Model order reduction of large-scale state-space models in fusion machines via Krylov methods

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This work presents a robust technique, based on Krylov subspace method, for the reduction of large-scale state-space models arising in many electromagnetic applications in fusion machines. The proposed approach, built on the Arnoldi algorithm, aims at reducing the number of states of the system and lowering the computational effort, with a negligible loss of accuracy in the numerical solution. A detailed performance study is presented on an ITER-like machine, addressing both 2D and 3D problems.

***Index Terms*—model order reduction, block Krylov subspace method, Arnoldi algorithm, fusion machines.**

I. Introduction

Electromagnetic fusion devices are large-scale machines, and their analysis and design involve the modeling of numerous complex subsystems and their coupling. The accuracy constraints together with the large geometrical size of these machines typically lead, through FEM formulations (here used to denote any computational method, including both differential and integral formulations), to the definition of high-dimensional state-space systems. Such arising problems are numerically difficult to treat because of the computational burden and memory requirements, and, moreover, show the limitation of the employment of standard models within a real-time control or estimation loop. It is clear that the coupling between FEM and model order reduction (MOR) is very interesting because it would make the resolution of very complex problems not only feasible, but also more affordable [1]. This is a particular key issue in the respect of fusion devices, where an algorithm is required to be not only accurate but also fast enough to allow real time implementation.

In this framework, the computation of passive conducting structures response and their effects (e.g. their contribution to the magnetic measurements) is directly linked to the identification of plasma shape and position, so it is a crucial target to implement real time plasma control. Since the description of both the plasma and the passive structures requires a fine discretization, such problem is characterized by a high number of states that are strongly correlated. It is clear that the adoption of MOR is strongly suggested to obtain compact and manageable models [2].

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MOR procedure has been widely studied and applied during the latest decades, in particular in association with FEM formulated problems. Among the different MOR techniques, this paper presents a method via Krylov subspace projection, built on the Arnoldi algorithm, which allows to avoid numerical instabilities when computing the reduced order model, and exploits both input/output Krylov methods.

Krylov-bsed MOR is summarized in Section II and the problem formulation is presented in Section III. A performance study on both 2D and 3D problems is presented in Section IV.

II. Model order reduction (MOR)

The description of the dynamics of many electromagnetic systems in terms of a high dimensional state vector $\mathbf{x}$ of size n (whose physical meaning changes depending on the problem) leads to a linear, stable, time invariant, multi-input multi-output system (input $\mathbf{u}$ of size m, output $\mathbf{y}$ of size p):

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \tag{1}$$

Krylov-based MOR considers the representation of the transfer function of (1) as a Laurent series and matches a given number of expansion coefficients (named *moments*) between the reduced order system of size $\tilde{n}$ and the full order system of size n to guarantee a suitable approximation of the input/output behaviour. The reduced order model $\Sigma(\mathbf{A_r}, \mathbf{B_r}, \mathbf{C_r}, \mathbf{D_r})$ can be obtained by projection of the system matrices:

$$\mathbf{A_r} = \mathbf{W^T A V} \quad \mathbf{B_r} = \mathbf{W^T B} \quad \mathbf{C_r} = \mathbf{C V} \quad \mathbf{D_r} = \mathbf{D} \tag{2}$$

where the columns of either $\boldsymbol{V}$ or $\boldsymbol{W}$ span alternatively the subspaces $\mathcal{K}_{in}, \mathcal{K}_{out}$ [3] and the other matrix $\boldsymbol{W}$ ($\boldsymbol{V}$) is chosen so that $\boldsymbol{A}_r$ results to be nonsingular:

$$\mathcal{K}_{in}(\boldsymbol{A}^{-1}, \boldsymbol{A}^{-1}\boldsymbol{B}) = span\{\boldsymbol{A}^{-1}\boldsymbol{B},\ \dots,\boldsymbol{A}^{-\tilde{n}}\boldsymbol{B}\} \tag{3}$$

$$\mathcal{K}_{out}(\boldsymbol{A}^{-T}, \boldsymbol{A}^{-T}\boldsymbol{C}^T) = span\{\boldsymbol{A}^{-T}\boldsymbol{C}^T,\ \dots,(\boldsymbol{A}^{-\tilde{n}})^T\boldsymbol{C}^T\} \tag{4}$$

As can be seen from equations (3), (4) there are two different projection subspaces onto which the transfer function can be projected, named respectively the *input Krylov* and the *output Krylov*, and the choice between them depends on the information contained in the input and output spaces, such as dimension and linear dependency.

For multi-input multi-output (MIMO) systems the expansion coefficients of the transfer function are a set of $m \times p$ matrices, and the number of matching parameters results to be $mp(\tilde{n}/m) = p\tilde{n}$ for the input Krylov method and $mp(\tilde{n}/p) = m\tilde{n}$ for the output Krylov method [4].

However, in a finite precision arithmetic, the computation of a basis of Krylov subspace through (3), (4) can lead to numerical issues [6] that can be avoided using the numerically robust Arnoldi algorithm based on modified Gram-Schmidt: a set of orthonormal vectors are iteratively computed to build a basis for a given Krylov subspace.

It can be proved [4] that the input/output behaviour of the reduced model is exactly the same for any basis of input/output Krylov subspaces. However, the representation of the reduced system remains the same of the full order model only for output Krylov reduction with a fixed $\boldsymbol{V}$, because through input Krylov subspace method the transfer function of the reduced order model changes when the system representation of the original system is changed. This can be a disadvantage, because the results depend on the representation of the original system. Moreover, this reduction technique has the drawback that there are no error bounds available [5]. As a matter of fact, moment matching grants the performance of the local approximation, while nothing can be inferred about the global behavior of the reduced transfer function.

III. Problem Formulation

In this work, the described approach is applied to derive a reduced model and solve the following time domain eddy current problems:

A. A 2D realistic problem: short-time scale variation (ms) of the plasma magnetic configuration (*L-H transition*), studied with a 2D axisymmetric model, see Figure 1.
B. A 3D benchmark problem: a toroidal massive conducting structure (see Figure 1, blue box, and its 3D model in Figure 2) excited by a uniform field (B_z) at $f = 50Hz$.

A. 2D axisymmetric problem

The sketch of a fusion device with *ITER-like* cross-section is shown in Figure 1. The field sources are:

- $n_s = 14$ active conductors (cyan), vector $\mathbf{I_s}$.
- $n_p = 100$ equivalent plasma currents (green), vector $\mathbf{I_p}$.
- $n_c = 1610$ passive elements (red), vector $\mathbf{I_c}$.

A dynamic model can be written in a state space form as (1), where the states are the magnetic fluxes (Ψ_c) linked to the n_c passive elements [2]. Since two different Krylov subspaces are available (*input* and *output* subspaces), and this choice depends on the number of input and output quantities, in this section we analyze the same problem as the size of input/output changes.

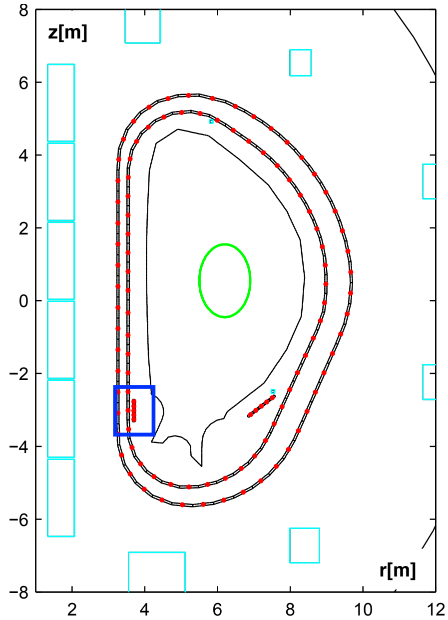

Fig. 1. ITER-like machine cross section and circuits representation.

This can be obtained by linear combinations of the input and output quantity to vary m, p while maintaining the problem basically unchanged.

Model 1: For this case $\mathbf{u} = [\mathbf{I_p^T}\ \mathbf{I_s^T}]^\mathbf{T}$ and $\mathbf{y} = \mathbf{I_c}$, so $m = 114$ and $p = n = 1610$. System (1) becomes:

$$\mathbf{A} = -\mathbf{R}_c\mathbf{M}_{cc}^{-1} \qquad \mathbf{C} = \mathbf{M}_{cc}^{-1}$$
$$\mathbf{B} = \mathbf{R}_c\mathbf{M}_{cc}^{-1}\begin{bmatrix}\mathbf{M}_{cp}\\ \mathbf{M}_{cs}\end{bmatrix} \qquad \mathbf{D} = -\mathbf{M}_{cc}^{-1}\begin{bmatrix}\mathbf{M}_{cp}\\ \mathbf{M}_{cs}\end{bmatrix}$$

where $\mathbf{R}_c$ is the passive structure resistance matrix, and $\mathbf{M}_{cc}$, $\mathbf{M}_{cp}$, $\mathbf{M}_{cs}$ are the mutual inductance matrices of the elements of the passive structure respectively with themselves, with the plasma, and with the active coils. The dimension p of the output vector depends on the discretization of the passive structure, and so it is clear that better accuracy leads to higher p. Then, since output Krylov subspace matches the first $\tilde{n}/p$ moments, and $p = n$, the minimum dimension $\tilde{n}$ of the reduced model suitable to obtain an integer number of matched moments is at least $\tilde{n} = n$, then output Krylov can't perform any reduction.

Model 2: For this case $\mathbf{u}$ is the same as for Model 1 and the output are chosen to be $\mathbf{y} = \mathbf{m_c}$, i.e. the magnetic measurements due to passive currents and provided by the pick-up coils that locally measure the flux density for a given direction, so $m = 114$ and $p = 36$. System (1) becomes:

$$\mathbf{A} = -\mathbf{R}_c\mathbf{M}_{cc}^{-1} \qquad \mathbf{C} = \mathbf{G}_c\mathbf{M}_{cc}^{-1}$$
$$\mathbf{B} = \mathbf{R}_c\mathbf{M}_{cc}^{-1}\begin{bmatrix}\mathbf{M}_{cp}\\ \mathbf{M}_{cs}\end{bmatrix} \qquad \mathbf{D} = -\mathbf{G}_c\mathbf{M}_{cc}^{-1}\begin{bmatrix}\mathbf{M}_{cp}\\ \mathbf{M}_{cs}\end{bmatrix}$$

$\mathbf{G}_c$ being the Green matrix linking the passive structure elements to the sensors. Despite it would be reasonable to expect a better accuracy from output Krylov due to the fact that $p < m$, nonetheless this will not happen because, about this specific problem, the input space has a higher dimension than the output one, but has more linear dependency. In fact, as it can be seen in the following model, the degrees of freedom are lower than the input dimension.

Model 3: For this case $\mathbf{y}$ is the same as for Model 2 and $\mathbf{u} = [\mathbf{q^T}\ \mathbf{I_s^T}]^\mathbf{T}$ where $\mathbf{q} = (\mathbf{q_0},\ \mathbf{q_1},\ \mathbf{q_2})$ are the first three moments of the toroidal current density, and so $m = 17$ and $p = 36$. System (1) becomes:

$$\mathbf{A} = -\mathbf{R}_c\mathbf{M}_{cc}^{-1} \qquad \mathbf{C} = \mathbf{G}_c\mathbf{M}_{cc}^{-1}$$
$$\mathbf{B} = \mathbf{P}\mathbf{R}_c\mathbf{M}_{cc}^{-1}\begin{bmatrix}\mathbf{M}_{cp}\\ \mathbf{M}_{cs}\end{bmatrix} \qquad \mathbf{D} = -\mathbf{P}\mathbf{G}_c\mathbf{M}_{cc}^{-1}\begin{bmatrix}\mathbf{M}_{cp}\\ \mathbf{M}_{cs}\end{bmatrix}$$

where $\mathbf{P}$ is the matrix linking $\mathbf{q}$ to an equivalent elliptic plasma current distribution placed at a fixed (*a priori* assigned) position. The exact relation $\mathbf{I_p} = \mathbf{P}\mathbf{q}$ between the equivalent plasma currents $\mathbf{I_p}$ and the first three toroidal current moments is extensively described in [7]. The degrees of freedom of the input subspace are lower than the output one. As a consequence, a better performance of input Krylov is expected.

B. 3D problem

The solution of three dimensional electromagnetic problems by means of either differential methods (e.g. FEM, DGA, FIT, etc.) or integral methods (e.g. surface integral, volume integral, PEEC, etc.) usually leads to an extremely high number of variables (DoFs), which are necessary to ensure an accurate description of the numerical domain. It is clear that finding a way to achieve low dimensional models without decreasing the accuracy would be advisable, because it would mean the possibility of solving very complex problems with limited memory and time requirements.

Here, the proposed Krylov-based MOR method is applied and validated against a 3D problem solved in the time domain by a novel Volume Integral Non COnformal formulation (VINCO) [8]. The unknowns are the circulations of the electric vector potential $\mathbf{T}$ on mesh edges and the *independent currents* $\mathbf{i}$ [9] introduced to treat the non trivial domain (the massive toroidal conducting structure shown in Figure 2). Then, the array containing the electric currents $\mathbf{I}$ on faces of the mesh is defined as

$$\mathbf{I} = \mathbf{C}(\mathbf{T} + \mathbf{H}\mathbf{i}) \qquad (5)$$

where $\mathbf{C}$ is the incidence matrix between face and edge pairs and $\mathbf{H}$ stores a set of representatives of the first cohomology group generators $H^1(\partial\mathcal{K}, \mathbb{Z})$ of the boundary of the conductor.

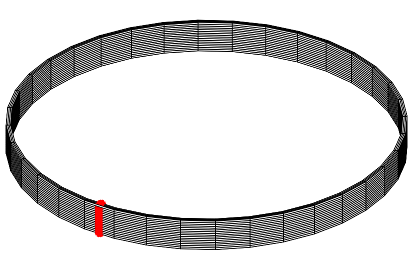

Fig. 2. Sketch of a massive toroidal conducting structure discretized into hundreds hexahedral elements.

Without going into details (see [8] for further description), the state space equation becomes:

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \text{with} \quad \mathbf{x} = \begin{bmatrix}\mathbf{T}\\ \hline \mathbf{i}\end{bmatrix} \qquad \mathbf{u} = \frac{d\tilde{\mathbf{A}}_s}{dt} \qquad (6)$$

where $\mathbf{x}$ and $\mathbf{u}$ are state and input variables, and

$$\mathbf{B} = \begin{bmatrix}\mathbf{C}^T\\ \hline (\mathbf{C}\mathbf{H})^T\end{bmatrix} \qquad \mathbf{A} = \mathbf{B}\mathbf{R}\mathbf{B}^T \qquad \mathbf{E} = \mathbf{B}\mathbf{M}\mathbf{B}^T \qquad (7)$$

where $\mathbf{R}$ and $\mathbf{M}$ are the resistance and inductance matrices, built as described in [8], and $\tilde{\mathbf{A}}_s$ is the circulation of the magnetic vector potential on dual edges due to a unit current source ($f = 50Hz$), computed by *Biot-Savart* law. Once the problem is solved, $\mathbf{I}$ is computed with (5); then the current density $\mathbf{J}$, uniform in each volume, can be easily retrieved. Lastly, the magnetic vector potential $\mathbf{a}(r)$ and the magnetic flux density $\mathbf{b}(r)$ are obtained in any field point r by an efficient implementation on GPU architecture of accurate closed-form expressions [10].

Krylov reduction has then been applied to (6) keeping the quantity $[\mathbf{T}^T\ \mathbf{i}^T]^T$ as an output, and then the resulting systems has $p = n$, the same as model 1 for the 2D case. Consequently, as we have already said before, the output Krylov reduction is not feasible because of the equality $m = n$, so only input Krylov reduction has been performed.

IV. Model validation and numerical results

A. 2D Axisymmetric problem

The validation of the described Krylov subspace method has been performed against a fast transition of the plasma from L-mode to H-mode in a ITER-like magnetic configuration. This transition concerns a sudden change of plasma position, current and internal profiles (β_p, l_i), and, consequently, strong eddy currents are induced in the passive conducting structures surrounding the plasma.

It is clear that the contribution of these eddy currents to the signals measured by magnetic sensors (e.g. pick-up coils) makes infeasible the extrapolation of plasma current information directly from the magnetic measurements, since the magnetic field is no longer produced by the plasma current only (active coils contribution is always supposed as known): in this situation an accurate evaluation of this eddy current is a key problem in order to reach the best possible knowledge of plasma measurements.

The experimental results show that for this specific problem output Krylov is very less suitable than input Krylov to perform the reduction, as can be shown in Table I. As also mentioned before, the output Krylov approach is even useless to the respect of Model 1, because the number of output is equal to the number of the states, and consequently it would be necessary to consider a reduced order model that is the full-order model itself. In addition to this, it seems that output space vectors contain less information than the input ones, as it can be appreciated in Table I. The input Krylov results are shown in Figure 3, in terms of time evolution of two different output quantities, together with the respective errors. As for the computational effort, this technique leads to a reduction of the simulation time of a factor 2 for *Model 1*, 3 times for *Model 2*, and around 30 times for *Model 3*.

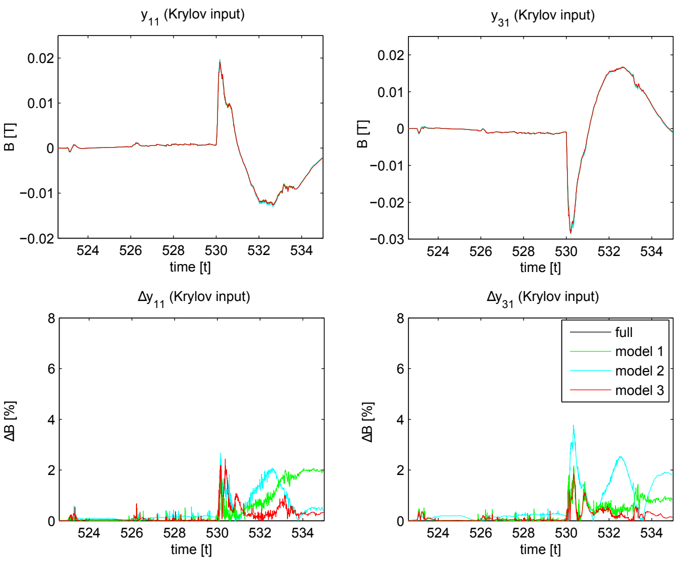

Fig. 3. Time evolution of two sample measurements output (m_{11} and m_{31}) obtained using Krylov input algorithm, with the respective percentage errors.

TABLE I
TIME AVERAGE OF MEAN ERROR $\bar{\bar{\epsilon}}$ AND MEAN STANDARD DEVIATION $\bar{\sigma}$

	input Krylov			output Krylov		
	$\tilde{n}$	$\bar{\bar{\epsilon}}$	$\bar{\sigma}$	$\tilde{n}$	$\bar{\bar{\epsilon}}$	$\bar{\sigma}$
case 1	70	1.09%	2.00%	–	–	–
case 2	50	0.42%	0.47%	50	3.55%	4.01%
case 3	20	0.54%	0.54%	50	3.53%	3.96%

B. 3D problem

Krylov reduction technique has then been tested against the 3D electromagnetic problem described in the previous section. It is clear that this problem is more challenging, from a computational point of view. As a matter of fact, the proposed approach enables the solution of extremely complex problems with reduced computational effort. Moreover, an accurate and compact model would be extremely useful for Real Time (RT) applications in Magnetic Confinement Fusion (MCF).

The 3D problem test can be set into this framework, considering that the structure represented in Figure 2 mimics one of the two conducting structures placed inside the vacuum vessel of the *ITER-like* machine (see Figure 1, blue box) with the aim of stabilizing (slowing down) possible plasma Vertical Displacement Events (VDEs) with thus a direct impact on RT-control for MCF.

The results presented in Figures 4 show that Krylov subspace can ensure high accuracy on current density $\mathbf{J}$ through element section reducing the system from $n = 2773$ to $\tilde{n} = 30$: for this particular case the error is less than 0.1%, and the computational time has been reduced by a factor of 70.

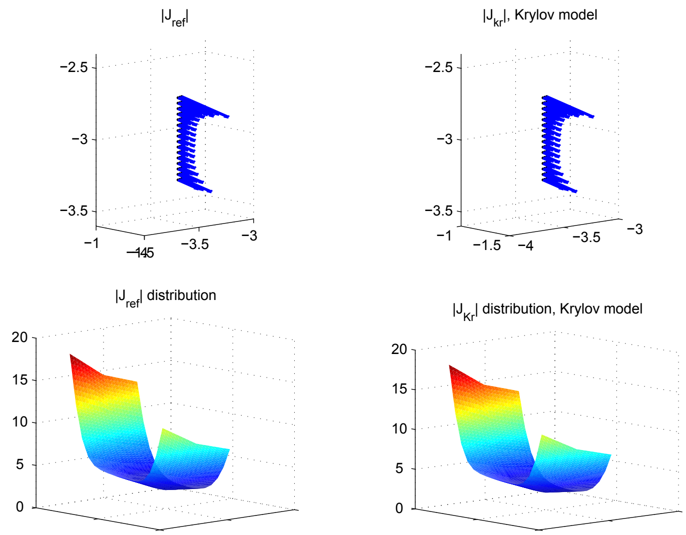

Fig. 4. Comparison between full model ($n = 2773$) output and Krylov-reduced one with $\tilde{n} = 30$. Upper plots show the current density vector field $\vec{J}$ through the section of the vertical structure, whereas the bottom figures show the amplitude $|\vec{J}|$ above the section.

This high accuracy can be explained taking into account that the source is a sinusoidal function at $f = 50Hz$, and a small number of expansion coefficients are needed to match the behavior of the system at such low frequency, and so a small $\tilde{n}$ is required. On the other hand, to simulate a real plasma scenario, a larger value for $\tilde{n}$ would be required, because of the broader frequency spectrum involved in such a problem.

V. CONCLUSIONS

It has been shown that Krylov subspace method is an effective tool to deal with both 2D and 3D electromagnetic formulations, by preserving desirable properties of the full models such as stability and passivity. Moreover, since time-domain formulation together with Krylov method could be very affordable, the entire transient study of the phenomenon can be performed retaining a fairly good accuracy with limited memory and time requirements.

VI. ACKNOWLEDGMENT

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