Control Laboratory: **a.a. 2015/2016**

Lezione 10 — 31 March

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10.1 Transfer function of a DC Motor

Starting from the equations derived from the DC motor model

$$\begin{cases} L_a \dfrac{di_a}{dt} + (R_a + R_s)\, i_a = u_{drv} - u_e & (10.1) \\ J_m \dfrac{d\omega_m}{dt} + B_m \omega_m = \tau_m - \dfrac{1}{N}\tau_l & (10.2) \\ J_l \dfrac{d\omega_l}{dt} + B_l \omega_l = \tau_l - \tau_d & (10.3) \\ u_e = k_e \omega_m & (10.4) \\ \tau_m = k_t i_a & (10.5) \\ N\omega_l = \omega_m & (10.6) \end{cases}$$

we want now to obtain the transfer function of the system from the driver input voltage u to the mechanical load position θ_l. In order to represent the differential equations of the system above as algebraic equations in the Laplace domain, we need to define the following Laplace transforms

$$\Omega_m(s) = \mathcal{L}\left[\omega_m\right]$$
$$\Omega_l(s) = \mathcal{L}\left[\omega_l\right]$$
$$T_m(s) = \mathcal{L}\left[\tau_m\right]$$
$$T_l(s) = \mathcal{L}\left[\tau_l\right]$$
$$T_d(s) = \mathcal{L}\left[\tau_d\right]$$
$$I_a(s) = \mathcal{L}\left[i_a\right]$$
$$U_{drv}(s) = \mathcal{L}\left[u_{drv}\right]$$
$$U_e(s) = \mathcal{L}\left[u_e\right]$$

and by applying the Laplace transform on both sides of each differential equation of the system we obtain:

$$\begin{cases} (L_a s + R_{eq})\, I_a(s) = U_{drv}(s) - K_e N \Omega_l(s) & (10.7) \\ (J_m s + B_m)\, N \Omega_l(s) = K_t I_a(s) - \dfrac{\tau_l(s)}{N} & (10.8) \\ (J_l s + B_l)\, \Omega_l(s) = T_l(s) - T_d(s) & (10.9) \end{cases}$$

where $R_{eq} = (R_a + R_s)$ is the equivalent resistance (notice that we combined 10.1, 10.2 and 10.3 with 10.4, 10.5 and 10.6). Now if we define

- $J_{eq} = \left(J_m + \dfrac{J_l}{N^2}\right)$ as the equivalent moment of inertia
- $B_{eq} = \left(B_m + \dfrac{B_l}{N^2}\right)$ as the equivalent friction constant

by combining 10.7, 10.8 and 10.9 we obtain:

$$\Omega_l(s) = \frac{k_t}{N} \frac{1}{(L_a s + R_{eq})(J_{eq} s + B_{eq}) + k_e k_t} U_{drv}(s) - \frac{1}{N^2} \frac{1}{(L_a s + R_{eq})(J_{eq} s + B_{eq}) + k_e k_t} T_d(s)$$

from which we can derive the transfer function from the output driver voltage u_{drv} to the mechanical load angular velocity ω_l:

$$\Omega_l(s) = P'(s) U_{drv}(s), \qquad P'(s) = \frac{k_t}{N} \frac{1}{(L_a s + R_{eq})(J_{eq} s + B_{eq}) + k_e k_t}$$

The denominator of $P'(s)$ can be expressed in the form

$$\frac{1}{(L_a s + R_{eq})(J_{eq} s + B_{eq}) + k_e k_t} = \frac{1}{R_{eq} B_{eq} (1 + s T_{elec})(1 + s T_{mech}) + k_e k_t}$$

where

- $T_{elec} = \dfrac{L_a}{R_{eq}}$ is the electrical time constant
- $T_{mech} = \dfrac{J_{eq}}{B_{eq}}$ is the mechanical time constant

and under the assumption that $T_{elec} \ll T_{mech}$ (this happens if L_a is very small), we can write

$$P'(s) = \frac{k_t}{N} \frac{1}{(R_{eq} J_{eq} s + R_{eq} B_{eq} + k_e k_t)}$$

Notice that the transfer function from the input driver voltage u to the output driver voltage u_{drv} can be simplified as follows (since $T_{drv} \ll T_{mech}$):

$$P_{drv}(s) = \frac{k_{drv}}{(1 + sT_{drv})} \simeq k_{drv}$$

Recalling that the mechanical load angle and the mechanical load angular velocity are related by

$$\omega_l = \frac{d\theta_l}{dt}$$

and by

$$\Omega_l(s) = s\Theta_l(s)$$

in the Laplace domain, we can lastly obtain the transfer function from the driver input voltage u to the mechanical load position θ_l:

$$\Theta_l(s) = P(s)U(s), \qquad P(s) = \frac{1}{s}P_{drv}(s)P'(s) = \frac{1}{s}\frac{1}{N}\frac{k_m}{(1 + sT_m)}$$

where

- $k_m = \dfrac{k_{drv}k_t}{R_{eq}B_{eq} + k_t k_e}$
- $T_m = \dfrac{R_{eq}J_{eq}}{R_{eq}B_{eq} + k_t k_e}$

10.2 PID configurations

10.2.1 PID standard configuration

The following configuration shows a classic feedback control system with a PID.

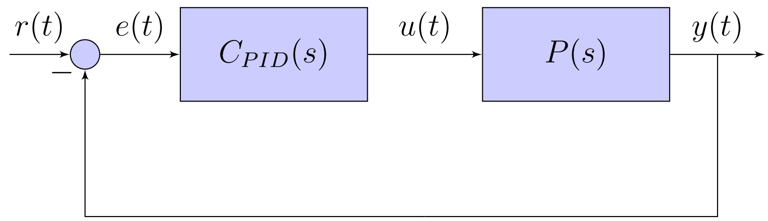

Figura 10.1. Process controlled by a PID.

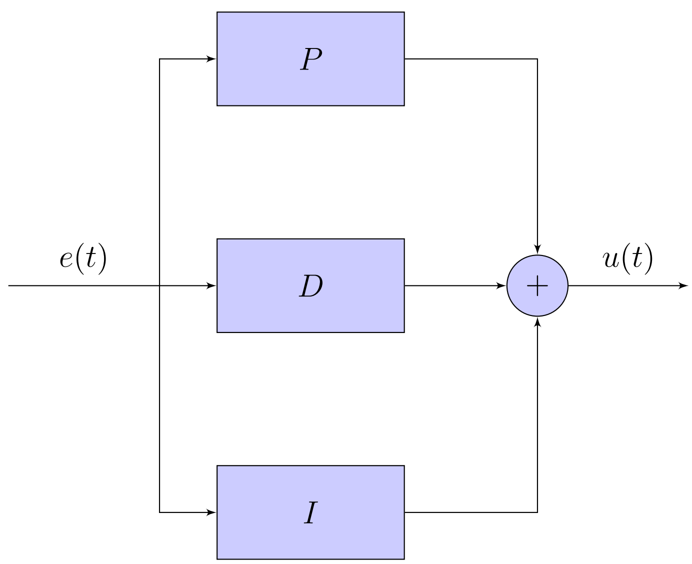

Figura 10.2. Parallel PID configuration.

The problem with this configuration is that at the initial time instant $t = 0$ we have $y(0) = 0$ and if we apply a step function as input signal, the derivative part of PID (with the add of an high frequency pole to make its transfer function proper) causes an impulse. In practice is the derivative action is implemented as $\frac{s}{1+\tau_L s}$, then it is not difficult to show via the Initial Value Theorem that the amplitude is proportional to $\frac{1}{\tau_L}$.

$$C_D(s) = \frac{K_D s}{1 + \tau_L s}$$

$$e(t) = r(t) = 1(t) \xrightarrow{\mathcal{L}} E(s) = \frac{1}{s}$$

$$U_D(s) = C(s)E(s) = \frac{K_D}{1 + \tau_L s} \xrightarrow{\mathcal{L}^{-1}} u_D(t) = \frac{K_D}{\tau_L} e^{-\frac{t}{\tau_L}}$$

In figure 10.3 we can notice that there is an impulse in the origin.

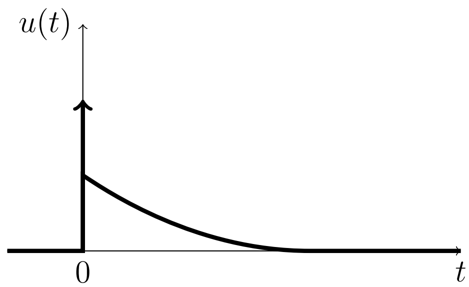

Figura 10.3. Step response: PID standard configuration.

10.2.2 PI-D configuration

With this configuration we can eliminate the impulse in the control signal. This can be very useful because the fatigue at the actuator input will be reduced.

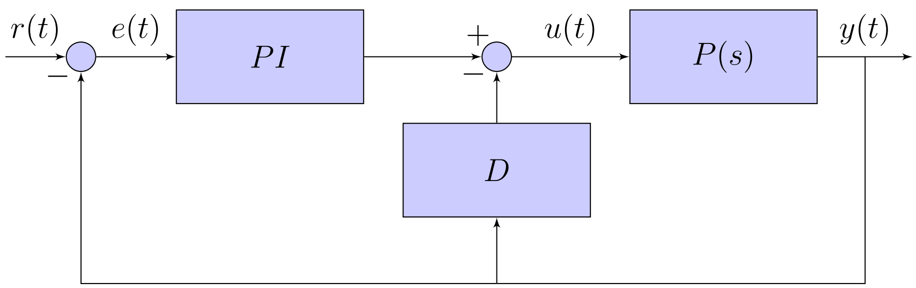

Figura 10.4. PI-D configuration.

How you can see in Figure 10.4, the derivative action is applied only at the output. With abuse of notation we can write:

$$u(t) = \left(K_P + \frac{K_I}{s}\right)(r(t) - y(t)) - K_D \frac{dy(t)}{dt} \tag{10.10}$$

With a step function as input signal (figure 10.5) we will obtain a control signal $u(t)$ with no impulses in the origin but still discontinuous.

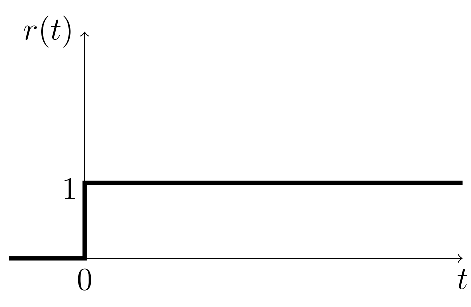

Figura 10.5. Input: step function.

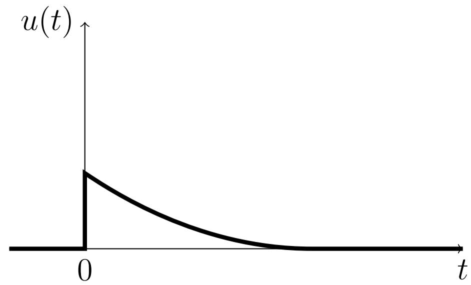

Figura 10.6. Step response: PI-D configuration.

10.2.3 I-PD configuration

With this configuration, the integrator action is predominant, but it increases the settling time. The control signal does not have a discontinuity and the eventual variation of the signal is limited. Considering the same step function input, we obtain the control signal in figure 10.8. This configuration is not useful to meet the settling time specification and it is used only if a quick system response is not needed.

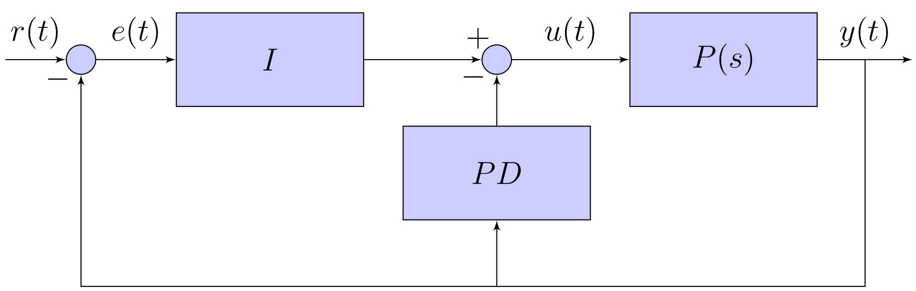

Figura 10.7. I-PD configuration.

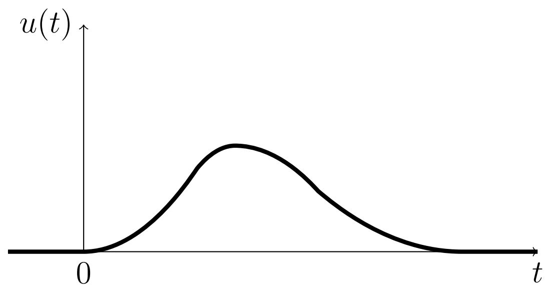

Figura 10.8. Step response:configuration I-PD

10.3 Anti wind-up

One of the most problematic source of non-idealities in control system design is the interaction between the integrator and actuators and it is due to the limitation of the control signal introduced by actuators. For example, by considering the voltage driver-DC motor system, the voltage driver output signal can not exceed (in absolute value) the maximum value of $6\ V$. The series connection between the integrator, contained in the PID-controller block ($C(s)$), and the actuator (represented by a saturator) is showed off in figure 10.9.

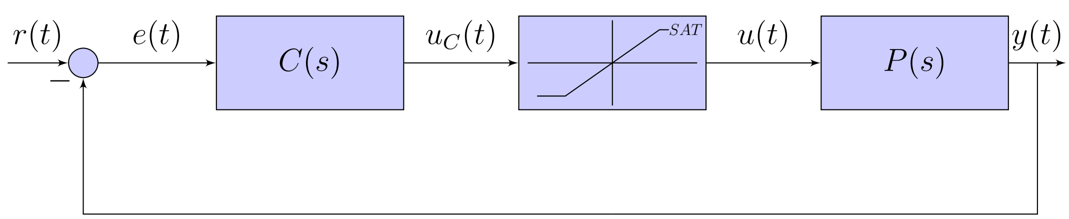

Figura 10.9. Series connection between the integrator and the actuator.

In order to make the comprehension easier, we split the PID-controller output signal ($u_C(t)$) into two components:

$$u_C(t) = u_I(t) + u_{PD}(t)$$

where $u_I(t)$ is the integral component and $u_{PD}(t)$ is the proportional-derivative one. The integral component can be rewritten as

$$u_I(t) = \int (r(t) - y(t))\ dt = \int e(t)\ dt$$

where $e(t) = r(t) - y(t)$ is the error, defined as the difference between the reference signal and the output signal (figure 10.10).

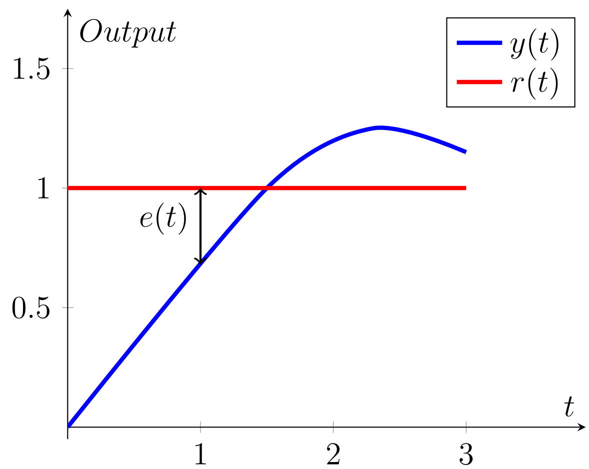

Figura 10.10. Error as the difference between $r(t)$ and $y(t)$.

The three temporal diagrams displayed in figure 10.11 show that the integral component ($u_I(t)$) of the control signal (i.e. the integral of the error), generated by the PID-controller, increases until the error $e(t)$ is positive, while it starts decreasing when the error turns negative (and this happens when the reference signal and the output signal assume the same value). As a consequence, the saturator output signal will mantain the saturation value (because of the high value assumed by the integral component) until the negative value of the error will produce a sufficiently large reduction of the integral term. The scenario here described will be referred to as **wind-up** problem.

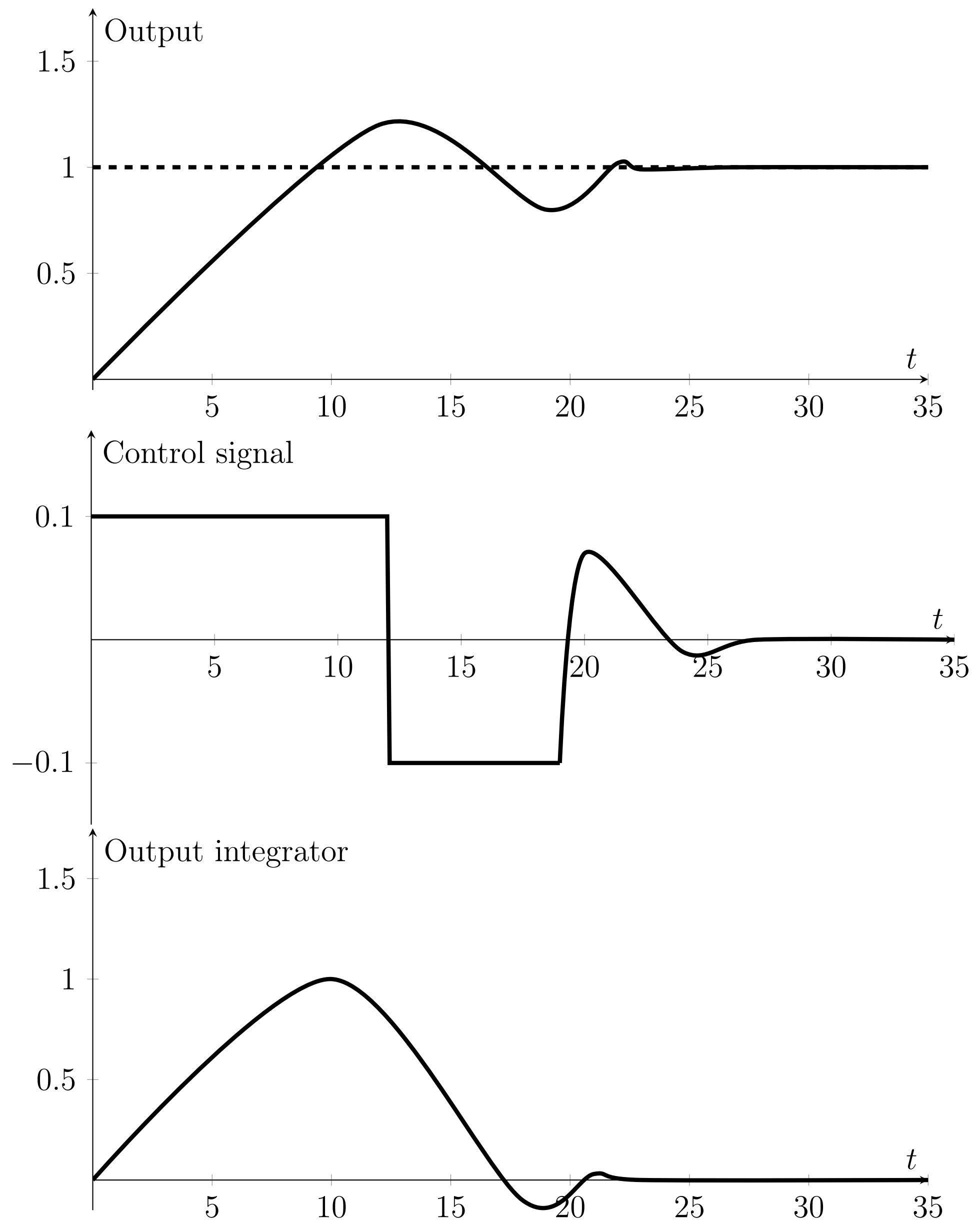

Figura 10.11. Effect of the saturation on the integral action (wind-up).

Realization of anti wind-up block

In order to avoid the wind-up problem, a system control model such as the one in figure 10.12 should be used.

Depending on the control signal $u_C(t)$ behaviour, we need to distinguish two different

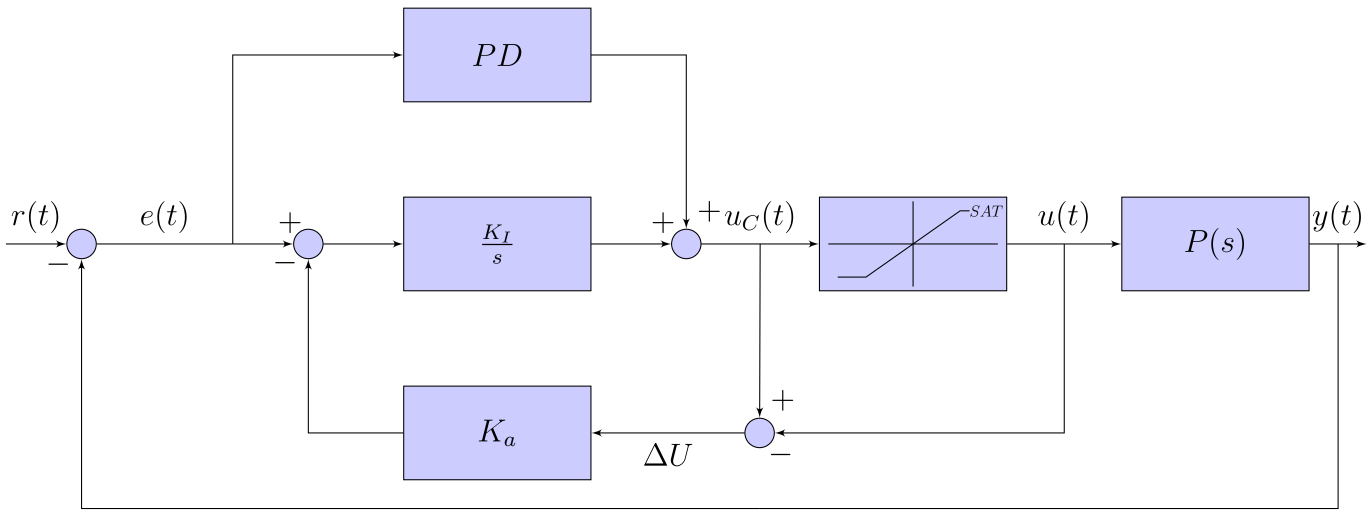

Figura 10.12. Control system model with anti wind-up.

cases:

- if $u_C(t) \leq U_{MAX}$ then $u(t) = u_C(t)$ and the system behaves as a standard PID configuration
- if $u_C(t) > U_{MAX}$ then $u(t) = U_{MAX}$ (i.e. the control signal $u(t)$ exhibits saturation problems). In this case let us consider the integral and anti wind-up parts of the controller as a first order system, obtained by unitary negative feedback of the system described by the transfer function

$$G(s) = \frac{K_a K_i}{s}$$

with U_{MAX} as input signal and $u_C(t)$ as output signal, as showed in figure 10.13 (the proportional-derivative control signal $u_{PD}(t)$ and the error $e(t)$ are treated as disturbances).
The closed-loop transfer function is

$$W(s) = \frac{K_a K_i}{s + K_a K_i}.$$

In order to satisfy the desired time-domain specifications we have to set the value of the parameter

$$T_a = \frac{1}{K_a K_i}$$

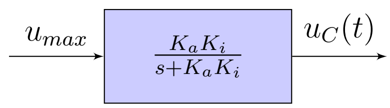

Figura 10.13. Equivalent first order system.

which is the time constant of the closed-loop created by the integrator and anti wind-up block. As a first guess we could set

$$T_a = \frac{t_s^*}{3}$$

where t_s^* is the desired settling time. Manual tuning of the gain K_a is however almost always needed in practice.