Control Laboratory:	**a.a. 2015/2016**

Lecture 12 — April, 6

Instructor: Luca Schenato — *Scribes: A. Babetto, L. Festa, A. Malaguti*

12.1 Reachability

Let us begin by summarizing all the equivalent conditions for the reachability of a system in the state space representation:

Def. The system (A, B, C, D) is reachable if and only if, for any $\mathbf{x}(0) \in \mathbb{R}^n$, $\mathbf{x}(T) \in \mathbb{R}^n$, $T \in \mathbb{R}$ positive, there exists u(t), $t \in [0, T]$, that drives $\mathbf{x}$(t) from $\mathbf{x}$(0) to $\mathbf{x}$(T).

$$\Updownarrow$$

Prop. 1 $\text{rank}(\mathcal{C}) = \text{rank}\begin{bmatrix} B & AB & .. & A^{n-1}B \end{bmatrix} = \text{n}$, where $\begin{bmatrix} B & AB & .. & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times np}$.

$$\Updownarrow$$

Prop. 2 $\text{rank(H)} = \text{rank}\begin{bmatrix} sI - A & B \end{bmatrix} = \text{n}, \forall s \in \Lambda(A)$, where $\begin{bmatrix} sI - A & B \end{bmatrix} \in \mathbb{R}^{n \times (n+p)}$.

$$\Updownarrow$$

Prop. 3 $\exists K \in \mathbb{R}^{p \times n}$ such that eigenvalues of $A_C = A - BK$ are in any arbitrary configuration $\{\lambda_1, .., \lambda_n\} \in \mathbb{C}$.

12.2 Observability

Now we introduce another major concept of modern control system theory, which is the observability of a system. This property describes whether the internal state variables of the system $\mathbf{x}$(t), which are not directly observable, can be externally measured from the knowledge of the control u(t) and the output y(t).

The observability conditions for a system in state space representation are expressed as follows:

Def. A system (A, B, C, D) is observable if $\forall \mathbf{x}(0) \in \mathbb{R}^n$, $\forall T > 0$, $\forall$ u(t) (control sequence), $t \in [0, T]$, there exists a procedure such that from u(t), y(t), $t \in [0, T]$ we can obtain $\mathbf{x}$(T). Moreover the property is even stronger, infact I can obtain the value $\mathbf{x}$(t), $t \in [0, T]$.

$$\Updownarrow$$

Prop. 1 rank $\begin{bmatrix} C \\ CA \\ .. \\ CA^{n-1} \end{bmatrix} = \text{n}$, where $\begin{bmatrix} C \\ CA \\ .. \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{nl \times n}$.

$$\Updownarrow$$

Prop. 2 rank $\begin{bmatrix} sI - A \\ C \end{bmatrix} = \text{n}$, $\forall s \in \Lambda(A)$, where $\begin{bmatrix} sI - A \\ C \end{bmatrix} \in \mathbb{R}^{(n+l) \times n}$.

$$\Updownarrow$$

Prop. 3 $\exists K \in \mathbb{R}^{n \times l}$ such that eigenvalues of $A_L = A - LC$ are in any arbitrary configuration $\{\lambda_1, .., \lambda_n\} \in \mathbb{C}$.

From the display of the definition and conditions of observability is easy to recognize that are very similar to the properties of a reachable system.

This similarity is expressed in the property of duality that exists between the concepts of observability and reachability:

Proposition (A, B, C, D) is observable $\Longleftrightarrow (A^T, C^T, B^T, D^T)$ is reachable.

Note: This result is very useful when you want to compute the L matrix that gives you a certain selection of complex eigenvalues, as said in Proposition 3. In Matlab there is no function that computes L, but that's not a problem: you can compute K using the function *place* or *acker*, with arguments $(A^T, C^T, \{\lambda_1, .., \lambda_n\})$ and then find the desired matrix as $L = K^T$.

12.3 Zeros of a system

We say that $\bar{s} \in \mathbb{C}$ is a *zero* of a system (A,B,C,D) if the following matrix

$$Z = \begin{bmatrix} \bar{s}I - A & -B \\ C & D \end{bmatrix} \in \mathbb{R}^{(n+l)\text{x}(n+p)}$$

has:

$$\text{rank}(Z) < \min(n + l, n + p) \tag{12.1}$$

For SISO system ($l = p = 1$), $Z \in \mathbb{R}^{(n+1)\text{x}(n+1)}$ is a square matrix. So, the condition (12.1) is equivalent to say:

$$\text{rank}(Z) < n + 1 \Longleftrightarrow \det(Z) = 0 \tag{12.2}$$

The determinant of a block matrix can be decomposed as follows:

$$\det \begin{bmatrix} \overline{s}I - A & -B \\ C & D \end{bmatrix} = \det(\overline{s}I - A) \cdot \det[D + C(\overline{s}I - A)^{-1}B] = d(\overline{s}) \cdot \frac{n(\overline{s})}{d(\overline{s})} = n(\overline{s}) = 0$$[1]

where

$$\det[D+C(\overline{s}I-A)^{-1}B] = D+C(\overline{s}I-A)^{-1}B = P(\overline{s}) = \frac{n(\overline{s})}{d(\overline{s})} \in \mathbb{C} \Longrightarrow \det \begin{bmatrix} \overline{s}I - A & -B \\ C & D \end{bmatrix} = n(\overline{s}) = 0$$

12.4 Property of asymptotically stable system

Consider a system (A, B, C, D) strictly stable

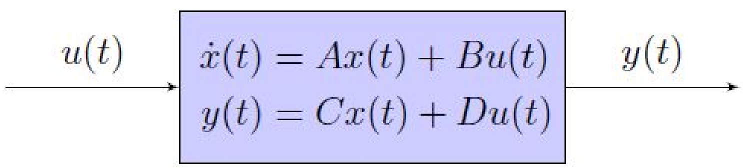

Figura 12.1. I/O model of a system in state space representation

For this system, we have that:

$$\mathrm{Re}\,[\lambda_i(A)] < 0, \quad \forall i \tag{12.3}$$

where $\lambda_i(A)$ indicates the spectrum of A.
We know that $\mathbf{x}(t)$ is given by:

$$\mathbf{x}(t) = \mathbf{x}_0(t) + \mathbf{x}_f(t)$$

where

$$\begin{cases} \mathbf{x}_0(t) = e^{At}\mathbf{x}(0) \\ \mathbf{x}_f(t) = \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau \end{cases} \tag{12.4}$$

[1] We know that:

$$\det \begin{bmatrix} F & G \\ H & J \end{bmatrix} = \det(F) \cdot \det(J - HF^{-1}G).$$

Suppose, now, to apply a constant signal $\mathbf{u}(t) = \mathbf{u}_{DC}\mathbf{1}(t)$ to the input of the system. For $t \longrightarrow \infty$, we have that:

$$\begin{array}{rcl} \mathbf{x}_0(t) & \to & 0, \quad \forall \mathbf{x}(0) \\ \mathbf{x}_f(t) & \to & \mathbf{x}_{DC} \in \mathbb{R}^n \quad (constant) \\ \mathbf{x}(t) & \to & \mathbf{x}_{DC} \quad \Rightarrow \quad \mathbf{y}(t) \to \mathbf{y}_{DC} \quad (constant) \\ \dot{\mathbf{x}}(t) & \to & 0 \end{array}$$

If we apply these results to the system of Figura(12.1), we get:

$$\begin{cases} \mathbf{0} = A\mathbf{x}_{DC} + B\mathbf{u}_{DC} \\ \mathbf{y}_{DC} = C\mathbf{x}_{DC} + D\mathbf{u}_{DC} \end{cases} \Longrightarrow \begin{cases} \mathbf{x}_{DC} = -A^{-1}B\mathbf{u}_{DC} \\ \mathbf{y}_{DC} = (D - CA^{-1}B)\mathbf{u}_{DC} \end{cases} \tag{12.5}$$

where we used the fact that A being asymptotically stable implies that there is no eigenvalue in zero and therefore it is invertible. So, if the input of a strictly stable system is constant, also the steady state output will be constant.
For SISO systems, we have:

$$\begin{array}{rl} P(s) & = D + C(sI - A)^{-1}B \\ P(0) & = D - CA^{-1}B \\ \Longrightarrow \quad \mathbf{y}_{DC} & = (D - CA^{-1}B)\mathbf{u}_{DC} = P(0)\mathbf{u}_{DC} \end{array}$$

12.5 Control design in state space

What's the approach for the control of a state space represented system?
We take the output $y(t)$ and we design, based on the observation of $y(t)$, a control input $u_c(t)$ that will go into the plant. We still want to have the freedom to choose the external input $u_{ext}(t)$, so the input of the plant will be $u(t) = u_{ext}(t) - u_c(t)$.

How do we design the F, G, H, J matrix of the dynamical system to stabilize the close loop system and sadisfie the requirements in terms of performance?

In reality the dynamical system is a little more complex, because we would like to use all the information that we have. That is why, in general, the inputs of the Dynamical System are three: the reference signal $r(t)$, the input $u(t)$ and the output $y(t)$.

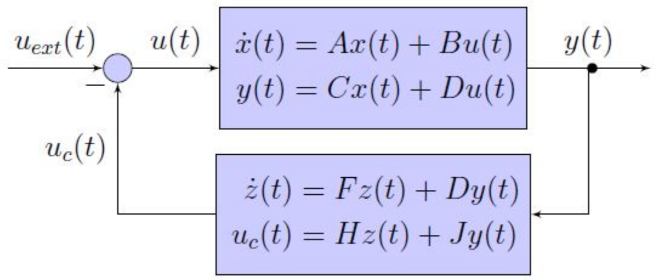

Figura 12.2. Feedback control system in state space representation

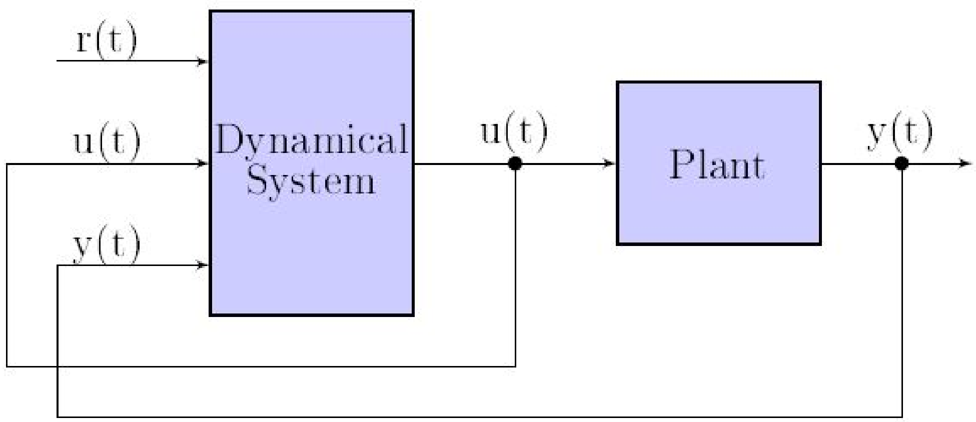

Figura 12.3. General state space representation

12.5.1 State Feedback

If we concentrate only on SISO system we have that $u_{ext}(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$ and $u_c(t) \in \mathbb{R}$. The state $x(t)$ is not accessible, i.e. it is not known. However, if we pretend that an estimate can be computed, then we can apply a linear feedback to it, as showed in Figure 12.5, where $K \in \mathbb{R}^{1\times n}$ is a row vector. Now we want to find what is the equivalent system in terms of external input $u_{ext}(t)$ and output $y(t)$, and we want to stabilize it.

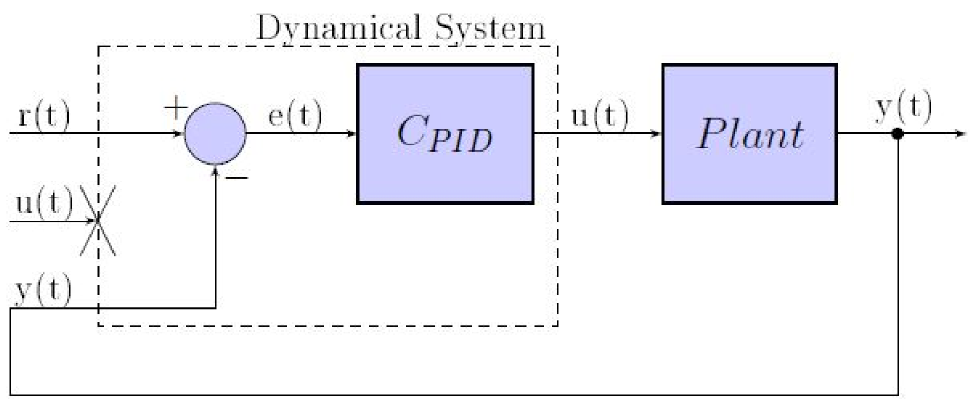

Figura 12.4. State space representation of a system with PID controller

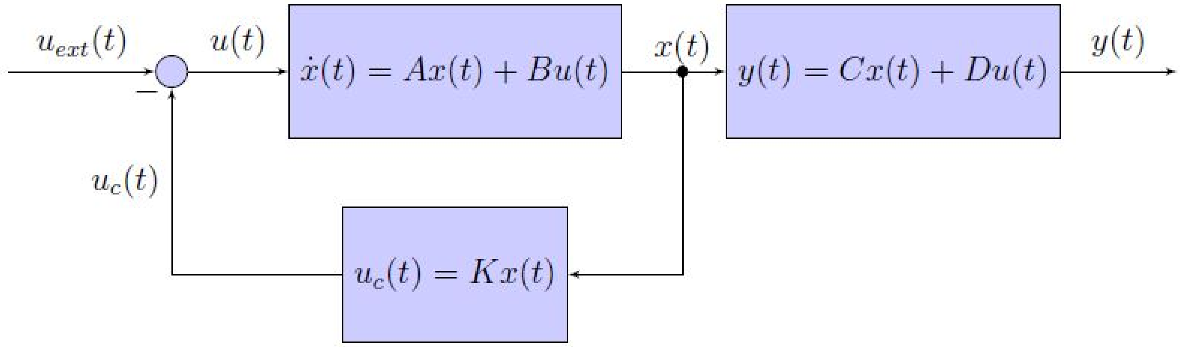

Figura 12.5. State feedback control system

The system is described by the following equations:

$$\dot{x}(t) = Ax(t) + By(t), \tag{12.6a}$$
$$u(t) = u_{ext}(t) - u_c(t), \tag{12.6b}$$
$$u_c(t) = Kx(t), \tag{12.6c}$$
$$y(t) = Cx(t) + Dy(t). \tag{12.6d}$$

By substituting the (12.6b) and (12.6c) equations in the (12.6a) equation, we can rewrite the dynamics of the state $x(t)$ as:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu_{ext}(t) - Bu_c(t) \\
&= Ax(t) + Bu_{ext}(t) - BKx(t) \\
&= (A - BK)x(t) + Bu_{ext}(t) \\
&= A_C x(t) + Bu_{ext}(t).
\end{aligned} \tag{12.7}
$$

Along the same lines we obtain the output $y(t)$:

$$
\begin{aligned}
y(t) &= Cx(t) + Du_{ext}(t) - Du_c(t) \\
&= Cx(t) + Du_{ext}(t) - DKx(t) \\
&= (C - DK)x(t) + Du_{ext}(t) \\
&= C_C x(t) + Du_{ext}(t).
\end{aligned} \tag{12.8}
$$

So we have a system in state space representation described by the equations (12.9a) and (12.10), and by the block diagram at Figure 12.6

$$\dot{x}(t) = A_C x(t) + Bu_{ext}(t), \tag{12.9a}$$
$$y(t) = C_C x(t) + Du_{ext}(t). \tag{12.9b}$$

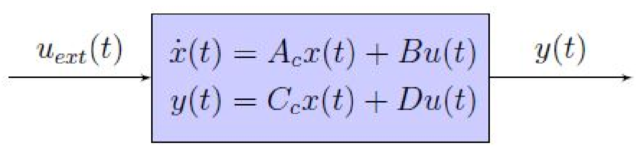

Figura 12.6. State space representation of the closed loop system

If the original system (A, B, C, D) is reachable, then there exist matrices K for which we have $\text{Re}[\lambda_i(A_C)] < 0$. Actually reachability is a stronger notion since it allows us to choose a matrix K such that the locations of the eigenvalues of A_C placeable are arbitrary. As so, an important question is to decide where to place the eigenvalues of K if we want to track a reference signal and to take care of performance too?

Basically we want to know what happens to the output $y(t)$ of the system if we apply a constant step function to the input u_{ext}. As seen before, we know that if we apply a constant input, we get a constant output, as equations 12.9a and 12.11 show us

$$u_{ext}(t) = u_{DC} 1(t) \tag{12.10}$$
$$y_{DC} = (D - C_C A_C^{-1} B) u_{DC} \tag{12.11}$$

Once we fix K, $(D - C_C A_C^{-1} B)$ will be equal to a scalar $\overline{N} \in \mathbb{R}$.
So, if we want y_{DC} to track a reference signal r_{DC}, we find

$$r_{DC} = y_{DC} = \overline{N} u_{DC} \tag{12.12}$$

That give us

$$u_{DC} = \frac{1}{\overline{N}} r_{DC} \tag{12.13}$$

Basically we got that once we design K than we can compute the controller in such a way that if we applay the step function, as given by equation 12.10, we will reach zero stady state error.

What we have done can be seen as a **feedforward control**, because we are trying to manipulate the reference signal $r(t)$ in a way to get the desired output $y(t)$. If we want we can see $\frac{1}{\overline{N}}$ as the simplest approximation of the dynamical system.

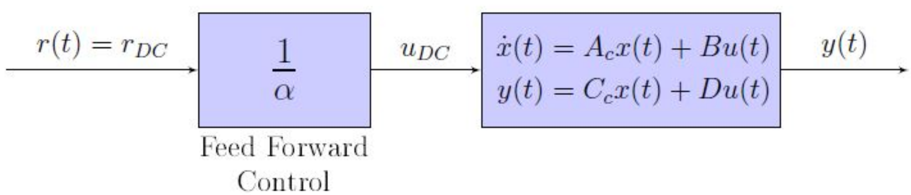

Figura 12.7. Feedforward control (nominal tracking) for reference tracking (Typo: α in the figure has to be replaced with $\overline{N}$).

12.5.2 State feedback by pole placement

At this point we have been able to design the closed loop in order to stabilize the system and we solved one problem in term of performance, which is the steady state error. The other problems are related to the transient and typically they include overshoot, settling time and rise time.

We know that the eigenvalues of matrix A_C correspond to the poles of the transfer function in closed loop and the performance metrics left are related to the eigenvalues of the system, which, under reachability assumption, we can place wherever I want. So now the question is: where should I place the eigenvalues in closed loop to take care of performance? One possible choice is to use the approximation of II order systems with dominant poles. This approach is based on the fact that, being the constraints on overshoot, settling time and rise time associated to a certain region in the complex plain, if I have a second order system and I place the poles in the region then I expect to satisfy the specifics on the transient.

Rarely I have a second order system, normally I have a much larger system, but the idea is to place the eigenvalues in closed loop in the desired region and hopefully the system will satisfy the specifications on s, t_s and t_r.

In practice it is a little more complicated because is not obvious where to place the poles in the region; this is the limitation of this approach, but in the next lectures we will see how optimal control will take care of the problem.

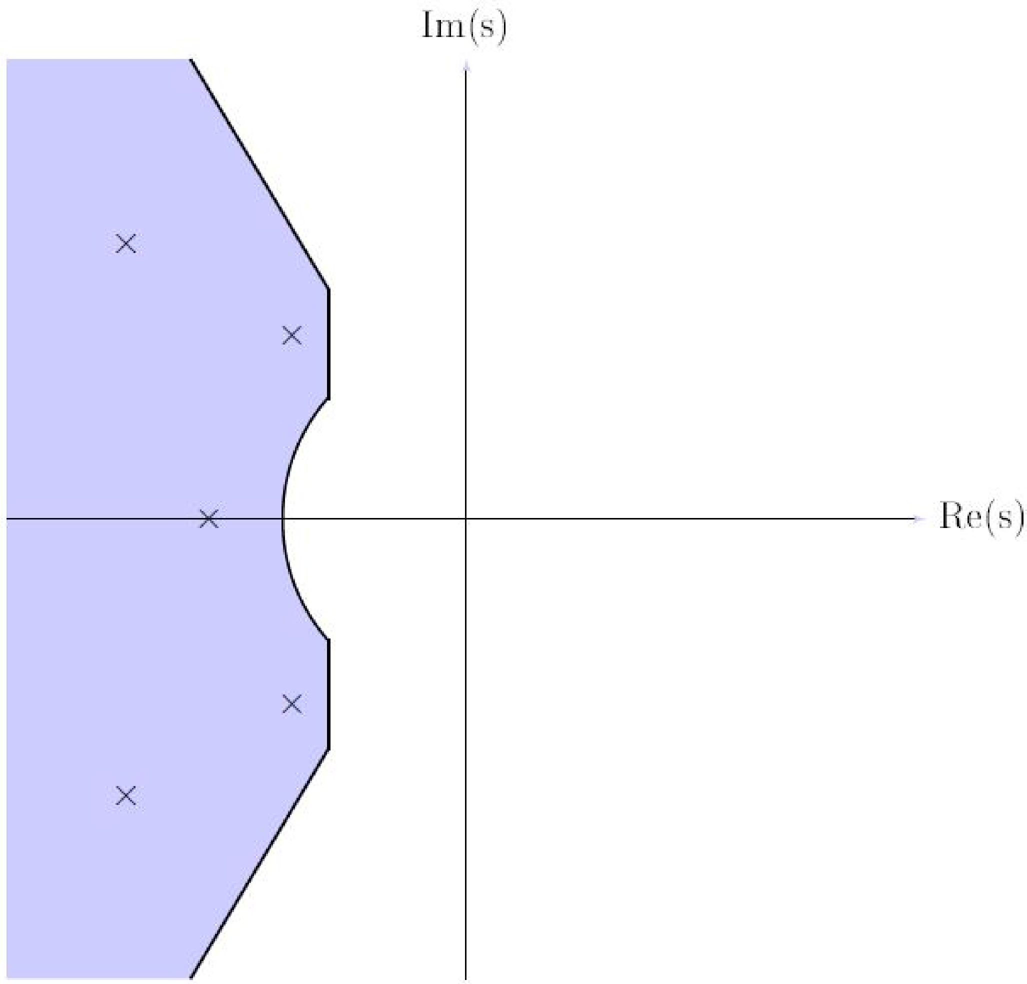

Figura 12.8. Pole placement region