Lezione 11 — 5^{th} April

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11.1 State space control (modern control)

In modern control theory, we give a different representation of the dynamics as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{11.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^l$. For $p = l = 1$ we have LTI SISO systems, but this representation is suitable also for general LTI MIMO systems where $p \geq 1$ and $l \geq 1$.

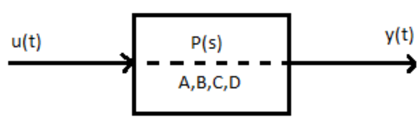

Figura 11.1. State Space representation

As mention previously, there are infinite state space representation for the same dynamical system, therefore a natural question is how to derive A,B,C,D? There are different ways to do that:

1. **Directly from basic physical equations**
   Example 11.1: Consider the physics of a DC motor (without the gears):

$$\begin{cases} u(t) = Ri_a(t) + K_e\dot{\theta}_m \\ \tau_m(t) = K_t i_a(t) \\ J_m\ddot{\theta}_m(t) = -b_m\dot{\theta}_m(t) + \tau_m(t) \\ u_m(t) = K_\theta\theta_m(t) \end{cases} \tag{11.2}$$

   where

   - $u(t)$ is the input voltage of the motor
   - τ_m is the torque applied to the motor
   - J_m is the rotor moment of inertia
   - u_m is the measured voltage at the potentiometer

- b_m is the friction coefficient
- K_θ is the transducer constant.

In the associated state space representation we can easily figure out which are the inputs and the outputs:

- $y(t) = u_m(t)$
- $u(t) = u(t)$
- $x(t) = \begin{bmatrix}\theta_m(t) & \dot{\theta}_m(t)\end{bmatrix}^T \Rightarrow \dot{x}(t) = \begin{bmatrix}\dot{\theta}_m(t) & \ddot{\theta}_m(t)\end{bmatrix}^T$

Note that different representations can be considered, but this one is straight-forward. From the equations of 11.1 we have:

$$J_m\ddot{\theta}_m(t) = -b_m\dot{\theta}_m(t) + K_t i_a(t)$$

$$i_a(t) = \frac{u(t)}{R} - \frac{K_e}{R}\dot{\theta}_m(t)$$

$$\Rightarrow J_m\ddot{\theta}_m(t) = -b_m\dot{\theta}_m(t) + K_t\frac{u(t)}{R} - K_t\frac{K_e}{R}\dot{\theta}_m(t)$$

$$\Rightarrow \ddot{\theta}_m(t) = -\left(\frac{b_m R + K_t K_e}{RJ_m}\right)\dot{\theta}_m(t) + \frac{K_t}{RJ_m}u(t)$$

Therefore, we obtain the following state space equations:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix}\dot{\theta}_m(t) \\ \ddot{\theta}_m(t)\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 \\ 0 & -\frac{b_m R + K_t K_e}{RJ_m}\end{bmatrix}}_{A}\begin{bmatrix}\theta_m(t) \\ \dot{\theta}_m(t)\end{bmatrix} + \underbrace{\begin{bmatrix}0 \\ \frac{K_t}{RJ_m}\end{bmatrix}}_{B}u(t) \\ y(t) = \underbrace{\begin{bmatrix}K_\theta & 0\end{bmatrix}}_{C}\begin{bmatrix}\theta_m(t) \\ \dot{\theta}_m(t)\end{bmatrix} + \underbrace{\begin{bmatrix}0\end{bmatrix}}_{D}u(t) \end{cases}$$

2. **Derive (A,B,C,D) from the transfer function P(s)**
   If a transfer function $P(s)$ is instead given, in this case the problem is that the representation is not unique. Hence, there are infinite choices of (A,B,C,D), all related through a linear transformation. Assume to have (A,B,C,D); then, $\exists T \in \mathbb{R}^{n\times n}$ which is invertible, i.e. $\exists T^{-1}$, that maps:

$$(A, B, C, D) \underset{T^{-1}}{\overset{T}{\rightleftarrows}} (A', B', C', D')$$

In particular, consider the following two systems:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{11.3}$$

and

$$\begin{cases} \dot{x}'(t) = A'x'(t) + B'u(t) \\ y(t) = C'x'(t) + D'u(t) \end{cases} \tag{11.4}$$

characterized by the same input and output but different states. Then, there exists T such that:

$$\underset{x(t)=T^{-1}z(t)}{\overset{z(t)=Tx(t)}{\Updownarrow}} \quad \Leftrightarrow \quad \underset{\dot{x}(t)=T^{-1}\dot{z}(t)}{\overset{\dot{z}(t)=T\dot{x}(t)}{\Updownarrow}}$$

Thus, we have:

$$\begin{cases} T^{-1}\dot{z}(t) = AT^{-1}z(t) + Bu(t) \\ y(t) = CT^{-1}z(t) + Du(t) \end{cases} \underset{\text{multiplying by T}}{\Rightarrow} \begin{cases} \dot{z}(t) = \underbrace{TAT^{-1}}_{A'} z(t) + \underbrace{TB}_{B'} u(t) \\ y(t) = \underbrace{CT^{-1}}_{C'} z(t) + \underbrace{D}_{D'} u(t) \end{cases}$$

Therefore, we obtain the following relations:

$$A' = TAT^{-1}, \quad B' = TB, \quad C' = CT^{-1}, \quad D' = D$$

Below, some of the more useful representations are listed:

- modal representation (Jordan form)
- observability canonical form
- controllability canonical form
- balanced realization.

11.1.1 Properties using state space represantation

The state space representation is characterized by the following properties:

1. it allows the analysis of MIMO systems
2. it describes the evolution of the system. The evolution of the state can be interpreted as the sum of two terms:

$$x(t) = \underbrace{x_0(t)}_{\text{natural state dynamic}} + \underbrace{x_f(t)}_{\text{forced state dynamic}}$$

computed as

$$x_0(t) = e^{At}x(0)$$

$$x_f(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

where the exponential of a matrix is defined as:

$$e^{At} = \sum_{h=0}^{\infty} \frac{(At)^h}{h!}$$

Therefore, the evolution of the output is given by:

$$\begin{aligned} y(t) &= C(x_0(t) + x_f(t)) + Du(t) \\ &= \underbrace{Ce^{At}x(0)}_{y_0(t)} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)}_{y_f(t)} \end{aligned}$$

Note that:

- x_0 and y_0 depend only on the initial conditions
- x_f and y_f depend only on the input.

3. It is possible to derive the transfer function P(s). Hence, to do so, it is sufficient to apply the Laplace transform:

$$\begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases} \tag{11.5}$$

From the first equation we have that:

$$(sI - A)X(s) = BU(s) \quad \Rightarrow \quad X(s) = (sI - A)^{-1}BU(s)$$

and substituting into the second equation:

$$Y(s) = \underbrace{\left[C(sI - A)^{-1}B + D\right]}_{=P(s)\in\mathbb{C}^{p\times l}} U(s)$$

Recall that for SISO systems $p = l = 1$ and, in particular, P(s) is the ratio between two polynomials:

$$P(s) = \frac{n(s)}{d(s)}.$$

For MIMO systems, instead, we have that $P(s)$ is given by the following matrix:

$$[P(s)]_{i,j} = \frac{n_{ij}(s)}{d(s)}$$

where all the elements (indexed by the pairs of numbers (i,j)) have the same denominator $d(s)$, defined as:

$$d(s) = \det(sI - A).$$

4. it is possible to study the dynamic of the system through the analysis of the eigenvalues and eigenvectors. In particular, $\lambda \in \mathbb{C}$ is an eigenvalue of A if there exists a vector $v \in \mathbb{C}^n$, $v \neq 0$, such that $Av = \lambda v$. Moreover, if $\lambda \in \mathbb{C}$ is an eigenvalue of A, then also $\bar{\lambda} \in \mathbb{C}$ is an eigenvalue of A.
   The eigenvalues of A are related to the poles of P(s). More specifically, if λ is eigenvalue, then $\det(\lambda I - A) = 0 = d(\lambda)$, which means that λ is a pole of $P(s)$. In general, the determinant of $(sI - A)$ is a polynomial of order n and we have that:
   $$\#\ \text{(number) of distinct eigenvalues} \leq n$$
   $$\det(sI - A) = \prod_{i=1}^{m} (s - \lambda_i)^{n_i} \quad \text{with} \quad \sum_{i=1}^{m} n_i = n$$
   The eigenvalues are also related to the natural response ($u(t) = 0$) of the system:
   $$\begin{aligned} x(t) = x_0(t) &= e^{At}x(0) \\ &= \sum_{i=1}^{m'} \sum_{j=0}^{n_i'-1} t^{j'} e^{\lambda_i t} v_j \end{aligned}$$
   If λ_i are all distinct, we have:
   $$x_0(t) = \sum_{i=1}^{n} e^{\lambda_i t} \alpha_i \bar{v}_i$$
   where $\{\bar{v}_i\}_{i=1}^n$ are the normalized eigenvectors and $\{\alpha_i\}_{i=1}^n$ are functions of $\{x_i(0)\}_{i=1}^n$. Thus, the (natural) output of the system is given by:
   $$\begin{aligned} y_0(t) &= Cx_0(t) \\ &= \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} \underbrace{C\bar{v}_i}_{\beta_i} \\ &= \sum_{i=1}^{n} \alpha_i \beta_i \underbrace{e^{\lambda_i t}}_{\text{modes}} \end{aligned}$$
   A system is asymptotically stable (AS) if and only if the eigenvalues of A have $\mathbb{R}e[\lambda_i] < 0, \quad \forall \lambda_i$. If a system is AS, then $y_0(t) \to 0$ as $t \to \infty$ for all initial conditions.

5. Controllability/Reachability of (A,B,C,D). Let us focus on reachability: the system (A,B,C,D) is reachable if $\forall x(0) \in \mathbb{R}^n$, $\forall x(T) \in \mathbb{R}^n$, $\forall T > 0, T \in \mathbb{R}$, then there exists $u(t), t \in [0, T]$ that drives x(t) from x(0) to x(T).

   Proposition 11.1. *: A system (A,B,C,D) is reachable if and only if the matrix $\mathcal{C} \in \mathbb{R}^{n \times (np)}$ given by*
   $$\mathcal{C} = [B|AB|\ldots|A^{n-1}B]$$
   has rank equal to n.

For SISO systems the latter condition implies that $\mathcal{C}$ is invertible, hence $\det(\mathcal{C}) \neq 0$.

Proposition 11.2. *(PBH TEST) : Let us define the matrix $H = [sI_n - A|B] \in \mathcal{C}^{n\times(n+p)}, \forall s \in \mathcal{C}$ .The system (A,B,C,D) is reachable if and only if $rank(H) = n, \forall s \in \mathcal{C}$. In particular, if s is not an eigenvalue of A, then*

$$s \neq \lambda_i \quad \Rightarrow \quad \text{rank}([sI_n - A]) = n$$

Therefore, we have to check that $rank(H) = n, \forall s = \lambda_i$.

Proposition 11.3. *: A system (A,B,C,D) is reachable if and only if $\forall\{\bar{\lambda}_1, \dots, \bar{\lambda}_n\} \in \mathcal{C}$, $\exists K \in \mathbb{R}^{p\times n}$ such that $A_c = A - BK \in \mathbb{R}^{n\times n}$ has eigenvalues equal to $\{\bar{\lambda}_1, \dots, \bar{\lambda}_n\}$.*

Summing up:

$$(A, B, C, D) \textit{ reachable} \Rightarrow \exists u(t), t \in [0, T] \text{ such that } x(t) \text{ goes from x(0) to x(T)}$$

$$\Leftrightarrow \text{ rank}[B|AB|\dots|A^{n-1}B] = n$$

$$\Leftrightarrow \text{ rank}([sI - A|B]) = n, \forall s \in \lambda_i(A)$$

$\Leftrightarrow \exists K$ such that the eigenvalues of $A_c = A - BK$ are in any arbitrary configuration.