

Bode meets Kuramoto: Synchronized Clusters in Oscillatory Networks

Chiara Favaretto, Danielle S. Bassett, Angelo Cenedese, and Fabio Pasqualetti

Abstract—In this paper we study cluster synchronization in a network of Kuramoto oscillators, where groups of oscillators evolve cohesively and at different frequencies from the neighboring oscillators. Synchronization is critical in a variety of systems, where it enables complex functionalities and behaviors. Synchronization over networks depends on the oscillators' dynamics, the interaction topology, and coupling strengths, and the relationship between these different factors can be quite intricate. In this work we formally show that three network properties enable the emergence of cluster synchronization. Specifically, weak inter-cluster connections, strong intra-cluster connections, and sufficiently diverse natural frequencies among oscillators belonging to different groups. Our approach relies on system-theoretic tools, and is validated with numerical studies.

I. INTRODUCTION

Synchronization is fundamental to a number of complex phenomena in natural, social, and man-made systems [1], [2]. Examples include coordinated flashing of fireflies [3], cohesive flocking of birds [4], orchestrated firing of neurons [5], [6], entrainment of circadian rhythms [7], and reliable energy production in power grids [8], [9]. While some systems rely on complete synchronization to function properly, recent studies have highlighted the importance of cluster synchronization, where different groups evolve cohesively but independently from one another. Cluster synchronization, for instance, may be responsible for several neural pathologies, such as Parkinson's [10] and Huntington's [11] diseases, and epilepsy [12]. Due to its ubiquitous relevance, complete synchronization has been a topic of extensive study in the last decades [13], [14], [15], [16]. Yet, fundamental mechanisms and conditions enabling partial or cluster synchronization have remained elusive.

In this paper we consider networks of Kuramoto oscillators [17], and we characterize topological and intrinsic conditions enabling cluster synchronization. Our choice of Kuramoto dynamics stems from its ability to characterize a variety of complex phenomena across scientific domains [16], [9], [6], [18], despite its simplicity and limited number of parameters. While cluster synchronization may be the result of intricate interactions, we show that three main properties are responsible for the emergence of clusters. Specifically,

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we show that oscillators may form a cluster when they are strongly interconnected among each other, with weak outside interconnections, and with natural frequencies that are sufficiently different from those of the connected units.

Related work Complete synchronization of Kuramoto oscillators has been the subject of extensive research [13], [19]. Results have been derived for different configurations, including infinite and finite-dimensional networks. For instance, it is now known that, when the oscillators' natural frequencies are heterogeneous, synchronization is achieved for sufficiently large coupling strength, which overcomes the differences between the intrinsic characteristics of each oscillator.

Cluster or partial synchronization has received considerably less attention than full synchronization. In [20] patterns and group synchronization are studied, together with their stability properties for different classes of dynamics. Similarly, in [21] the authors describe a network of oscillators where clustered dynamics are due to the nonidentical dynamical behaviors of different clusters, and focus on the relationship between cluster synchronization and the topology of the underlying unweighted graph. More recently, the relation between cluster synchronization and network symmetry is studied in [22]. In [23] the authors propose a general technique to study stability of each cluster. In particular they exploited symmetry methods to find all possible clusters in networks of Laplacian-coupled oscillators. Cluster synchronization in networks with general topologies is the topic of [24], where the authors use the graph-theoretical notion of external equitable partitions to find clusters of oscillators. In this work we propose the use of system-theoretic tools to study cluster synchronization in networks of Kuramoto oscillators, and we characterize key properties enabling the emergence and stability of clusters.

Paper contributions The contribution of this paper is as follows. First, we formalize the problem of cluster synchronization in a network of Kuramoto oscillators, and we provide a condition to ensure that the phases of a group of oscillators remain within a certain angle from each other. This condition quantifies the importance of inter- and intra-cluster connections. That is, a cluster requires strong coupling within the oscillators and weak coupling with the neighboring oscillators outside the cluster. Second, we use a series of approximations and tools from frequency analysis of linear systems to show that, independently of the strength of the interconnections, a group of oscillators remains cohesive when their natural frequencies are sufficiently different from

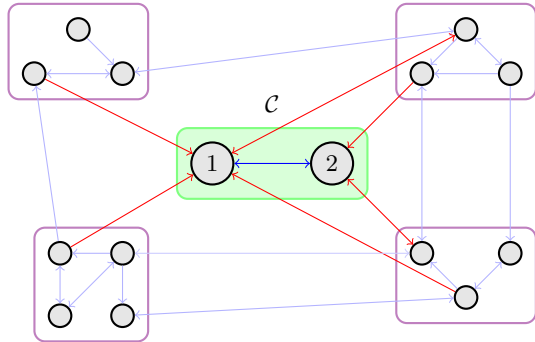


Fig. 1. This Figure shows a network of oscillators, where clusters of synchronized oscillators emerge. The topics of the paper focus on the analysis of the properties of the cluster $\mathcal{C} = \{1, 2\}$.

the natural frequencies of the neighboring oscillators outside the cluster. Finally, we provide numerical results to validate our findings and motivate further studies.

Paper organization The rest of the paper is organized as follows. Section II contains our setup and some preliminary results. Section III contains our conditions for the emergence of cluster synchronization. Finally, Section IV contains our numerical studies, and Section V concludes the paper.

II. PROBLEM SETTING AND PRELIMINARY NOTIONS

We consider a network of oscillators represented by the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denote the set of oscillators and their interconnections, respectively. Let $A = [a_{ij}]$ be the weighted adjacency matrix of \mathcal{G} , with $a_{ij} \in \mathbb{R}_{>0}$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise, and let $\theta_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ be the map describing the phase of the i -th oscillator. We assume that the phase θ_i evolves as

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^n a_{ij} \sin(\theta_j - \theta_i), \quad i = 1, \dots, n \quad (1)$$

where $\omega_i \in \mathbb{R}_{\geq 0}$ is the natural frequency of the i -th oscillator. The dynamics (1) are a generalized version of the classic Kuramoto model [17].

Depending on the interconnection pattern and weights, networks of Kuramoto oscillators exhibit a variety of synchronization behaviors [25], [26]. In this paper, we are particularly interested in characterizing clustered synchronization, where the oscillators can be grouped based on their oscillation phase. To formalize this concept, we introduce the following definition.

Definition 1: (Cluster of oscillators) The set of oscillators $\mathcal{C} \subseteq \mathcal{V}$ is a cluster if there exists an angle $0 \leq \gamma \leq \pi$ such that, whenever $|\theta_i(0) - \theta_j(0)| \leq \gamma$, then $|\theta_i(t) - \theta_j(t)| \leq \gamma$, for all $i, j \in \mathcal{C}$ and at all times $t \geq 0$. \square

While complete synchronization is a well-studied problem in the literature of Kuramoto networks [13], [14], [15], [16], the emergence of synchronized clusters is a complex phenomena whose explanation has remained elusive. In this paper, we derive topological and intrinsic conditions on the network of oscillators that facilitate the formation of

clustered dynamics. In particular, we show that synchronized clusters may emerge as the result of three independent network features: weak inter-cluster connections, strong intra-cluster connections, and sufficiently diverse natural frequencies among oscillators belonging to different groups. To better convey this message and for ease of presentation, we focus on the analysis of the properties of the cluster $\mathcal{C} = \{1, 2\}$ shown in Fig. 1. Furthermore, and without loss of generality, we assume $\omega_1 = \omega_2 = 0$. We remark that the ideas and methods developed in this paper apply in fact to more general network configurations, as we illustrate in our numerical studies in Section IV.

III. CLUSTER SYNCHRONIZATION IN KURAMOTO NETWORKS

Consider the network configuration in Fig. 1. The dynamics of the clustered oscillators read as

$$\begin{aligned} \dot{\theta}_1 &= a_{12} \sin(\theta_2 - \theta_1) + \sum_{j \neq 2}^n a_{1j} \sin(\theta_j - \theta_1), \\ \dot{\theta}_2 &= a_{21} \sin(\theta_1 - \theta_2) + \sum_{j \neq 1}^n a_{2j} \sin(\theta_j - \theta_2). \end{aligned} \quad (2)$$

A simple, yet conservative, bound on the clustering angle γ between the phases of the oscillators can readily be obtained.

Lemma 3.1: (Cluster condition based on edges weight) Consider the dynamics (2), and let

$$|\theta_1(0) - \theta_2(0)| \leq \arcsin \left(\sum_{j=3}^n \frac{a_{1j} + a_{2j}}{a_{12} + a_{21}} \right) =: \gamma.$$

Then, at all times $t \in \mathbb{R}_{\geq 0}$,

$$|\theta_1(t) - \theta_2(t)| \leq \gamma,$$

that is, $\mathcal{C} = \{1, 2\}$ is a cluster with respect to the angle γ .

Proof: To prove the positive invariance w.r.t. γ , we have to show that, if $|\theta_1(t) - \theta_2(t)| = \gamma$ at an instant of time t , then the phase difference will not increase in absolute value or, equivalently, the derivative of the absolute value of the difference function is non-positive. The derivative of $|\theta_1 - \theta_2|$ at time t is the following:

$$\frac{d}{dt} |\theta_1 - \theta_2| = -(a_{12} + a_{21}) \sin(\gamma) + u \quad (3)$$

where $\gamma = |\theta_1 - \theta_2|$ and

$$u := \pm \sum_{j \neq 2}^n a_{1j} \sin(\theta_j - \theta_1) \mp \sum_{j \neq 1}^n a_{2j} \sin(\theta_j - \theta_2)$$

Because the sine function is bounded, it is $|u| \leq \sum_{j=3}^n (a_{1j} + a_{2j})$. Moreover, the hypothesis of Lemma 3.1 ensures that:

$$\gamma = \arcsin \left(\sum_{j=3}^n \frac{a_{1j} + a_{2j}}{a_{12} + a_{21}} \right),$$

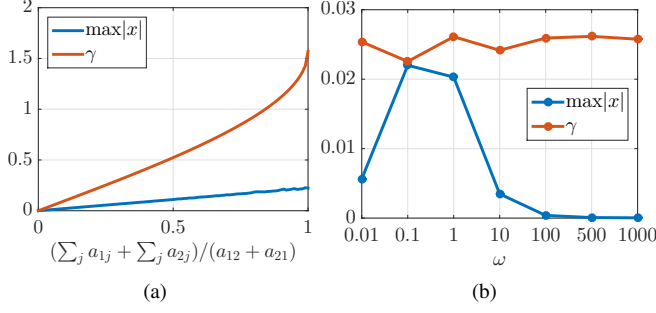


Fig. 2. This Figure shows $\max|x| := \max_t |\theta_1 - \theta_2|$ and the bound γ as a function of the couplings strengths and the oscillators' natural frequencies.

which implies that the derivative (3) is always non positive. This fact guarantees the positive invariance of \mathcal{C} w.r.t. γ . ■

Lemma 3.1 highlights an important property of a cluster, that is, nodes inside the cluster are strongly connected while nodes outside the cluster interact only through weak connections. In fact, consistent with our intuition, the clustering angle γ converges to zero when the ratio of the edges entering the cluster (namely, $\sum_{j=3}^n (a_{1j} + a_{2j})$ in Lemma 3.1) to the edges within the cluster ($(a_{12} + a_{21})$ in Lemma 3.1) approaches zero, and the bound becomes exact in this limit (see Fig. 2(a)). Yet, as we show in Fig. 2(b), the bound in Lemma 3.1 can be very conservative, especially when the natural frequencies of the oscillators within the cluster are far apart from the natural frequencies of the connected oscillators outside the cluster.

To capture this behavior and obtain a more accurate bound, we proceed as follows. Let $x := \theta_1 - \theta_2$, and notice that

$$\dot{x} = -(a_{12} + a_{21}) \sin(x) + \sum_{j=3}^n (a_{1j} u_{1j} - a_{2j} u_{2j}), \quad (4)$$

where $u_{ij} = \sin(\theta_j - \theta_i)$ for $i \in \{1, 2\}$. The following result uses two linear systems to find a bound for the evolution of x .

Lemma 3.2: (Linear bound) Let x , u_{1j} , and u_{2j} be as in Equation (4), and γ as in Lemma 3.1, respectively. If $|x_l(0)| \leq |x(0)| \leq |x_u(0)|$, then

$$|x_1| \leq |x| \leq |x_u|,$$

where x_1 and x_u satisfy

$$\begin{aligned} \dot{x}_1 &= -(a_{12} + a_{21})x_1 + \sum_{j \neq 2}^n a_{1j} u_{1j} - \sum_{j \neq 1}^n a_{2j} u_{2j}, \\ \dot{x}_u &= -\frac{(a_{12} + a_{21}) \sin(\gamma)}{\gamma} x_u + \sum_{j \neq 2}^n a_{1j} u_{1j} - \sum_{j \neq 1}^n a_{2j} u_{2j}. \end{aligned}$$

Proof: By hypothesis and Lemma 3.1, we know that $|x(t)| \leq \gamma$ and $|x_u(t)| \leq \gamma$ for every t . If $x, x_u \geq 0$, we have that $\dot{x} \leq \dot{x}_u$, indeed:

$$-\frac{\sin(\gamma)}{\gamma} x_u \geq -\frac{\sin(\gamma)}{\gamma} \gamma = -\sin(\gamma). \quad (5)$$

By using the *Comparison Lemma* in [27], it follows that

$$x(t) \leq x_u(t), \quad \text{if } x, x_u \geq 0, \forall t \geq 0. \quad (6)$$

If $x, x_u \leq 0$, we can consider $\tilde{x} := -x$, $\tilde{x}_u := -x_u$, which are both non-negative and the *Comparison Lemma* still proves that $\tilde{x}(t) \leq \tilde{x}_u(t)$, for all $t \geq 0$, which means

$$x(t) \geq x_u(t), \quad \text{if } x, x_u \leq 0, \forall t \geq 0. \quad (7)$$

Equations (6) and (7) together prove the right side of the thesis $|x| \leq |x_u|$.

In order to prove that $|x_1| \leq |x|$, we can follow the same reasoning as above, by reversing the inequalities. ■

Lemma 3.2 exploits the *Comparison Lemma* [27] to bound the evolution of the nonlinear differential dynamics of x with linear dynamics. These expressions can be further used to quantify how the oscillators' natural frequencies influence the angle deviations of nodes within the same cluster. In particular, we approximate the differential angle x with its upper bound x_u , which obeys linear dynamics. Then, we use Bode analysis to show that, in its linear approximation, the cluster behaves as a *low pass filter* with respect to the difference of the natural frequencies of the oscillators within and outside the cluster. Thus, when the natural frequencies of the oscillators outside the cluster increase, the inputs u_{ij} have an increasingly smaller effect on the dynamics of the cluster. This allows us to explain the behavior highlighted in Fig. 2(b), and to derive a better bound – both qualitatively and quantitatively – on the invariance properties of the cluster. Let i be the imaginary unit, and let $a \lesssim b$ denote that the value of a is approximately less than the value of b .

Theorem 3.3: (Cluster condition based on edges weight and oscillators' natural frequency) Consider the dynamics (2), and let $|\theta_1(0) - \theta_2(0)| \leq \gamma$. Then,¹

$$|\theta_1(t) - \theta_2(t)| \lesssim \sum_{j=3}^n (a_{1j} + a_{2j}) |G(i\omega_j)| := \beta, \quad (8)$$

where G is the transfer function defined as

$$G(s) = \left(s + \frac{(a_{12} + a_{21}) \sin(\gamma)}{\gamma} \right)^{-1}.$$

Moreover, as ω_j increases to infinity for $j = \{3, \dots, n\}$,

$$\lim_{t \rightarrow \infty} |\theta_1(t) - \theta_2(t)| = 0.$$

Proof: From (4), define

$$y_{1j} := \theta_j - \theta_1, \quad y_{2j} := \theta_j - \theta_2, \quad u_{ij} := \sin(y_{ij}). \quad (9)$$

At first, let us show that $\theta_j(t) \rightarrow \omega_j t$ as ω_j goes to infinity, for each $j = 3, \dots, n$. If we consider that each θ_j is coupled with other K_j nodes in addition to θ_1 and θ_2 , with coupling weights w_{kj} , we can evaluate $\dot{\theta}_j/\omega_j$ as:

$$\frac{\dot{\theta}_j}{\omega_j} = 1 - \sum_{i=1}^2 \frac{a_{ij}}{\omega_j} \sin(y_{ij}) + \sum_{k=1}^{K_j} \frac{w_{jk}}{\omega_j} \sin(\theta_k - \theta_j),$$

¹In other words, $\max |\theta_1(t) - \theta_2(t)| - \sum_{j=3}^n (a_{1j} + a_{2j}) |G(i\omega_j)| \approx 0$.

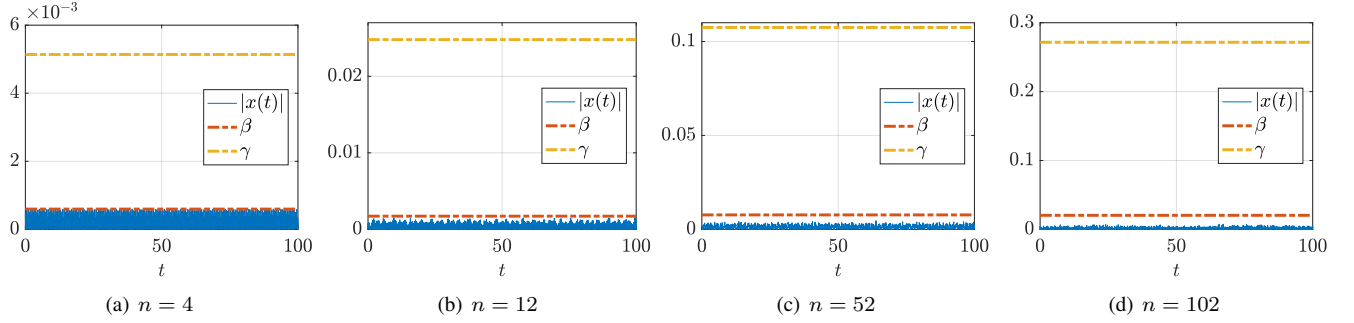


Fig. 3. Example of simulation with $\omega_1 = \omega_2 = 0$, $\omega_i \in [0, 100]$ for $i = 3, \dots, n$, $a_{12} = a_{21} = 1$ and $a_{ij} \in [0, 0.05]$ for the other couplings, except for the case $n = 102$ (in this case $a_{ij} \in [0, 0.02]$). The blue line is the dynamics of $|x| = |\theta_1 - \theta_2|$, the yellow one indicates the value of γ from Lemma 3.1, whereas the red one is the tighter bound β given by Theorem 3.3.

which tends to 1 as ω_j tends to infinity. As a consequence:

$$\dot{\theta}_j(t) \longrightarrow \omega_j \quad \text{and} \quad \theta_j(t) \longrightarrow \omega_j t. \quad (10)$$

By Lemma 3.2, we know that x_u is an upper bound for $\theta_1 - \theta_2$, if γ obeys the assumptions. The transfer functions w.r.t. inputs u_{ij} for the linearized system x_u , for each $i = 1, 2$ and $j = 3, \dots, n$, are the following:

$$G^{(ij)}(s) = a_{ij}G(s), \quad (11)$$

$$G(s) = \left(s + \frac{(a_{12} + a_{21}) \sin(\gamma)}{\gamma} \right)^{-1}.$$

Note that $G^{(ij)}$ plays the role of a low pass filter (see Fig. 4). Therefore, if we take into account the frequency of the inputs u_{ij} , we can reduce the conservative bound given by Lemma 3.1, by evaluating the maximum Bode magnitude of each input signal:

$$\left\| G^{(ij)} \right\|_{\infty} = \max_{\omega} \left| G^{(ij)}(i\omega) \right| \approx a_{ij} |G(i\omega_j)|.$$

Equation (10) states that this approximation improves as the natural frequencies tend to infinity.

The superposition principle for linear systems allow us to define an approximation for the upper bound for the influence of the inputs u_{ij} on $|x(t)|$ as the sum of every contribution:

$$|x(t)| \lesssim \sum_{j=3}^n (a_{1j} + a_{2j}) |G(i\omega_j)|,$$

which corresponds exactly to equation (8).

From (9) we have that for all $j = 3, \dots, n$ and $i = 1, 2$, it holds $\sin(\theta_j - \theta_i) \rightarrow \sin(\omega_j t - \theta_i)$, as $\omega_j \rightarrow +\infty$.

Therefore, there is:

$$\lim_{\{\omega_j\}_{j=3}^n \rightarrow \infty} \dot{x} = -(a_{12} + a_{21}) \sin(x) + \sum_{j=3}^n \dot{v}_{1j} - \dot{v}_{2j},$$

where $\dot{v}_{ij} := a_{ij} \sin(\omega_j t - \theta_i)$.

As each ω_j tends to infinity, then every v_{ij} tends to zero. Indeed, for $3 \leq j \leq n$ and $i = 1, 2$:

$$\begin{aligned} v_{ij}(t) &= \int_0^t a_{ij} \sin(\omega_j \tau - \theta_i) d\tau \\ &= a_{ij} \int_0^t \sin(\omega_j \tau) \cos(\theta_i) - \cos(\omega_j \tau) \sin(\theta_i) d\tau. \end{aligned}$$

If we define:

$$f_1(t) := \frac{1}{\omega_j} \cos(\omega_j t), \quad g_1(t) := -\cos(\theta_i(t)), \quad (14a)$$

$$f_2(t) := \frac{1}{\omega_j} \sin(\omega_j t), \quad g_2(t) := -\sin(\theta_i(t)), \quad (14b)$$

it follows that

$$\begin{aligned} v_{ij} &= a_{ij} \int_0^t (df_1 g_1 + df_2 g_2) d\tau \\ &= a_{ij} \left(f_1 g_1 \Big|_0^t - \int_0^t f_1 dg_1 d\tau + f_2 g_2 \Big|_0^t - \int_0^t f_2 dg_2 d\tau \right). \end{aligned}$$

From equation (2), we have that $\dot{\theta}_i$, $i = 1, 2$, is $O(\omega_j)$, if ω_j tends to infinity and hence:

$$dg_1 = \dot{\theta}_i \sin(\theta_i) \in O(\omega_j), \quad dg_2 = -\dot{\theta}_i \cos(\theta_i) \in O(\omega_j),$$

which implies that each term of v_{ij} tends to zero as ω_j goes to infinity, and hence

$$\lim_{\{\omega_j\}_{j=3}^n \rightarrow \infty} \dot{x} = -(a_{12} + a_{21}) \sin(x). \quad (17)$$

To prove the thesis, we can show that $x^* = 0$ is a globally asymptotically stable [27] equilibrium point for (17). Let us consider the Lyapunov function $V(x) := \frac{x^2}{2} > 0$, $\forall x \neq x^*$. In particular, we have:

$$V(x) \rightarrow +\infty \text{ as } \|x\| \rightarrow +\infty, \quad (18a)$$

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x} \cdot \dot{x} = -(a_{12} + a_{21}) x \sin(x). \quad (18b)$$

$\dot{V}(x)$ is definite negative for all x s.t. $|x| \leq \gamma \leq \pi/2$. By the *Lyapunov Theorem for Global Asymptotic Stability* [27] the global asymptotic stability of x^* is proved. ■

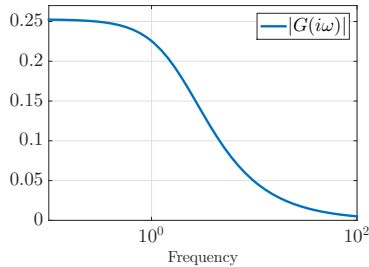


Fig. 4. Bode diagram of the magnitude (in absolute value) of the transfer function that characterizes the linear system \dot{x}_u , which plays the role of a bound for the dynamics of x w.r.t. each input u_{ij} . This linear system acts as a low-pass filter, which cuts off the high frequencies of the input.

Theorem 3.3 quantifies how the natural frequencies of the oscillators connected to a cluster affect the cluster stability. In particular, the larger the difference between the frequencies of the node within and outside the cluster, the smaller the effect of the outside nodes on the evolution of the cluster, independently of the weight of the interconnection edges. In the limit when these frequencies grow to infinity, the cluster becomes practically disconnected from the neighboring nodes and achieves phase synchronization [13], [19].

To conclude this section, we show that the bounds in Lemma 3.1 and Theorem 3.3 coincide when the natural frequencies of the oscillators outside the cluster approach the natural frequencies of those inside the cluster. In particular, it is important to note that in this situation both the bounds are conservative, since a synchronization among all the nodes within the network is reached [13], [19].

Corollary 3.4: (Equivalence of bounds when the natural frequencies coincide) Let $\omega_j = 0$ for all $j = 1, \dots, n$. Then

$$\sum_{j=3}^n (a_{1j} + a_{2j}) |G(i\omega_j)| = \gamma,$$

that is, the bounds in Lemma 3.1 and Theorem 3.3 coincide.

Proof: As defined in (11), $G(s)$, evaluated in $s = i0$ becomes:

$$G(i0) = \left(\frac{(a_{12} + a_{21}) \sin(\gamma)}{\gamma} \right)^{-1},$$

which implies

$$\sum_{j=3}^n (a_{1j} + a_{2j}) |G(i0)| = \frac{\gamma}{\sin(\gamma)} \underbrace{\sum_{j=3}^n \frac{a_{1j} + a_{2j}}{a_{12} + a_{21}}}_{\sin(\gamma)} = \gamma.$$

Therefore, the two bounds coincide. \blacksquare

IV. NUMERICAL EXAMPLES

In this section we validate our theoretical findings and assumptions through a number of numerical studies. Fig. 3 and Fig. 5(a) show the behavior of the phase difference x with respect to the bounds derived in Lemma 3.1 and Theorem 3.3. In particular, Fig. 3 shows that the bound

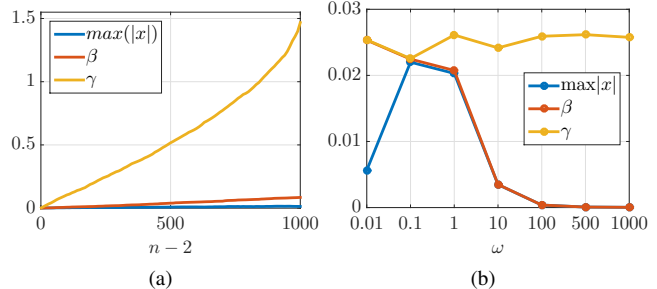


Fig. 5. (a) Comparison between the growth of the two bounds γ (related to Lemma 3.1) and β (obtained by Theorem 3.3) w.r.t. the number of nodes n . (b) Comparison between the growth of the bounds β and the real value of $\max|x|$ w.r.t. the values of ω_j .

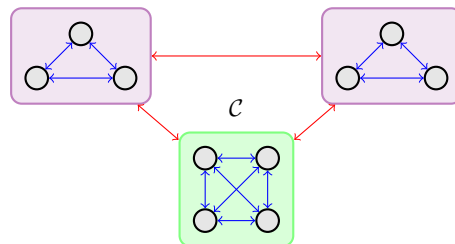


Fig. 6. Example of configuration of a network, which nodes are represented by clusters of oscillators.

γ derived in Lemma 3.1 is more conservative than the bound β computed in Theorem 3.3, because it does not account for the frequencies of the neighboring oscillators. The simulations show that both bounds become more and more conservative as the number of neighboring oscillators increases, suggesting that the properties of a cluster may also depend on the number of neighboring oscillators. Fig. 5 summarizes the results in Fig. 3 showing how the bounds γ and β and the phase difference x behave as a function of the number of neighboring oscillators and their frequencies. It should be observed that, while γ is a bound for the phase difference x , the quantity β is only approximately greater than x . In other words, β is an approximation for the largest phase difference x over all possible network configurations.

The analysis in this paper is limited to the case of a cluster with two oscillators with natural frequency equal to zero. Yet, the clustering mechanisms apply to more general network configurations. Consider for instance the network of oscillators represented in Fig. 6. Fig. 7 shows the largest difference of the phases of the oscillators within the cluster \mathcal{C} , as a function of the coupling strength and the natural frequencies of the neighboring oscillators outside the cluster. Consistent with our analysis for a cluster with two nodes, the cluster \mathcal{C} is more and more cohesive as the coupling strength with neighboring nodes decreases, or the difference with the neighboring natural frequencies increases.

Finally, we consider the case where the natural frequencies of the nodes within the cluster are nonzero ($\omega_1 = \omega_2 = \omega_0 \neq 0$). As shown in Fig. 8(a) and 8(b), all our results and bounds still hold, as they depend on the difference between

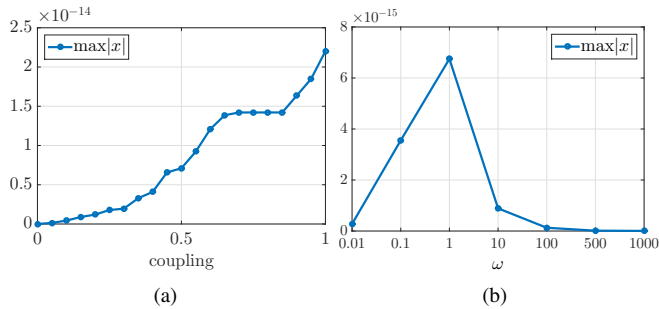


Fig. 7. Maximum difference between the nodes within the cluster w.r.t. (a) the coupling with the nodes outside the cluster and w.r.t. (b) the differences between the natural frequencies, in the configuration of Fig. 6, where a cluster \mathcal{C} of $n = 4$ nodes is analysed. Results obtained as the mean of 20 simulations per each value of the considered parameter. Specifically, the coupling strength a_{ij} is chosen in the range $[0, 1]$ (with $a_{12} = a_{21} = 1$) and the natural frequencies are in the set $\{0.01, 0.1, 1, 10, 100, 500, 1000\}$.

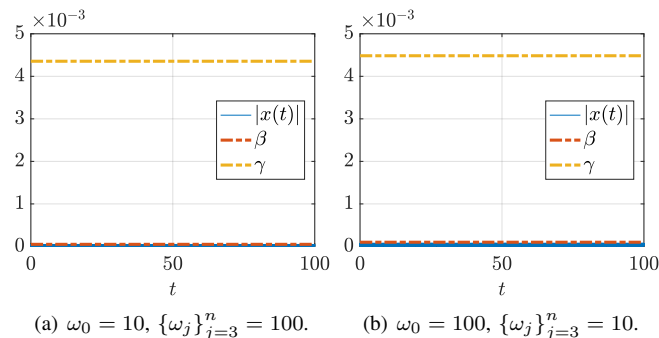


Fig. 8. Examples of simulation of a network with $n = 4$ nodes and the natural frequencies of the clustered oscillators $\omega_0 \neq 0$. The two simulations are symmetric, in terms of difference between ω_0 and the natural frequencies of the external nodes.

the natural frequencies of the nodes within and outside the cluster, not their absolute values.

V. CONCLUSION AND FUTURE WORK

In this work we characterize cluster synchronization in networks of Kuramoto oscillators. In particular, we unveil conditions on the interaction network and oscillators natural frequencies that enable the emergence of groups of oscillators that evolve cohesively independently of the neighboring oscillators. We use tools from linear and nonlinear systems theory to quantify that cluster synchronization depends primarily on three properties: strong intra-cluster coupling, weak inter-cluster coupling, and sufficiently heterogeneous oscillators' natural frequencies. Several directions are left as the subject of future investigation, including the study of more complex clusters configurations and network topologies, and the design of systematic procedures to predict clusters.

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