Krylov Subspace Methods for Model Order Reduction in Computational Electromagnetics ??

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Abstract: This paper presents a model order reduction method via Krylov subspace projection, for applications in the field of computational electromagnetics (CEM). The approach results to be suitable both for SISO and MIMO systems, and is based on the numerically robust Arnoldi procedure. We have studied the model order reduction as the number of inputs and outputs changes, to better understand the behavior of the reduction technique. Relevant CEM examples related to the reduction of finite element method models are presented to validate this methodology, both in the 2D and in the 3D case.

Keywords: Model order reduction (MOR), Krylov subspace method, Arnoldi algorithm, Computational Electromagnetics.

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1. INTRODUCTION

In the last decade, computational modeling and simulation has grown as a proper discipline and can be nowadays considered an approach for both the analysis and the synthesis of physical systems that complements theory and experiment. Computer simulations are now performed routinely for several kinds of processes, and, in particular, numerical simulation plays a fundamental role in the study of the complex dynamical phenomena in Electromagnetism (see for example, **?**).

In this framework, the model description often comes out from a Finite Element Method (FEM) formulation either in 2D or 3D (**??**), where the input/output/state variables describe the physical relations in the mesh elements of the geometrical discretization of the problem domain (e.g. active and passive structures). The high complexity and level of detail of such a description might be necessary to ensure a certain precision but this directly translates into a high-dimensional state-space representation that can be numerically difficult to treat, because of a high computational and memory cost.

To cope with this problem, the fundamental idea is to derive models of reduced order, capable of giving an accurate description of the real system and at the same time allowing to simplify the design of the controller (**?????**). Following this procedure, the system behavior can be described only considering a small group of dominant states, and the main issue is to identify this set to obtain a lower order model that matches (or better approximates) the full order model behavior.

This paper presents a Model Order Reduction (MOR) method via Krylov subspace projection, particularly suitable for strong order reduction of FEM models. The approach is based on the Laurent series expansion of the transfer function of the full order model Σ, to obtain a reduced order model Σ_q that matches the first q expansion coefficients of the original transfer function, so-called *moments* (**?**). In order to avoid numerical problem when reducing Σ to Σ_q the classical Arnoldi algorithm is used. The features of this MOR procedure in terms of system behavior are discussed both for Single Input Single Output (SISO) and Multi Input Multi Output (MIMO) systems, and with relevant examples related to 2D and 3D FEM models.

2. PRINCIPLES OF MOR

2.1 System representation

Let $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be a n-th order continuous-time, MIMO, linear, time-invariant state space model with m inputs and p outputs:

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \tag{1}$$

with state $\mathbf{x}(t) \in \mathbb{R}^n$, input vector $\mathbf{u}(t) \in \mathbb{R}^m$, output vector $\mathbf{y}(t) \in \mathbb{R}^p$, and where $\mathbf{E} \in \mathbb{R}^{n\times n}$, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, $\mathbf{D} \in \mathbb{R}^{p\times m}$.

The transfer function of the system is:

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{2}$$

relating the inputs to the outputs by $\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s)$.

If the number of states n is large (from hundreds to tens of thousands as in FEM models), a computer simulation of the system can be costly (and often unfeasible) in terms of CPU time and memory used. Therefore, when dealing with such high-order models, it is mandatory to try and derive approximated models Σ_q of order $q \ll n$:

$$\begin{cases} \mathbf{E}_q\dot{\mathbf{x}}_q(t) = \mathbf{A}_q\mathbf{x}_q(t) + \mathbf{B}_q\mathbf{u}(t) \\ \mathbf{y}_q(t) = \mathbf{C}_q\mathbf{x}_q(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \tag{3}$$

where $\mathbf{E}_q \in \mathbb{R}^{q\times q}$, $\mathbf{A}_q \in \mathbb{R}^{q\times q}$, $\mathbf{B}_q \in \mathbb{R}^{q\times m}$, $\mathbf{C}_q \in \mathbb{R}^{p\times q}$. In a MOR by projection the system matrices take the form:

$$\mathbf{E}_q = \mathbf{W}^T\mathbf{E}\mathbf{V} \qquad \mathbf{A}_q = \mathbf{W}^T\mathbf{A}\mathbf{V} \tag{4}$$

$$\mathbf{B}_q = \mathbf{W}^T\mathbf{B} \qquad \mathbf{C}_q = \mathbf{C}\mathbf{V} \tag{5}$$

2.2 Moments of the transfer function

The reduction of Σ to Σ_q is expressed as a relationship between the transfer function of the full order system and that of the reduced one. Set $s = \sigma + s_i$ and expand (**??**) in Laurent series around s_i to obtain:

$$(s\mathbf{E} - \mathbf{A})^{-1} = [\sigma\mathbf{E} + (s_i\mathbf{E} - A)]^{-1} \tag{6}$$

$$= [\sigma(s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{E} + \mathbf{I}]^{-1}(s_i\mathbf{E} - \mathbf{A})^{-1} \tag{7}$$

$$= \sum_{k=0}^{\infty}(-1)^k\sigma^k[(s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}]^k(s_i\mathbf{E} - \mathbf{A})^{-1} \tag{8}$$

so (**??**) can be written as:

$$\mathbf{H}(\sigma) = \sum_{k=0}^{\infty} \mathbf{M}_k(s_i)\sigma^k \tag{9}$$

the term $\mathbf{M}_k$ is a $p \times m$ matrix, and is called *moment* of order k of the transfer function:

$$\mathbf{M}_k(s_i) = (-1)^k \mathbf{C}[(s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}]^k (s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \tag{10}$$

This means that, using this technique, we can choose both the accuracy (with a suitable dimension q) and the frequency of the transfer function where moment matching is required. For the most common case of steady state matching ($s_i = 0$), we get:

$$\mathbf{M}_k(0) = (-1)^k \mathbf{C}[-\mathbf{A}^{-1}\mathbf{E}]^k (-\mathbf{A})^{-1}\mathbf{B} \tag{11}$$

$$= \; -\mathbf{C}[\mathbf{A}^{-1}\mathbf{E}]^k \mathbf{A}^{-1}\mathbf{B} \tag{12}$$

Then the q-order approximated transfer function results:

$$\mathbf{H}_q(\sigma) = \sum_{k=1}^{q} \mathbf{M}_k(s_i)\sigma^k = \mathbf{C}_q(\sigma\mathbf{E}_q - \mathbf{A}_q)^{-1}\mathbf{B}_q \tag{13}$$

This is a general result that can be obtained with a Laurent expansion of the transfer function, and is not referred to more specific reduction technique. Next section will describe how to build the matrices of the reduced system Σ_q.

3. STANDARD BLOCK KRYLOV SUBSPACES METHOD

3.1 Block Krylov subspaces and properties

Let $\mathbf{x} = \mathbf{V}\mathbf{x}_q$ be a change of variable and the matrix $\mathbf{W}$ a suitable matrix that will be described in the following part of the section. With these positions, a reduced order model can be found as:

$$\begin{cases} \mathbf{W}^T\mathbf{E}_q\mathbf{V}\dot{\mathbf{x}}_q(t) = \mathbf{W}^T\mathbf{A}_q\mathbf{V}\mathbf{x}_q(t) + \mathbf{W}^T\mathbf{B}_q\mathbf{u}(t) \\ \mathbf{y}_q(t) = \mathbf{C}_q\mathbf{V}\mathbf{x}_q(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \tag{14}$$

and it can be proved that, if $\mathbf{V}$, $\mathbf{W}$ are the basis of a block Krylov subspace referred to the full order system Σ, the moment matching up to the order q is reached.

The *block Krylov subspace* is defined as follows. Let $\mathbf{F} \in \mathbb{R}^{n\times n}$, $\mathbf{G} \in \mathbb{R}^{n\times m}$ be two matrices, then the block Krylov subspace $\mathcal{K}_q(\mathbf{F}, \mathbf{G})$ is defined as

$$\mathcal{K}_q(\mathbf{F}, \mathbf{G}) = colspan\{\mathbf{G}, \mathbf{F}\mathbf{G}, \mathbf{F}^2\mathbf{G}, ..., \mathbf{F}^{q-1}\mathbf{G}\} \tag{15}$$

In particular, we choose the two matrices above such that

$$\mathbf{F} = (s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{E} \qquad \mathbf{G} = (s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \tag{16}$$

and we write the zero order moment

$$\mathbf{M}_{q,0} = -\mathbf{C}_q\mathbf{A}_q^{-1}\mathbf{B}_q = -\mathbf{C}\mathbf{V}(\mathbf{W}^T\mathbf{A}_q\mathbf{V})^{-1}\mathbf{W}^T\mathbf{B}_q \tag{17}$$

It can be proved that for a particular matrix $\mathbf{V}$ basis of $\mathcal{K}_{q1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B})$ and $\mathbf{W}$ chosen such that $\mathbf{A}_q$ is non-singular, the zero order moment of the reduced model $m_{q,0}$ matches the moment of the full model m_0.

The generalization of this result for a general order q is summarized in the following theorems, which give an idea about one of the main strength of this reduction technique: namely, two choices of the matrices $\mathbf{V}$, $\mathbf{W}$ are available starting from two different Krylov subspaces $\mathcal{K}_{q1}$, $\mathcal{K}_{q2}$ that depend on particular features of the input and output spaces.

Theorem 1. If the matrix $\mathbf{V}$ used in (**??**) is a basis of Krylov subspace $\mathcal{K}_{q1}((s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}, (s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{B})$ with rank q and and $\mathbf{W}$ is chosen such that the matrix $\mathbf{A}_q$ is non-singular, then the first q/m moments (around s_i) of the original and reduced order systems match.

Theorem 2. If the matrix $\mathbf{W}$ used in (**??**) is a basis of Krylov subspace $\mathcal{K}_{q2}((s_i\mathbf{E} - \mathbf{A})^{-T}\mathbf{E}, (s_i\mathbf{E} - \mathbf{A})^{-T}\mathbf{C}^T)$ with rank q and and $\mathbf{V}$ is chosen such that the matrix $\mathbf{A}_q$ is non-singular, then the first q/p moments (around s_i) of the original and reduced order systems match.

The proof of Theorem **??** (respectively **??**) can be obtained by writing moments (**??**) as linear combinations of the columns of $\mathbf{V}$ (respectively $\mathbf{W}$) basis of subspace $\mathcal{K}_{q1}$ (respectively $\mathcal{K}_{q2}$). For further details see **?**. These theorems show that the presented approach is able to solve MOR in a flexible way, meaning that we can tackle different order reduction problems with the most suitable subspace. In addition to this, it follows directly from Theorems **??**–**??** that moment matching is obtained for *any* basis of input or output Krylov subspaces used for order reduction. Depending on the choice of $\mathbf{V}, \mathbf{W}$ we will say respectively *input-Krylov* and *output-Krylov subspace.*

Another important property can be shown by comparing two different reduced models with $\mathbf{V}_1$, $\mathbf{W}_1$ and $\mathbf{V}_2$, $\mathbf{W}_2$, both couple of matrices satisfying theorems **??**, **??**:

$$\begin{cases} \mathbf{W}_1^T\mathbf{E}_q\mathbf{V}_1\dot{\mathbf{x}}_q(t) = \mathbf{W}_1^T\mathbf{A}_q\mathbf{V}_1\mathbf{x}_q(t) + \mathbf{W}_1^T\mathbf{B}_q\mathbf{u}(t) \\ \mathbf{y}_q(t) = \mathbf{C}_q\mathbf{V}_1\mathbf{x}_q(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \tag{18}$$

$$\begin{cases} \mathbf{W}_2^T\mathbf{E}_q\mathbf{V}_2\dot{\mathbf{x}}_q(t) = \mathbf{W}_2^T\mathbf{A}_q\mathbf{V}_2\mathbf{x}_q(t) + \mathbf{W}_2^T\mathbf{B}_q\mathbf{u}(t) \\ \mathbf{y}_q(t) = \mathbf{C}_q\mathbf{V}_2\mathbf{x}_q(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \tag{19}$$

since $\mathbf{V}_1$, $\mathbf{W}_1$ can be written as a linear combination of $\mathbf{V}_2$, $\mathbf{W}_2$ (and vice versa) it is clear that (**??**) and (**??**) have the same transfer function, and so they yield the same realization.

On the other hand, this reduction technique has the drawback that there are no error bounds known (**?**). This happens because with moment matching we know the approximation quality only locally in the points used, and nothing can be said about the global behavior of the reduced transfer function.

Remark 1. Theorems **??**–**??** are valid for MIMO systems, where each moment is a $m \times p$ matrix, thereby, the number of matching scalar characteristic parameters is $mp(q/m) = pq$ for input Krylov subspace and $mp(q/p) = mq$ for output Krylov subspace. On the other hand, for SISO systems each moment is a scalar quantity, and the number of matched moments is q both for input and output Krylov subspaces.

3.2 Arnoldi algorithm

There are numerical limitations to the construction of the bases $\mathbf{V}$ and $\mathbf{W}$, because the simple computation of the following blocks

$$[\mathbf{G}, \mathbf{F}\mathbf{G}, \mathbf{F}^2\mathbf{G}, ..., \mathbf{F}^{q-1}\mathbf{G}] \tag{20}$$

to generate the bases of $\mathcal{K}_{q1}$, $\mathcal{K}_{q2}$ (Theorems **??**–**??**) from definition (**??**) is not a viable numerical procedure. The reason is that, in a finite precision arithmetic, as the power $\mathbf{F}^i\mathbf{G}$ increases, the blocks in (**??**) quickly converge to the dominant eigenvector of $\mathbf{F}$ (**?**). It is required instead to employ a suitable subspace generation algorithm that build up numerically better basis vectors for the subspaces associated to the matrix $\mathcal{K}_q$.

To derive the Arnoldi algorithm we start from (**??**) to write:

$$\mathbf{F}\mathcal{K}_q = [\mathbf{F}\mathbf{G}, \mathbf{F}^2\mathbf{G}, ..., \mathbf{F}^q\mathbf{G}] = \mathcal{K}_q\mathbf{H} \tag{21}$$

where $\mathbf{H} = [\mathbf{e}_2, \mathbf{e}_3, ..., \mathcal{K}_q^{-1}\mathbf{F}^q\mathbf{G}]$ is a upper-Hessenberg matrix (this is the reduction of $\mathbf{F}$ to a Hessenberg form $\mathbf{F}$ to $\mathbf{F} = \mathcal{K}_q\mathbf{H}\mathcal{K}_q^T$). Let now consider only a part of the system (**??**), with the notation $\mathcal{K}_i = [\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_i]$, $i < q$, it follows that:

$$\mathbf{F}\mathcal{K}_i = \mathbf{k}_{i+1}\mathbf{H}_i \tag{22}$$

than, the $i-th$ column of (**??**) gives:

$$\mathbf{F}\mathbf{k}_i = h_{1,i}\mathbf{k}_1 + \ \dots \ + h_{i+1,1}\mathbf{k}_{i+1} \tag{23}$$

$$\Rightarrow \ \mathbf{k}_{i+1} = \frac{\mathbf{F}\mathbf{k}_i - \sum_{j=1}^{i} h_{j,i}\mathbf{k}_i}{h_{i+1,i}} \tag{24}$$

that is an iterative formulation (Arnoldi algorithm) to compute $\mathbf{k}_{i+1}$ column of the matrix $\mathcal{K}_q$ up to $i = q$.

In other words, Arnoldi iteration is nothing but the orthogonal projection of $\mathbf{F}$ onto $\mathcal{K}_q$, with the vectors $[\mathbf{k}_1, ..., \mathbf{k}_q]$ as basis, and so it can be used to find a set of q ortonormal vectors to have the basis $\mathbf{V}$, $\mathbf{W}$ of Theorems **??**–**??**.

The presented algorithm is able to work with more than one starting vector (MIMO systems), and is called *block* Arnoldi (see Algorithm **??**). In the reported pseudocode, the notation $\mathbf{G}^{(i-m)}$ means the $(i-m)$-th column of the matrix $\mathbf{M}$.

It is possible to compute the vectors $[\mathbf{k}_1, ..., \mathbf{k}_q]$ also using the QR decomposition,

Remark 2. The expansive step in Krylov methods is the operation $\hat{\mathbf{v}} = \mathbf{A}_1\mathbf{V}^{i-m}$, that requires to solve two $O(N^2)$ operations:

$$\hat{v}_h = \sum_{k=1}^{N} a_{h,k}v_k$$

It is possible to reduce the computational cost to $O(N)$ at the expense of a small error tolerance ($\sim 10^{-6}$) using *fast Gauss transform* (FGT) algorithms (**?**).

4. MODEL VALIDATION

4.1 Problem formulation - 2D case

Krylov MOR techniques for CEM applications have been first validated on the state space system representing the axisymmetric coil-plate configuration, whose 2D cross section is shown in Fig. **??**. The coil section (black in Fig. **??**)

Algorithm 1 Block Arnoldi with Modified Gram-Schmidt

procedure ARNOLDI
  $\widehat{\mathbf{G}} \leftarrow \text{licols}(\mathbf{G})$ ▷ Remove linearly dependent cols.
  $\mathbf{b}_1 \leftarrow \widehat{\mathbf{G}}^{(1)}$ ▷ Starting vector
  $\mathbf{v}_1 \leftarrow \mathbf{b}_1/\|\mathbf{b}_1\|_2$
  $\mathbf{V}^{(1)} \leftarrow \mathbf{v}_1$
  $m \leftarrow \text{size}(\mathbf{G}, 2)$
  for $i \leftarrow 2, q$ **do**
    if $i \le m$ **then**
      $\widehat{\mathbf{v}}_i = \widehat{\mathbf{G}}^{(i)}$
    else
      $\widehat{\mathbf{v}}_i = \mathbf{F}\widehat{\mathbf{G}}^{(i-m)}$
    end if
    for $j \leftarrow 1, i-1$ **do**
      $h = \mathbf{V}^{(i)T}\widehat{\mathbf{v}}_j$
      $\widehat{\mathbf{v}}_i = \widehat{\mathbf{v}}_i - h\mathbf{v}_j$ ▷ Orthogonalization
    end for
    $\mathbf{V}^{(i)} = \widehat{\mathbf{v}}_i/\|\widehat{\mathbf{v}}_i\|_2$ ▷ Normalization
  end for
end procedure

is described as $n_s = 120$ parallel conductors, and the plate section (gray in figure **??**) is described as $n_c = 480$ parallel conductors, making a total of $n = 600$ elements.

The state space system of the problem results to be stable (since it is composed of passive elements) and can be formulated either as a SISO or a MIMO:

input since n_c conductors are short-circuited, the input quantity is the voltage $\mathbf{v}_s$ of every conductor of the source coil, and can be set arbitrarily, so we can write the input quantity as $\mathbf{v} = [\mathbf{v}_s\ 0]$. Then the dimension m can be set $m = 1$ by considering all the parallel source conductors at the same voltage, or $m = n_s = 120$ by considering every conductor at a different voltage.

output the output quantity is the magnetic flux measured by an arbitrary set of sensors placed below the conducting plate. Since the position of the probes is arbitrary, two different cases has been investigate, first using a single sensor

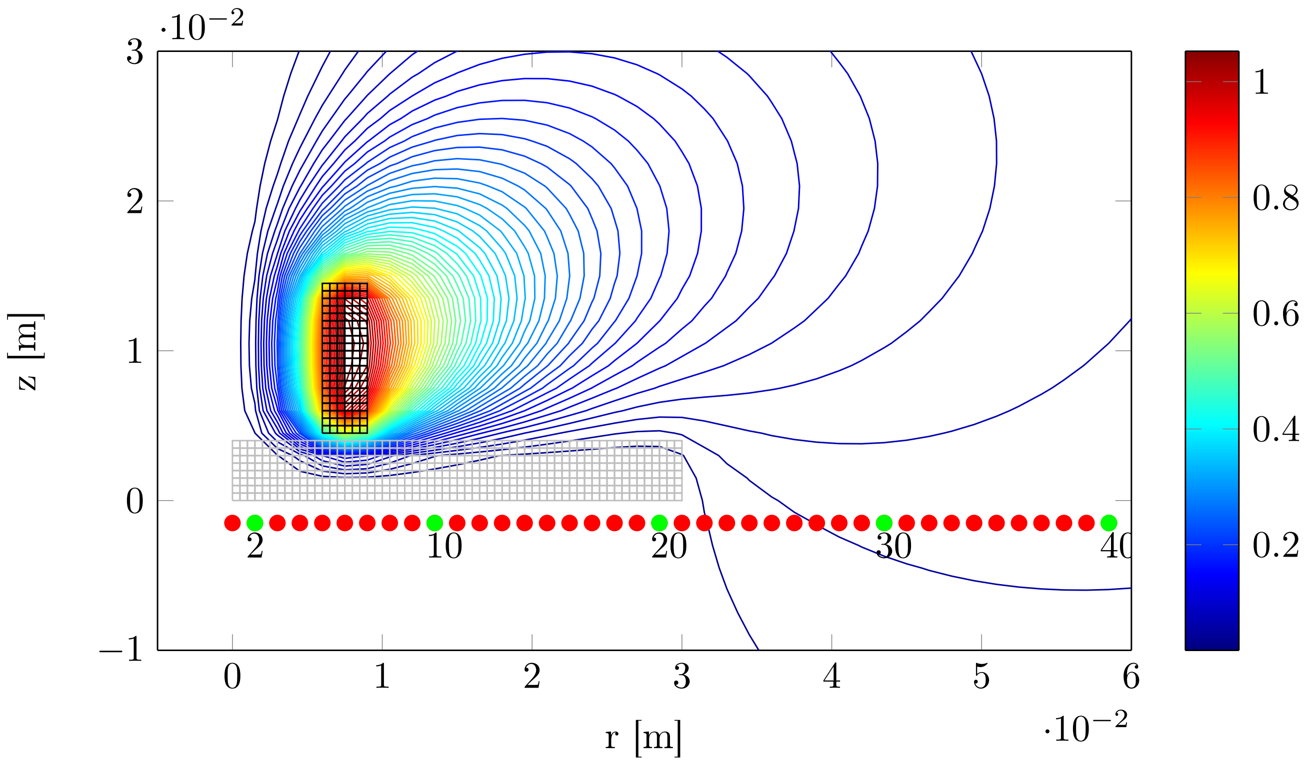

Fig. 1. Radial section of axisymmetric coil-plate problem.

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to have $p = 1$ (one of the sensors in figure ??), and then using a multiple set of sensors (aligned sensors shown as red/green dots in Fig. ??).

Both SISO and MIMO systems behavior w.r.t. the MOR technique are hereafter presented and analyzed.

The problem can be cast as follows. The voltage equation for a general set of conductors can be written as

$$\dot{\mathbf{\Psi}}(t) + \mathbf{RI}(t) = \mathbf{v}(t) \tag{25}$$

where $\mathbf{\Psi}$ is the magnetic flux linked by the elements of the plate. The first n_s equations refer to the source coil, while the remaining n_c equations refer to the conducting plate: then $\mathbf{\Psi} = [\mathbf{\Psi}_s\ \mathbf{\Psi}_c]$, $\mathbf{I} = [\mathbf{I}_s\ \mathbf{I}_c]$, and:

$$\mathbf{R} = \left[\begin{array}{c|c} \mathbf{R}_s & 0_{ns\times nc} \\ \hline 0_{nc\times ns} & \mathbf{R}_c \end{array}\right]$$

We can write equation (??) taking current $\mathbf{I}$ as the state vector, and we obtain:

$$\mathbf{L}\dot{\mathbf{I}}(t) + \mathbf{RI}(t) = \mathbf{v}(t) \tag{26}$$

and the output equation gives the magnetic flux $\mathbf{\Psi}_p$ measured by a probe is $\mathbf{\Psi}_p = \mathbf{G}_p\mathbf{I}$, where $\mathbf{G}_p = [\mathbf{G}_{sp}\ \mathbf{G}_{cp}]$ is the Green matrix that gives the magnetic flux measured by the sensor in axisymmetric configuration (?, ?). The state space model can hence be written as

$$\begin{cases} \dot{\mathbf{I}} & = \mathbf{AI} + \mathbf{Bv} \\ \mathbf{\Psi} & = \mathbf{CI} \end{cases} \tag{27}$$

where $\mathbf{A} = -\mathbf{RL}^{-1}$, $\mathbf{C} = \mathbf{G}_p$ and

$$\mathbf{B}_{\text{SISO}} = \mathbf{L}^{-1}\left[\begin{array}{c} \mathbf{I}_{ns\times 1} \\ \hline 0_{nc\times 1} \end{array}\right] \qquad \text{for the SISO case} \tag{28}$$

$$\mathbf{B}_{\text{MIMO}} = \mathbf{L}^{-1}\left[\begin{array}{c} \mathbf{I}_{ns\times ns} \\ \hline 0_{nc\times ns} \end{array}\right] \qquad \text{for the MIMO case.} \tag{29}$$

Remark 3. Since this a reduction by projection method, the physical meaning of the states is lost. This can be a problem in such models where the physical meaning is important. If it is necessary to know the behaviour of one or some original states, the solution to avoid this drawback can be obtained recalling that $\mathbf{x}_q = \mathbf{V}^T\mathbf{x}$, and then $\mathbf{x} = \mathbf{V}\mathbf{x}_q$. This allows to have the entire evolution of the original set of states, but it can be done only *a posteriori*.

4.2 Problem formulation - 3D case

The aim of this section is to verify the behavior of the Krylov-based approach with a more complex case of 3D-FEM. 3D-FEM employment is always more frequent, and most of the time the main problem related to this technique is the extremely high number of variables, which are necessary to ensure an accurate description of the domain. It is clear that to try and find a way to achieve low dimensional models without decreasing the accuracy would be advisable, because it would mean the possibility of solving very complex problems with a small amount of resources.

To this respect, we have tested Krylov method against a 3D volume integral formulation, to solve eddy current problem on massive conductors discretized with polyhedral meshes: this integral approach has several advantages, in particular it allows a fast and robust mesh generation and easier techniques for adaptive mesh refinement and de-refinement, as well as non overlapping domain decomposition with non-matching grids.

The considered problem is the evaluation of the steady state eddy currents induced inside massive copper sphere by a pulsed magnetic field at $f = 50$Hz (see Fig. ??). Without going into details (see ? for further description), the state space equation of (??) becomes:

$$\mathbf{C}^T\mathbf{MC}\dot{\mathbf{T}} = -\mathbf{C}^T\mathbf{RCT} - \mathbf{C}^T\dot{\mathbf{A}}_s \tag{30}$$

where $\mathbf{T}$ is the electric vector potential on mesh edges, $\mathbf{R}$ and $\mathbf{M}$ are the resistance and inductance matrices, $\mathbf{C}$ is the incidence matrix between faces and edge pairs and $\mathbf{A}_s$ is the circulation of magnetic vector potential on dual edges due to the source currents, and the chosen input quantity is $\mathbf{A}_s = \sin(\omega t)$ with $f = 50$Hz oscillation. Once the electric vector potential $\mathbf{T}$ is known, the current crossing a dual face comes from Ampere's law and can be written as

$$\mathbf{I} = \mathbf{CT} \tag{31}$$

the related current density $\mathbf{J}$ results (see ?):

$$\mathbf{J} = \frac{1}{|v|}\sum_{i=1}^{R}(I_s)_i^r\tilde{r}_i \tag{32}$$

We underline that since the current is supposed to be perfectly parallel to the coil path, the current across the lateral faces of the volume is zero.

Krylov reduction has then been applied to (??) keeping the quantity $\mathbf{T}$ as an output, and then the resulting systems has $p = n$, the same as model 1 for the 2D case. Consequently, as we have already said before, the output Krylov reduction

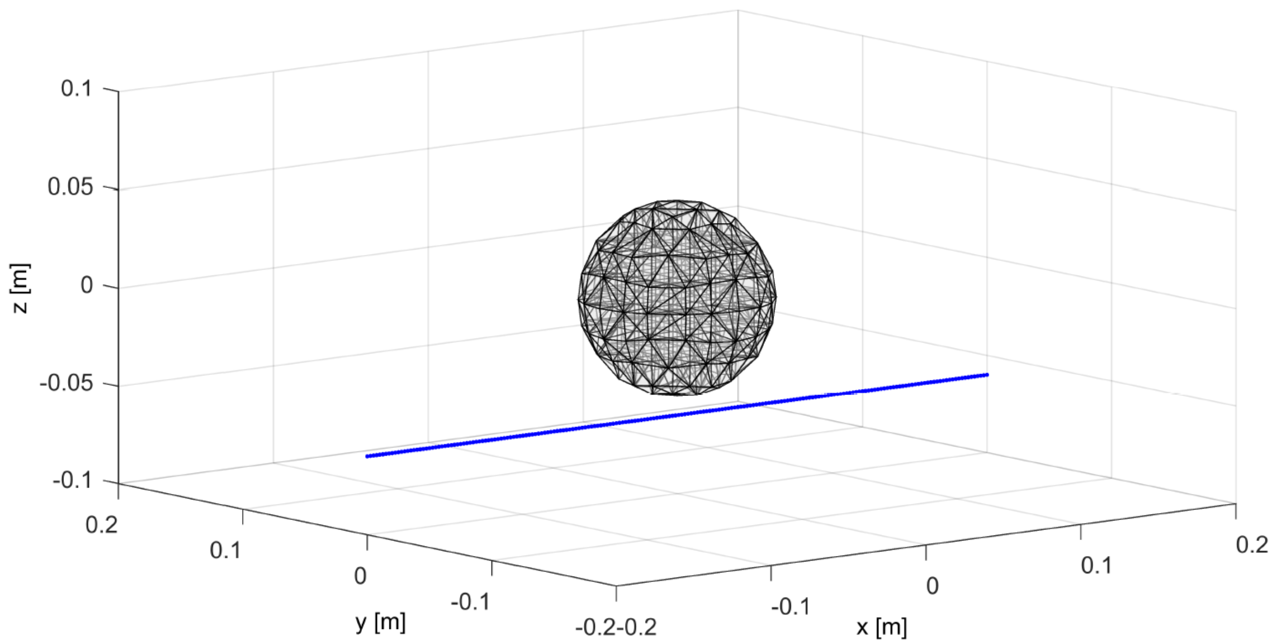

Fig. 2. Overview of the 3D-FEM model of the sphere and the array of sensors (blue line).

is not feasible because of the equality $m = n$, so only input Krylov reduction has been performed. When the system reaches the steady state condition the current density $\mathbf{J}$ has been computed as complex quantity and then compared to the reference one arising from the full order model.

4.3 Results and discussion

2D SISO case The first numerical example refers to a SISO system, as described in paragraph **??**. The voltage $\mathbf{v}_s$ of every parallel conductor of the source coil is the unitary step function $u(t) = H(t)$, and so $m = 1$.

The comparison between full-order and reduced-order model results is shown in Fig. **??**. Both Krylov input and Krylov output works with the same precision, and the number of states can be strongly reduced from $n = 600$ respectively to $q_{in} = 6,\ q_{out} = 5$. The comparison between the transfer functions $\mathbf{H}$ and $\mathbf{H}_q$ amplitude in both cases is shown in Fig. **??**.

2D MIMO case In order to obtain a MIMO system out of model **??** we set different values of the voltage of the coil conductors. The input vector is then $\mathbf{u}(t) = [U_1 \sin(\omega t)\ ...\ U_{ns} \sin(\omega t)] = \mathbf{U} \sin(\omega t)$, with randomly chosen maximum amplitude vector $\mathbf{U}$ to obtain $m = 120$. The output quantity are chosen to be the flux measured by $p = 40$ sensor placed as shown in Fig. **??**.

The time evolution of some output (green sensors in Fig. **??**) for Input Krylov are reported in Fig. **??**, where we can see the full order response (solid lines) compared with the reduced order response (star marker lines). As we can easily understand from the figure, the two responses are very close one to the other.

The comparison between input and output Krylov results trough percentage error and standard deviation is shown in Tab. **??**. We can see, for this problem, that input Krylov subspace projection is more suitable than output Krylov. We can read this result by assuming that the input quantities are more linearly dependent than the output ones, and there are less degrees of freedom, so the related input subspace can lead to a good description even if it is smaller than the output one.

	Krylov input ($q = 10$)		Krylov output ($q = 55$)	
	ϵ	σ	ϵ	σ
y_2	0.032%	0.021%	7.39%	3.85%
y_{10}	0.023%	0.015%	3.88%	2.06%
y_{20}	0.012%	0.012%	0.76%	0.44%
y_{30}	0.009%	0.007%	0.54%	0.26%
y_{40}	0.009%	0.007%	0.61%	0.30%

Table 1. MIMO case. Comparison between input and output Krylov methods.

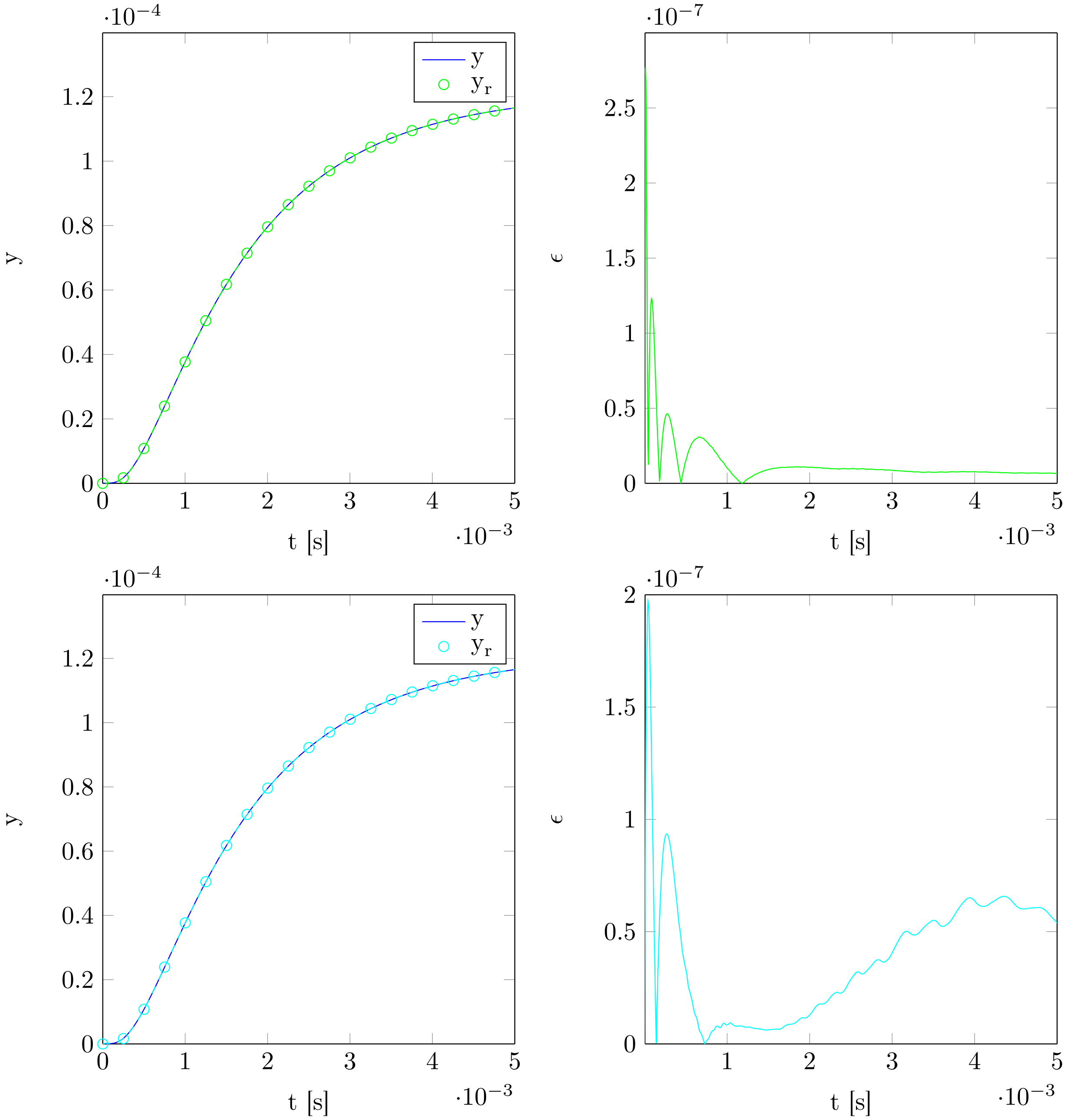

Fig. 3. 2D SISO case. Output and error with Krylov input $q = 6$ (top) and Krylov output $q = 5$ (bottom).

3D MIMO case For the 3D MIMO case, Fig. **??** shows the eddy currents induced inside the sphere volume. The arrows represent the current density vector field, respectively its real and immaginary part (not to scale). Starting from a number of states $n = 549$ for the full system Σ, the dimension has been decreased to $q = 50$ for Σ_q.

To better appreciate the effects of the MOR, a comparison between the full model and the reduced one in Fig. **??** it is represented the real and the imaginary parts of the magnetic fields as measured by a fixed set of sensors (see Fig. **??**). It can be appreciated that the reduction procedure does not affect to much the amplitude of the magnetic field, even if the phase seems to be less accurate.

5. CONCLUSIONS

In this paper a method to reduce the order of a state space model that is based upon Krylov subspaces projection is presented, with a focus on electromagnetic applications. This method has been validated against both SISO and MIMO systems, and it has proved to be effective in strongly reducing the dimension of both state space models.

In particular, for the considered SISO system this technique is able to reduce the order from $n = 600$ to $q = 5$ with an error less than 0.2%. The reduction of MIMO system is more difficult than that of the SISO but remains effective.

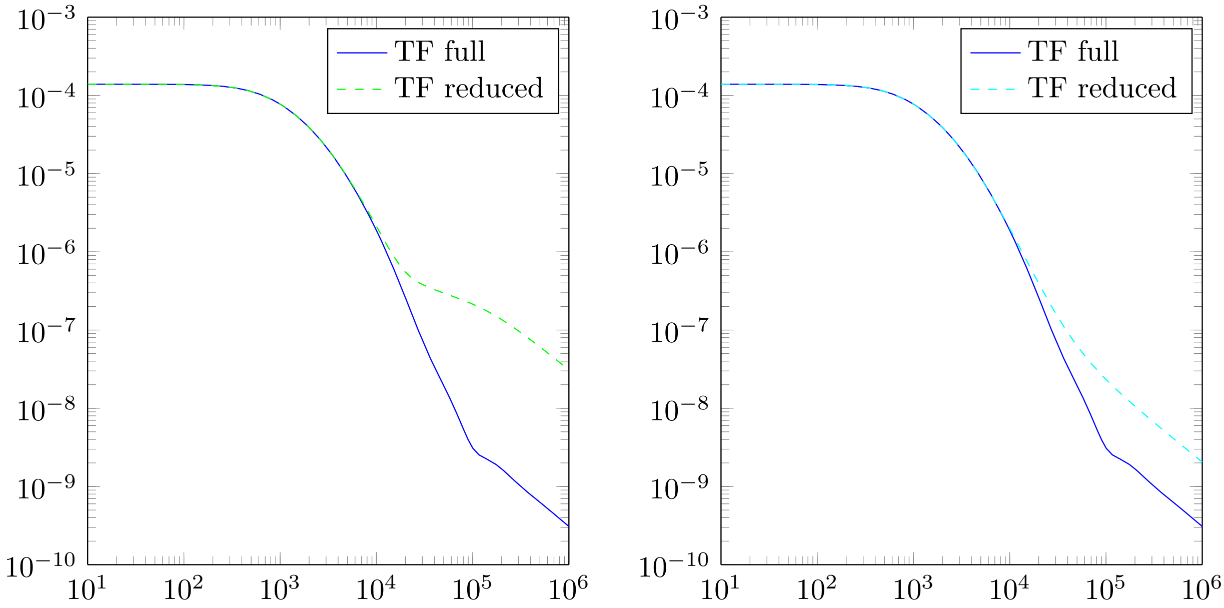

Fig. 4. 2D SISO case. Transfer function comparison (amplitude), Krylov input (left) and Krylov output (right).

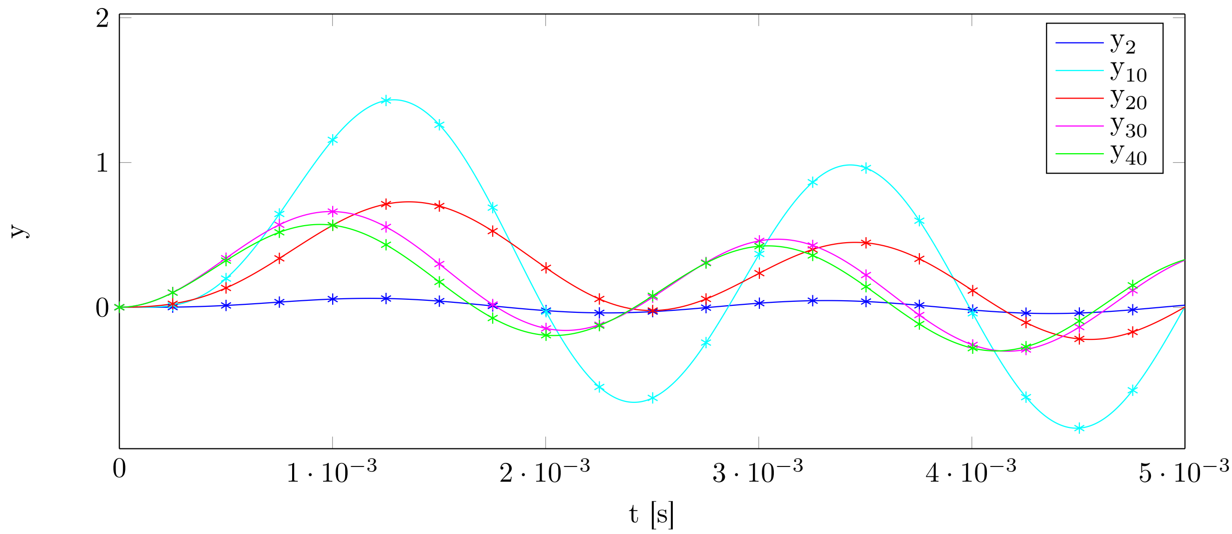

Fig. 5. 2D MIMO case. Some Krylov input outputs, $q = 10$ (solid=reference, star=reduced).

However, for this particular problem the choice of using output Krylov subspace may seem not the optimal solution, because the output error is bigger that obtained using the input Krylov approach.

More interestingly for state of the art applications of computational electromagnetics, it has been shown that this approach provides good results when applied to complex 3D-FEM models. To this respect, a further observation is in order, also for a deeper study on the subject: it has to be taken into account that FE formulation in the frequency domain is less computationally difficult than time domain resolution, but it is only able to show the steady state behaviour. On the other hand, a time domain formulation would provide also the entire transient study of the phenomenon, and could be also numerically less demanding if coupled with a suitable reduction technique, such as Krylov subspace methods.

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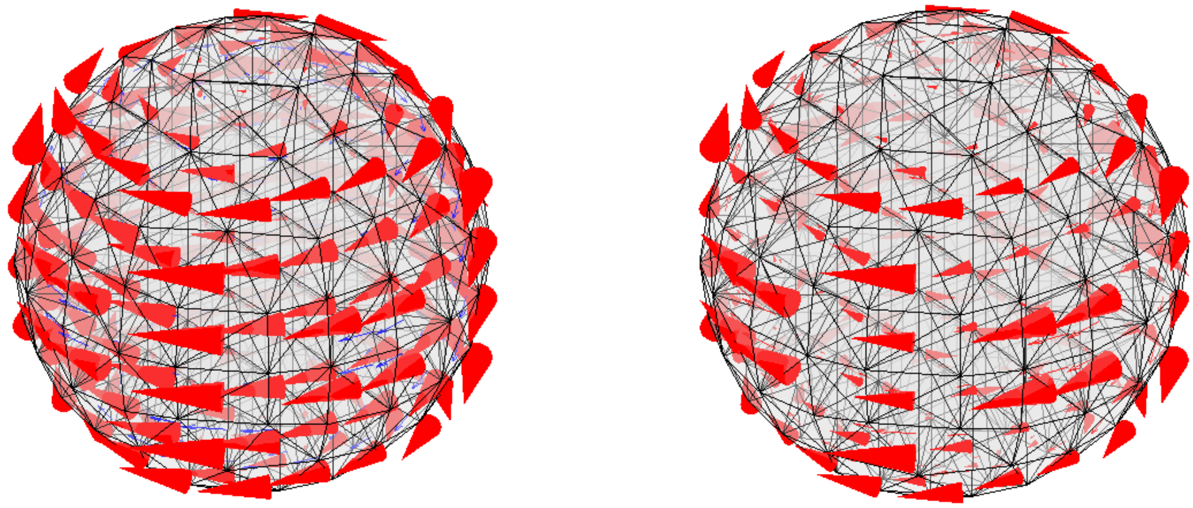

Fig. 6. 3D MIMO case. 3D plot of real and imaginary part of current vector field.

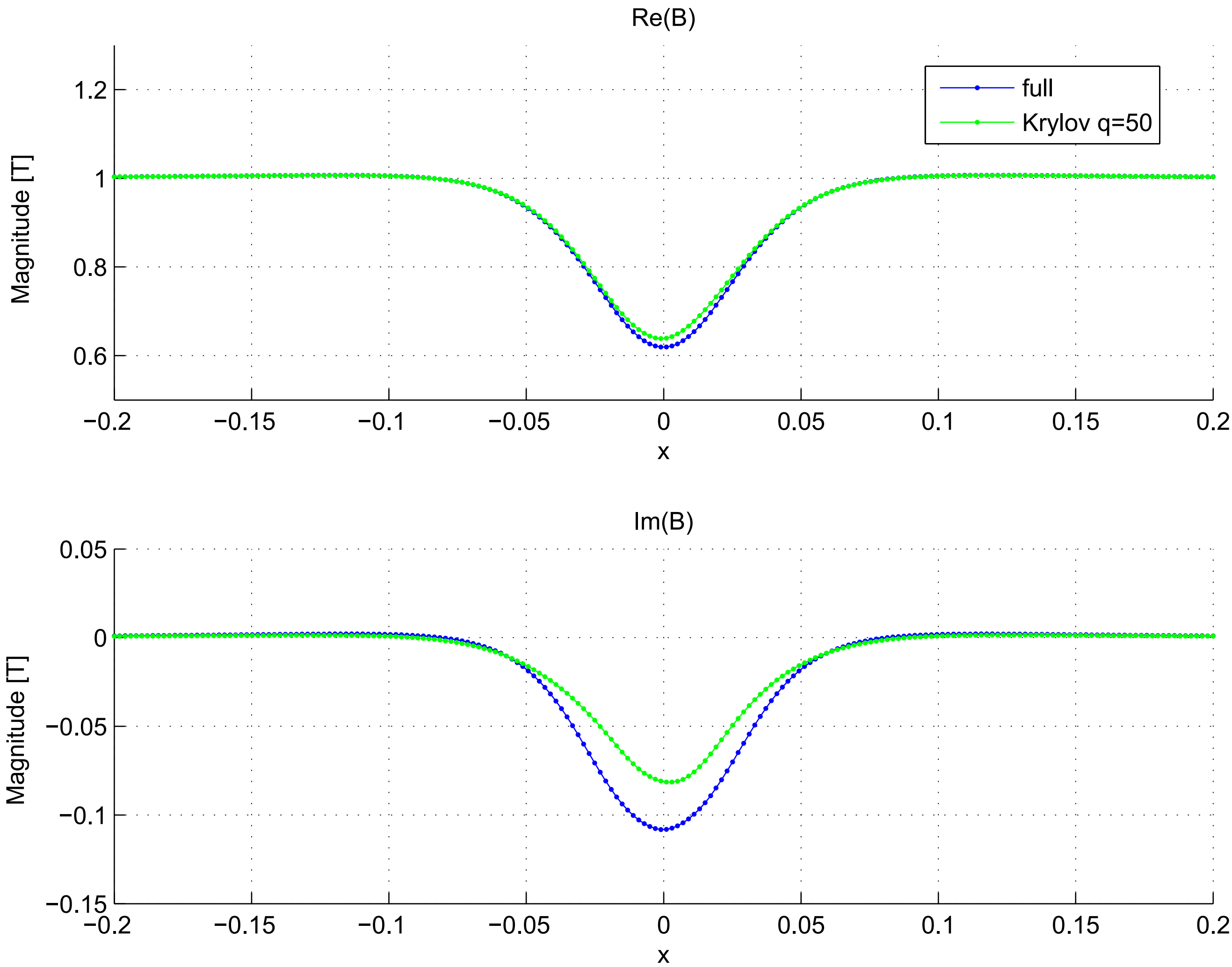

Fig. 7. 3D MIMO case. Comparison between reference full order (blue) and reduced order (green) models: real and imaginary part of magnetic field along the fixed set of sensors.

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