin{bmatrix} x_I(t) \\ \hline x(t) \end{bmatrix}}_{z(t)} = -K_I x_I(t) - Kx(t) = -K_z z(t) \tag{14.3}$$

$$-K_I x_I(t) = -K_I \int (y(t) - r(t))dt = K_I \int (r(t) - y(t))dt$$

Let us now focus on the dynamics of the extended state vector $z(t)$ defined in 14.3, in order to study the behaviour of the system with input reference signal $r(t)$ and input disturbance $w(t)$ of the kind

$$r(t) = r_{DC}1(t)$$
$$w(t) = w_{DC}1(t)$$

where $1(t)$ is the unitary step function.

$$\begin{bmatrix} \dot{x}_I(t) \\ \dot{x}(t) \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} 0 & C \\ \hline 0 & A \end{array}\right]}_{A_z} \underbrace{\begin{bmatrix} x_I(t) \\ x(t) \end{bmatrix}}_{z(t)} + \left[\begin{array}{c|c|c} 0 & 0 & -1 \\ \hline B & B & 0 \end{array}\right] \begin{bmatrix} u_{in}(t) \\ w(t) \\ r(t) \end{bmatrix} \tag{14.4}$$

$$= A_z z(t) + \left[\begin{array}{c|c} 0 & -1 \\ \hline B & 0 \end{array}\right] \begin{bmatrix} w(t) \\ r(t) \end{bmatrix} + \underbrace{\left[\begin{array}{c} 0 \\ \hline B \end{array}\right]}_{B_{in}} u_{in}(t) \tag{14.5}$$

The closed loop system is described by the following dynamics:

$$\begin{bmatrix} \dot{x}_I(t) \\ \dot{x}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A_z - B_{in}K_z \end{bmatrix}}_{A_c} \begin{bmatrix} x_I(t) \\ x(t) \end{bmatrix} + \left[\begin{array}{c|c} 0 & -1 \\ \hline B & 0 \end{array}\right] \begin{bmatrix} w(t) \\ r(t) \end{bmatrix}$$

As we have already seen, if A_c is asymptotically stable, then $z(t) \longrightarrow z_{DC}$, which implies $\dot{z}(t) \longrightarrow 0$: in particular $\dot{x}_I \longrightarrow 0$. Therefore $e(t) \longrightarrow 0$ by definition, so the signal tracking at steady state is well performed.
Our aim is to determine under which conditions A_c is asymptotically stable: in other words, we want to check when (A_z, B_{in}) is reachable[1]. To do that, we refer to the PBH test, which states that

$$\text{The system } (A_z, B_{in}) \text{ is reachable } \Leftrightarrow \text{rank}\left[\begin{array}{c|c} sI - A_c & B_{in} \end{array}\right] = n+1 \;\; \forall\, s \in \mathbb{C}$$

Therefore we have to see when, $\forall\, s \in \mathbb{C}$,

$$\text{rank}\left[\begin{array}{c|c} sI_{n+1} - \left[\begin{array}{c|c} 0 & C \\ \hline 0 & A \end{array}\right] & \left[\begin{array}{c} 0 \\ \hline B \end{array}\right] \end{array}\right] = n+1$$

$$\Rightarrow \text{rank}\left[\begin{array}{c|c|c} s & -C & 0 \\ \hline 0 & sI_n - A & B \end{array}\right] = n+1$$

At this point we see that two cases have to be taken into account:

[1]Actually it would hold also for a stabilisable system: we choose this stricter condition for the sake of simplicity

1. $s \neq 0$
   Here the first row is surely independent from the others: therefore we have to impose that
   $$\text{rank}\left[\begin{array}{c|c} sI_n - A & B \end{array}\right] = n$$
   We notice that this submatrix can be interpreted as a PBH matrix for the system (A, B): then in this case (A_z, B_{in}) is reachable $\Leftrightarrow$ (A, B) is reachable.

2. $s = 0$
   Now the first column has only zero elements. In this case we have to see when
   $$\text{rank}\left[\begin{array}{c|c} -C & 0 \\ \hline sI_n - A & B \end{array}\right] = n + 1$$
   As row and column permutations do not affect the matrix rank, by some operations we get
   $$\text{rank}\left[\begin{array}{c|c} sI_n - A & B \\ \hline -C & 0 \end{array}\right] = n + 1$$
   Also changing the sign of some rows or columns does not affect the rank of a matrix, therefore we have
   $$\text{rank}\left[\begin{array}{c|c} sI_n - A & -B \\ \hline C & D \end{array}\right] = n + 1$$
   we know that, if $\exists\, \bar{s}$ such that the matrix above loses rank, then $\bar{s}$ is a zero for the system (A, B, C, D): then, in order to have (A_z, B_{in}) reachable we must have that $s = 0$ is not a zero for (A, B, C, D).

Let us consider the meaning of this last statement for a SISO system. If it is described by the quadruple (A, B, C, D), then we know that its transfer function is:

$$P(s) = C(sI - A)^{-1}B + D$$

The statement is equivalent to having $P(0) \neq 0$: a valid transfer function may then be

$$P(s) = \frac{(s + 1)}{(s - 2)(s + 3)}$$

On the contrary, this next case would not allow to position the eigenvalues of A_c in any arbitrary position:

$$P(s) = \frac{s}{(s - 2)(s + 3)}$$

14.2 Feedforward Control: example for the DC motor

Now consider the DC motor seen in laboratory. We know that the relation that connects $\Theta_l(s)$ to $U(s)$ is

$$\Theta_l(s) = P(s)U(s)$$

furthermore the transfer function is:

$$P(s) = \frac{K}{s(s+p)}$$

If we want to use a feed-forward control, we have to decide an input $u_{in}(t)$ where the expression of $x(t)$ is

$$x(t) = \begin{bmatrix} \theta_l(t) \\ \dot{\theta}_l(t) \end{bmatrix}$$

so we can write $u_{in}(t)$ as:

$$u_{in}(t) = -Kx(t) + \overline{N}r(t) = -K_1\theta_l(t) - K_2\dot{\theta}_l(t) + \overline{N}$$

where $r(t) = 1(t)$. See *Figure* 14.2. We have to move the poles of the original plant in the closed loop poles whatching to the specs. We must consider that a feed-forward approach implies that disturbances cause a steady state error. Furthermore we can have steady state errors also if we do not know exactly the system parameters, so even without a disturbance we could have a steady state error using only feed-forward approach. To avoid steady state error we can use an integrator.

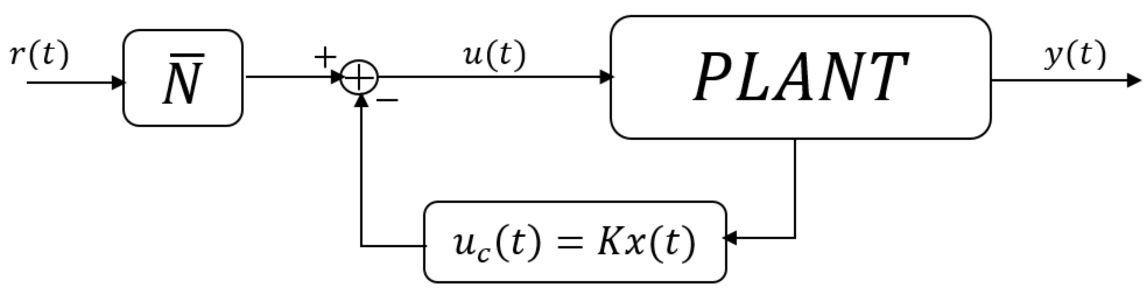

Figura 14.2.

Using an integrator we'll have $u_{in}(t)$ as:

$$u_{in}(t) = K_I x_I(t) - Kx(t) = -K_1\theta_l(t) - K_2\dot{\theta}_l(t) - K_I \int e(t)dt$$

We will start from a different configuration compared to the previous; we will have two poles in the origin and one in $-p$. The design region is the same because the specs are the same[2]. With the last approach to eliminate steady state error with position control of the motor we can see that robust tracking, in this case, corresponds to a I-PD control. In more complicated systems we will have always the I term, but the "PD" term will be replaced by something more complicated. This is a scenario which shows a direct link between classical and modern control.

14.3 Internal Model Principle

Internal model principle is a generalization of integral control. Suppose to have a MISO linear system with disturbance $w(t)$:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Gw(t) \\ y(t) = Cx(t) \end{cases}$$

Suppose that:

- $u \in \mathbb{R}^p$
- $x \in \mathbb{R}^n$
- $w \in \mathbb{R}^d$
- $r \in \mathbb{R}$
- $y \in \mathbb{R}$

Very common cases are:

- $r(t)$ e $w(t)$ are constant signals (but not known), that is $\dot{r}(t) = 0, \dot{w}(t) = 0,\ \forall t \geq 0$. Note that these signals correspond to the modes of a dynamic system where poles are given by the equation
$$s = 0$$
.
- r e w sinusoidal of frequency ω_0, which of course must satisfy the differential equations $\ddot{r}(t) + \omega_0^2 r(t) = 0, \ddot{w}(t) + \omega_0^2 w(t) = 0,\ \forall t \geq 0$. See that these signals correspond to the modes of a dynamical system where the poles are given by the equation
$$s^2 + \omega_0^2 = 0$$

[2] k_1, k_2 and k_I are determined if we choose where to place the poles

- It is possible to generalize the problem for periodic signals for which we do not know amplitude and phase. We can find the associated polynomial multiplying the polynomials of the different frequencies, assuming that the signal is of the type

  $$r(t) = a_0 + a_1 \sin(\omega_1 t + \phi_1) + a_2 \sin(\omega_2 t + \phi_2)$$

  where $(a_0, a_1, a_2, \phi_1, \phi_2)$ are not known and the polynomial is

  $$s(s^2 + \omega_1^2)(s^2 + \omega_2^2) = s^5 + (\omega_1^2 + \omega_2^2)s^3 + \omega_1^2\omega_2^2 s = 0$$

  equivalent to the differential equation:

  $$r^{(5)}(t) + (\omega_1^2 + \omega_2^2)r^{(3)}(t) + ... + \omega_1^2\omega_2^2 r(t) = 0$$

We want to analyse the generic problem and build a controller that guarantees $y(t) \to r(t)$ at steady state. We consider that the reference signals satisfy the next differential equations where the parameters $\{\alpha_i\}_{i=0}^{m-1}$ are known:

$$r^{(m)} + \alpha_{m-1}r^{(m-1)} + ... + \alpha_0 r = 0 \tag{14.6}$$

$$w^{(m)} + \alpha_{m-1}w^{(m-1)} + ... + \alpha_0 w = 0 \tag{14.7}$$

where $r(t)$ and $w(t)$ have the same coefficients.

We define $e = y - r = Cx - r$ as the tracking error.

So we find $e^{(m)} = Cx^{(m)} - r^{(m)}$ $\forall m$ and so

$$r^{(m)} = Cx^{(m)} - e^{(m)} \tag{14.8}$$

From 14.6 e 14.8 we find that:

$$Cx^{(m)} - e^{(m)} + \alpha_{m-1}(Cx^{(m-1)} - e^{(m-1)}) + ... + \alpha_0(Cx - e) = 0$$

and so:

$$e^{(m)} + \alpha_{m-1}e^{(m-1)} + ... + \alpha_0 - C(\underbrace{x^{(m)} + \alpha_{m-1}x^{(m-1)} + ... + \alpha_0 x}_{\xi \in \mathbb{R}^n}) = 0$$

The new state is:

$$\xi \triangleq x^{(m)} + \alpha_{m-1}x^{(m-1)} + ... + \alpha_0 x$$

Whence:

$$
\begin{aligned}
\dot{\xi} &= x^{(m+1)} + \alpha_{m-1}x^{(m)} + ... + \alpha_0 x^{(1)} \\
&= (Ax^{(m)} + Bu^{(m)} + Gw^{(m)}) + ... + \alpha_0(Ax + Bu + Gw) \\
&= A(x^{(m)} + \alpha_{m-1}x^{(m-1)} + ... + \alpha_0 x) + B(\underbrace{u^{(m)} + \alpha_{m-1}u^{(m-1)} + ... + \alpha_0 u}_{u_\xi \in \mathbb{R}^p}) + \\
&\quad + G(w^{(m)} + \alpha_{m-1}w^{(m-1)} + ... + \alpha_0 w) \\
&= A\xi + Bu_\xi
\end{aligned}
$$

Where we have used 14.7 in the last equation to eliminate the disturbance. Now we will consider an augmented state where we add the error and its derivative up to its $(m-1)$-th order and ξ.

So the new state is $z \triangleq \begin{bmatrix} e \\ \vdots \\ e^{(m-1)} \\ \hline \xi \end{bmatrix} \in \mathbb{R}^{(n+m)}$

Whence:

$$
\dot{z} = \begin{bmatrix} e^{(1)} \\ \vdots \\ e^{(m)} \\ \hline \dot{\xi} \end{bmatrix} = \overbrace{\left[\begin{array}{ccccc|c} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{m-1} & C \\ \hline 0 & 0 & 0 & \cdots & 0 & A \end{array}\right]}^{A_z} \begin{bmatrix} e \\ \vdots \\ e^{(m-1)} \\ \hline \xi \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \hline \mathrm{B} \end{bmatrix}}^{B_z} u_\xi
$$

So $\dot{z} = A_z z + B_z u_\xi$. See that in this new state representation the reference signal $r(t)$ and the input disturbance vector $w(t)$ have disappeared. Now we want to find $u_\xi(t) = -k_z z(t)$ that stabilizes the closed loop system. We'll see it in the next lecture.