Control Laboratory:	**a.a. 2015/2016**
Lezione 13 — 07 April	
Instructor: Luca Schenato	*Scribes: Andrea Franceschetti, Patrick Magon, Elisa Feltre*

13.1 Approximation of II order system & Integral control

13.1.1 Approximation of II order systems with dominant poles

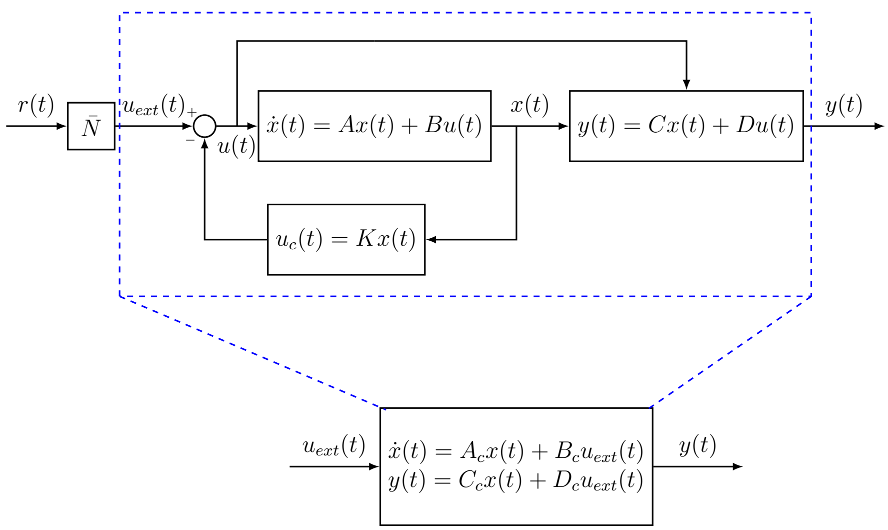

Figura 13.1. System with state feedback K

In the previous lecture, we proposed to modify the dynamics of the system by adding a static feedback on the state: placing a suitable scalar term $\bar{N}$ and properly designing a controller K, under the assumption of (A, B) reachable, the closed-loop dynamics $A_c = A - BK$ is asymptotically stable with arbitrary eigenvalue positions. We found that, assuming $D = 0$ for the sake of simplicity, the suitable scalar is

$$\bar{N} = \frac{1}{-CA_c^{-1}B} = -\frac{1}{C(A - BK)^{-1}B} \tag{13.1}$$

This way, if $r(t) = \mathbf{1}(t)$ is the step function, the corresponding output will be able to track it, i.e. $y(t) \to \mathbf{1}(t)$, which means that the steady state error is null.

How does the transient behave? The closed-loop dynamics can be written as a function of the reference:

$$\begin{cases} \dot{x}(t) = A_c x(t) + B u_{ext}(t) = A_c x(t) + B\bar{N} r(t) \\ y(t) = Cx(t) \end{cases} \tag{13.2}$$

$\lambda(A_c)$, eigenvalues of the closed-loop system, correspond to the poles of the closed-loop transfer function, which is

$$W(s) = C(sI - A_c)^{-1} B\bar{N} \tag{13.3}$$

The whole system can be seen as a SISO system, in fact both input and output are scalar. We can place $\lambda(A_c)$ arbitrarily, under the reachability assumption, so we place $\lambda(A_c)$ according to the dominant pole region determined by performance specifications (t_r, t_s, M_P).

The problem of state feedback control will reduce to the problem of placing the poles of the closed-loop system in the coloured region of Figure 13.2.

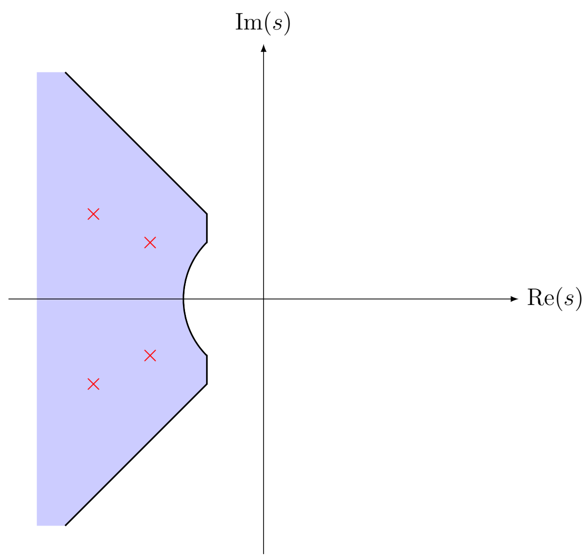

Figura 13.2. Performance region

This is just an approximation, so it is not guaranteed that the closed-loop system satisfy the requirements.

With this approach, there are mainly two problems:

1. *Reachability is a Yes/No answer*: intuitively, there should be systems which are more reachable, whose poles are "easier" to move from one location to another, but reachability does not provide a notion of "how much" reachable the system is.

2. *Role of the zeros of $W(S)$*: in fact, the approximation is determined only by the poles, but the zeros are important too.

Example 1 Consider the following system:

$$A = \begin{bmatrix} 0 & \varepsilon \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = [0]$$

It has two eigenvalues, one on 0 and the other in 1, so it is evident that it is unstable.
The reachability matrix has full rank if and only if its determinant is not equal to zero.

$$\mathcal{C} = [B|AB] = \begin{bmatrix} 0 & \varepsilon \\ 1 & 1 \end{bmatrix} \qquad \det \begin{bmatrix} 0 & \varepsilon \\ 1 & 1 \end{bmatrix} = -\varepsilon$$

The system is reachable if and only if $\varepsilon \neq 0$.

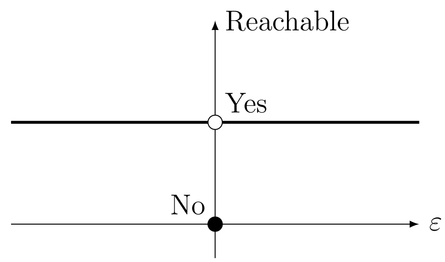

Figura 13.3. Reachability map

Therefore, we know that there exists

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \in \mathbb{R}^{1\times 2}$$

such that $\det(sI - A_c) = s^2 + 2\xi\omega_n s + \omega_n^2$, which means that, for any ξ and ω_n, I can find a suitable K to place the poles in the desired position.
We want to find an explicit relation between k_1, k_2 and ξ, ω_n:

$$\begin{aligned}
\det(sI - A_c) &= \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} 0 & \varepsilon \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) \right) = \\
&= \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & \varepsilon \\ -k_1 & 1-k_2 \end{bmatrix} \right) = \\
&= \det \begin{bmatrix} s & -\varepsilon \\ -k_1 & s-1+k_2 \end{bmatrix} = \\
&= s(s-1+k_2) + \varepsilon k_1 = s^2 + (k_2-1)s + \varepsilon k_1
\end{aligned}$$

For the two polynomials to be the same:

$$\begin{cases} k_2 - 1 = 2\xi\omega_n \\ \varepsilon k_1 = \omega_n^2 \end{cases} \qquad \Rightarrow \qquad \begin{cases} k_1 = \frac{\omega_n^2}{\varepsilon} \\ k_2 = 1 + 2\xi\omega_n \end{cases}$$

So, given the position of the poles in closed-loop, these equations let me compute the values of K. In Matlab, this task can be performed using `place` or `acker`.
Observations:

- $|\varepsilon| \to 0 \Rightarrow k_1 \to +\infty$
  Remembering that $u_c = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = k_1 x_1(t) + k_2 x_2(t)$, if the poles are close to the region where the system is not reachable, it is harder to control the system and the control signal gets larger.

- $\omega_n \to +\infty \Rightarrow k_1, k_2 \to +\infty$
  A faster system (indeed, if $\omega_n \to +\infty$, then $t_s \simeq \frac{cost}{\omega_n} \to 0$) implies that we want to move from one point to the other more quickly and to do so we need to push the control harder and use very large gains.

In conclusion, the poles must be placed in the suitable region, but they cannot be too negative, otherwise we are likely to pay in terms of control effort.

Example 2 Consider the dynamical system

$$P(s) = \frac{s - z_0}{(s+4)(s+3)} \qquad z_0 \in \mathbb{R}$$

A state space representation of the dynamical system may be

$$A = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -z_0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = [0]$$

Let's verify that the transfer function is $P(s)$:

$$\begin{aligned} P(s) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(sI - \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ -z_0 \end{bmatrix} = \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+7 & -1 \\ 12 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -z_0 \end{bmatrix} = \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\text{Adj} \begin{bmatrix} s+7 & -1 \\ 12 & s \end{bmatrix}}{s^2 + 7s + 12} \begin{bmatrix} 1 \\ -z_0 \end{bmatrix} = \frac{s - z_0}{s^2 + 7s + 12} \end{aligned}$$

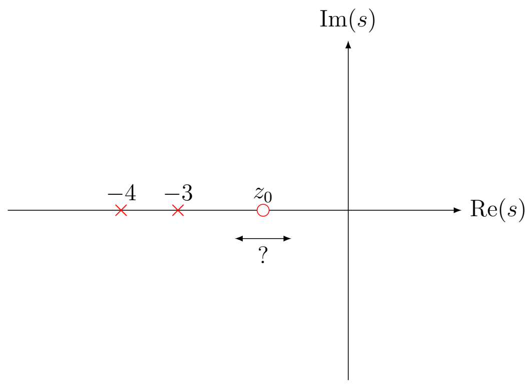

Figura 13.4. Zero/Pole map of $P(s)$

Now, let us check if the system is controllable:

$$\det \mathcal{C} = \det[B|AB] = \det \begin{bmatrix} 1 & -7-z_0 \\ -z_0 & -12 \end{bmatrix} = -12 + z_0(-7-z_0) = -12 - 7z_0 - z_0^2 \neq 0$$

The determinant is different from 0 if and only if z_0 is not equal to 3 or 4 (no zero/pole cancellation). If this happens, the system is reachable and the poles can be placed anywhere we want. $\det(sI - A_c)$ is computed as follows:

$$\det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -z_0 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right)\right) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -7-k_1 & 1-k_2 \\ -12+z_0k_1 & z_0k_2 \end{bmatrix}\right) = \dots$$
$$= (s+7+k_1)(s-z_0k_2) - (k_2-1)(12-z_0k_1)$$
$$= s^2 + (7+k_1-z_0k_2)s - 7z_0k_2 - 12k_2 + 12 - z_0k_1$$

$$\begin{cases} 7+k_1-z_0k_2 = 2\xi\omega_n \\ -7z_0k_2 - 12k_2 + 12 - z_0k_1 = \omega_n^2 \end{cases} \quad \Rightarrow \quad \begin{cases} k_1 = \frac{z_0(14\xi\omega_n - 37 - \omega_n^2) + 12(2\xi\omega_n - 7)}{(z_0+3)(z_0+4)} \\ k_2 = \frac{z_0(7-2\xi\omega_n) + 12 - \omega_n^2}{(z_0+3)(z_0+4)} \end{cases}$$

From the previous expression, it is clear that if we place the zero too close to the poles of the transfer function, k_1 and k_2 go to infinity.
In conclusion, it is more difficult to move the open-loop poles away from zeros located nearby. Sometimes, we get a better performance leaving the pole near to the zero rather than trying to move it in the performance region of the complex plane.

So, where should the closed-loop poles be placed? Rules of thumb:

1. Do not move poles (eigenvalues in state space) if they already are inside the dominant pole region.

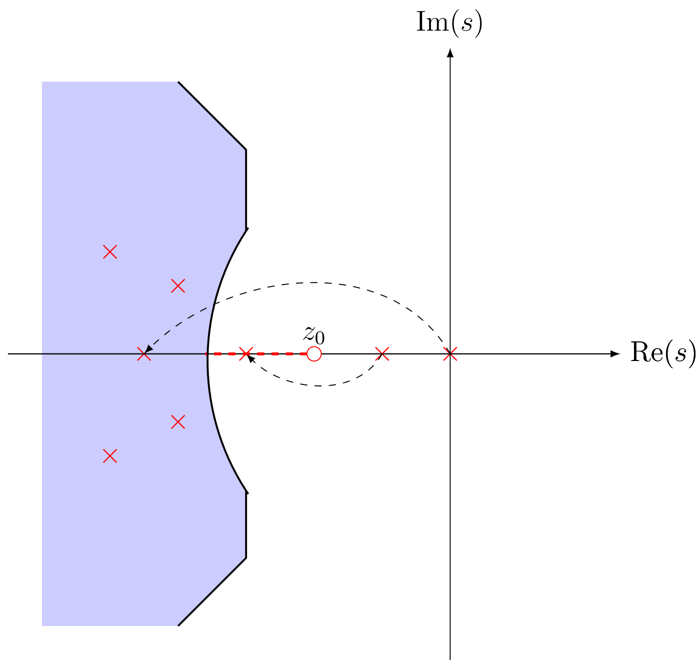

Figura 13.5. Rule of thumbs visual representation

2. Do not move poles too much, place them inside the dominant pole region moving them as little as possible.

3. If $P(s)$ has zeros with negative real part, place the poles which are not in the dominant pole region in a position between the zero and the border of the dominant pole region.In fact, it is hard to move poles away from the zero, so we can try to place the pole closer to the dominant pole region to make it more negative, in order to avoid inputs to become too large (for such a situation might cause saturation and therefore lead to poor performance).

13.1.2 Tracking control

The following scheme (already seen in Figure 13.1) is called *nominal tracking*.

The output $y(t)$ tends to the reference $r(t)$ only if the nominal plant values A, B and C are exactly known (if they are not, then the steady state error it likely to be not null). This type of tracking does not compensate for unknown input disturbances and this leads once again to a steady state error different from zero.

These two problems can be taken care with integral control.

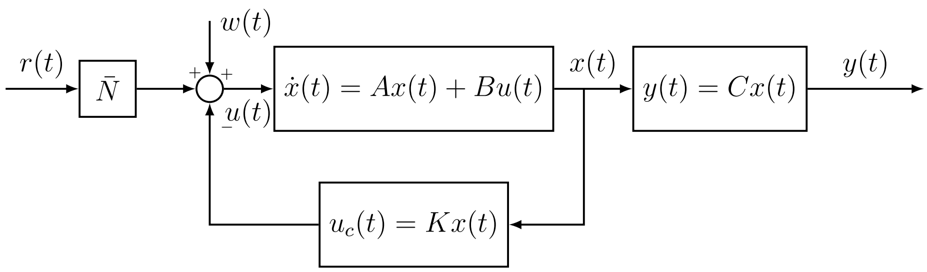

Figura 13.6. Nominal tracking block diagram.

13.1.3 Integral control/Robust tracking

The idea is to exploit the information contained in the error in order to design a better control.

$$e(t) = y(t) - r(t) = Cx(t) - r(t) \tag{13.4}$$

We add another variable which corresponds to the integral of the error:

$$x_I \in \mathbb{R}: \quad \dot{x}_I(t) := e(t) \tag{13.5}$$

We assume that $x(t)$ in accessible and, to simplify the notation, we suppose $D = 0$. Besides, we will assume that the external disturbance and the reference input are constant.

$$w(t) = w_{DC}\mathbf{1}(t) \qquad\qquad r(t) = r_{DC}\mathbf{1}(t) \tag{13.6}$$

We want $y(t)$ to converge to $r(t)$ with no steady state error for any w_{DC} and r_{DC}.

Let us define a new state with one extra variable: this state corresponds to the state of the original system plus the new variable x_I previously defined.

$$z = \begin{bmatrix} x_I \\ x \end{bmatrix} \in \mathbb{R}^{n+1} \tag{13.7}$$

If the error does not go to zero, x_I will keep on growing, therefore it can be used if the steady state error in not zero. Input and output can be rewritten as

$$u_z = \begin{bmatrix} u_{in} \\ w \\ r \end{bmatrix} \qquad\qquad y_z = \begin{bmatrix} y \\ x_I \end{bmatrix} \tag{13.8}$$

We will not actually use the output because we assumed to have access to $x(t)$. The system is rewritten as

$$\begin{cases} \dot{z}(t) = A_z z(t) + B u_z(t) \\ y_z(t) = C_z z(t) + D_z u_z(t) \end{cases} \tag{13.9}$$

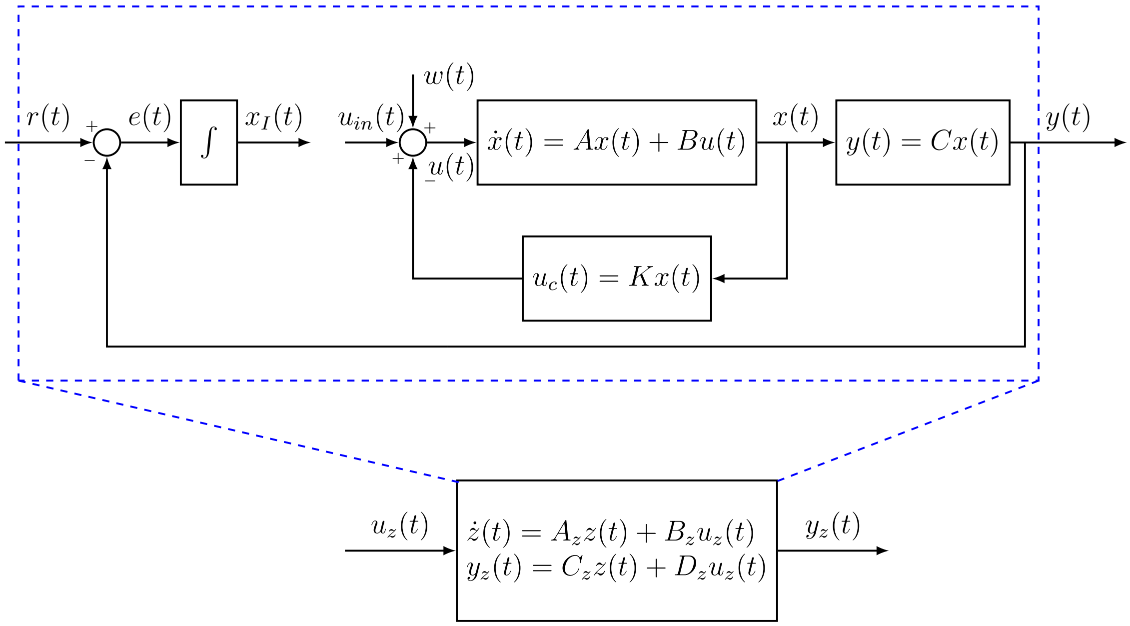

Figura 13.7. Integral control block diagram. (There is a typo in the figure: the input to the integrator is $-e(t)$ and not $e(t)$.

$$\dot{z} = \begin{bmatrix} \dot{x}_I \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix} \begin{bmatrix} x_I \\ x \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \\ B & B & 0 \end{bmatrix} \begin{bmatrix} u_m \\ w \\ r \end{bmatrix} \tag{13.10}$$

We want to design a controller which is a linear state feedback:

$$u_{in}(t) = -K_z z(t) = -\begin{bmatrix} K_I & K \end{bmatrix} \begin{bmatrix} x_I \\ x \end{bmatrix} = -K_I x_I - Kx \tag{13.11}$$

The feedback is similar to before, but now I'm using the information added by the integral of the error.

$$\dot{z}(t) = \underbrace{(A_z - B_u K_z)}_{A_z^c} z(t) + \begin{bmatrix} 0 & -1 \\ B & 0 \end{bmatrix} \begin{bmatrix} w \\ r \end{bmatrix} \tag{13.12}$$

Let's assume (A_z, B_u) reachable, where $B_u = \begin{bmatrix} 0 \\ B \end{bmatrix}$ is the first column of the second matrix in (13.10). Then, there exists a K_z such that A_z^c is asymptotically stable (that is, $\mathrm{Re}[\lambda(A_z^c)] < 0$). If we apply a constant input in an asymptotically stable system, we know that the output will reach a steady state value: $z(t) \to z_{DC} \Rightarrow y(t) \to y_{DC}$. We do not know the value y_{DC} yet, but it will be a constant for sure. So, we can conclude that

$$\dot{z}(t) = 0 \Rightarrow \dot{x}_I(t) \to 0 \Rightarrow e(t) \to 0 \Rightarrow y(t) \to r_{DC} \quad \forall w_{DC}$$

and the tracking of the reference signal is perfect. The only thing left to do is to demonstrate that (A_z, B_u) is reachable. We will show that (A_z, B_u) is reachable if and only if $P(s)$ has no zeros in $s = 0$, i.e. if and only if $P(0) \neq 0$.