Stability, Stabilizability and Control of certain classes of Positive Systems

Irene Zorzan

Ph.D. Advisor: Prof. Maria Elena Valcher

Ph.D. School in Information Engineering - University of Padova

February 22, 2018

Outline of the talk

- Positive Systems: definition and motivations

Outline of the talk

- Positive Systems: definition and motivations
- Control of Positive Multi-Agent Systems
- Optimal control of Positive Bilinear Systems
- Stability and stabilizability of Compartmental Switched Systems

Outline of the talk

- Positive Systems: definition and motivations
- Control of Positive Multi-Agent Systems
- Optimal control of Positive Bilinear Systems
- Stability and stabilizability of Compartmental Switched Systems
- Feedback stabilization of Compartmental Systems

Positive Systems

Positive Systems

nonnegative state and output variables
for every nonnegative initial condition
and every nonnegative input

Positive Systems

Positive Systems

nonnegative state and output variables
for every nonnegative initial condition
and every nonnegative input

pressures
absolute temperatures
concentrations of substances
population levels
probabilities

Positive Systems

Positive Systems

nonnegative state and output variables
for every nonnegative initial condition
and every nonnegative input

pressures
absolute temperatures
concentrations of substances
population levels
probabilities

economics
system biology
congestion control
pharmacokinetics
power networks

Positive Systems: state-space representation

Consider a continuous-time linear system described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{1}$$

Positive Systems: state-space representation

Consider a continuous-time linear system described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{1}$$

where

- $\mathbf{x}$ is the n-dimensional state vector

Positive Systems: state-space representation

Consider a continuous-time linear system described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{1}$$

where

- $\mathbf{x}$ is the n-dimensional state vector
- $\mathbf{u}$ is the m-dimensional input vector

Positive Systems: state-space representation

Consider a continuous-time linear system described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{1}$$

where

- $\mathbf{x}$ is the n-dimensional state vector
- $\mathbf{u}$ is the m-dimensional input vector
- A, B are matrices of consistent dimensions

Positive Systems: state-space representation

Consider a continuous-time linear system described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{1}$$

where

- $\mathbf{x}$ is the n-dimensional state vector
- $\mathbf{u}$ is the m-dimensional input vector
- A, B are matrices of consistent dimensions

System (1) is a Positive System if and only if

Positive Systems: state-space representation

Consider a continuous-time linear system described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{1}$$

where

- $\mathbf{x}$ is the n-dimensional state vector
- $\mathbf{u}$ is the m-dimensional input vector
- A, B are matrices of consistent dimensions

System (1) is a Positive System if and only if

- A is a Metzler matrix
  A square matrix is Metzler if all its off-diagonal entries are nonnegative.

Positive Systems: state-space representation

Consider a continuous-time linear system described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \qquad (1)$$

where

- $\mathbf{x}$ is the n-dimensional state vector
- $\mathbf{u}$ is the m-dimensional input vector
- A, B are matrices of consistent dimensions

System (1) is a Positive System if and only if

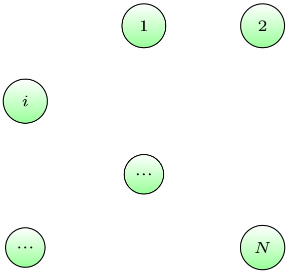
- A is a Metzler matrix
  A square matrix is Metzler if all its off-diagonal entries are nonnegative.
- B is a positive matrix
  A matrix is positive if all its entries are nonnegative.

Positive consensus problem: set-up

Positive consensus problem: set-up

N identical Positive Single-Input Systems:

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t), \quad i \in [1, N]$$



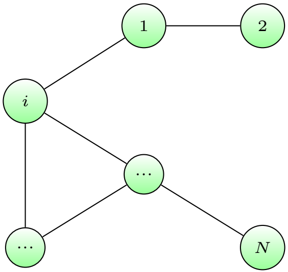
Positive consensus problem: set-up

N identical Positive Single-Input Systems:

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t), \quad i \in [1, N]$$

Undirected, weighted and connected communication graph:

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$$



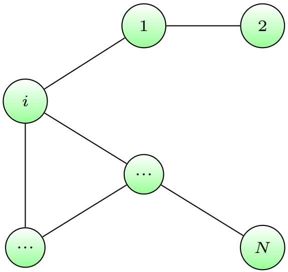
Positive consensus problem: set-up

N identical Positive Single-Input Systems:

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t), \quad i \in [1, N]$$

Undirected, weighted and connected communication graph:

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$$

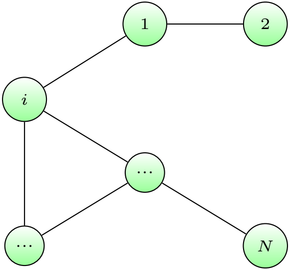


Assume that A is non-Hurwitz and that (A, B) is stabilizable.

Positive consensus: problem formulation

Each agent adopts the state-feedback control law

$$u_i(t) = K \sum_{j=1}^{N} \mathcal{A}_{ij} \left[\mathbf{x}_j(t) - \mathbf{x}_i(t) \right]$$

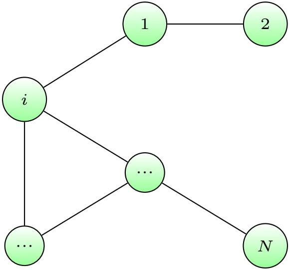


Positive consensus: problem formulation

Each agent adopts the state-feedback control law

$$u_i(t) = K \sum_{j=1}^{N} \mathcal{A}_{ij} \left[\mathbf{x}_j(t) - \mathbf{x}_i(t) \right]$$

where $K \in \mathbb{R}^{1\times n}$ is a matrix to be designed.

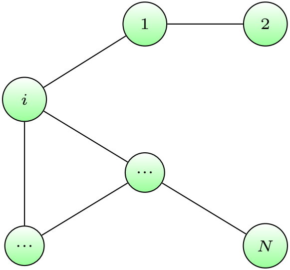


Positive consensus: problem formulation

Each agent adopts the state-feedback control law

$$u_i(t) = K \sum_{j=1}^{N} \mathcal{A}_{ij} \left[\mathbf{x}_j(t) - \mathbf{x}_i(t) \right]$$

where $K \in \mathbb{R}^{1\times n}$ is a matrix to be designed.



Positive consensus problem:

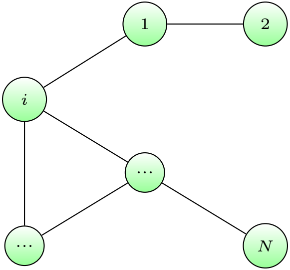
determine a feedback matrix $K \in \mathbb{R}^{1\times n}$ such that:

Positive consensus: problem formulation

Each agent adopts the state-feedback control law

$$u_i(t) = K \sum_{j=1}^{N} \mathcal{A}_{ij} \left[\mathbf{x}_j(t) - \mathbf{x}_i(t) \right]$$

where $K \in \mathbb{R}^{1 \times n}$ is a matrix to be designed.



Positive consensus problem:
determine a feedback matrix $K \in \mathbb{R}^{1 \times n}$ such that:

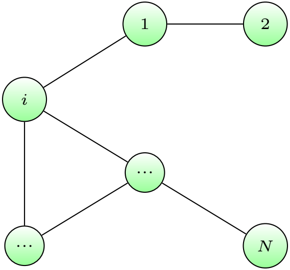
i) positivity of the overall dynamics is preserved

Positive consensus: problem formulation

Each agent adopts the state-feedback control law

$$u_i(t) = K \sum_{j=1}^{N} \mathcal{A}_{ij} \left[\mathbf{x}_j(t) - \mathbf{x}_i(t) \right]$$

where $K \in \mathbb{R}^{1\times n}$ is a matrix to be designed.



Positive consensus problem:
determine a feedback matrix $K \in \mathbb{R}^{1\times n}$ such that:

i) positivity of the overall dynamics is preserved

ii) all the agents achieve consensus, namely for every $i, j \in [1, N]$ it holds $\lim_{t\to+\infty} x_i(t) - x_j(t) = 0$

Positive consensus problem: motivations

In several contexts where consensus problems arise, the positivity of the overall controlled system is an intrinsic feature that should be taken into consideration when formalizing the problem.

Positive consensus problem: motivations

In several contexts where consensus problems arise, the positivity of the overall controlled system is an intrinsic feature that should be taken into consideration when formalizing the problem.

This is the case, for instance, when dealing with

- sensor networks for greenhouse monitoring

Positive consensus problem: motivations

In several contexts where consensus problems arise, the positivity of the overall controlled system is an intrinsic feature that should be taken into consideration when formalizing the problem.

This is the case, for instance, when dealing with

- sensor networks for greenhouse monitoring

  CO_2 concentration
  humidity level
  light intensity

Positive consensus problem: motivations

In several contexts where consensus problems arise, the positivity of the overall controlled system is an intrinsic feature that should be taken into consideration when formalizing the problem.

This is the case, for instance, when dealing with

- sensor networks for greenhouse monitoring
  - CO_2 concentration
  - humidity level
  - light intensity
- fleet of cooperating hybrid vehicles

Positive consensus problem: motivations

In several contexts where consensus problems arise, the positivity of the overall controlled system is an intrinsic feature that should be taken into consideration when formalizing the problem.

This is the case, for instance, when dealing with

- sensor networks for greenhouse monitoring

  CO_2 concentration
  humidity level
  light intensity

- fleet of cooperating hybrid vehicles

  CO_2 emission level
  electric/combustion based propulsion

Positive Bilinear Systems

Positive Bilinear Systems

Consider a Positive Bilinear System described by

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \left(A + \sum_{i=1}^{M} u_i D_i\right)\mathbf{x}(t) + \mathbf{w}(t)\\ \mathbf{y}(t) &= C\mathbf{x}(t)\end{aligned}$$

Positive Bilinear Systems

Consider a Positive Bilinear System described by

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^{M} u_i D_i\right) \mathbf{x}(t) + \mathbf{w}(t)$$

$$\mathbf{y}(t) = C\mathbf{x}(t)$$

where

- $\mathbf{x}$ is the n-dimensional state vector

Positive Bilinear Systems

Consider a Positive Bilinear System described by

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^{M} u_i D_i\right)\mathbf{x}(t) + \mathbf{w}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

where

- $\mathbf{x}$ is the n-dimensional state vector
- $\mathbf{w}$ is an arbitrary n-dimensional external disturbance

Positive Bilinear Systems

Consider a Positive Bilinear System described by

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^{M} u_i D_i \right) \mathbf{x}(t) + \mathbf{w}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

where

- $\mathbf{x}$ is the n-dimensional state vector
- $\mathbf{w}$ is an arbitrary n-dimensional external disturbance
- $\mathbf{y}$ is the p-dimensional output vector

Positive Bilinear Systems

Consider a Positive Bilinear System described by

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^{M} u_i D_i\right)\mathbf{x}(t) + \mathbf{w}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

where

- $\mathbf{x}$ is the n-dimensional state vector
- $\mathbf{w}$ is an arbitrary n-dimensional external disturbance
- $\mathbf{y}$ is the p-dimensional output vector
- u_i, $i \in [1, M]$, are the scalar control inputs

Positive Bilinear Systems

Consider a Positive Bilinear System described by

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^{M} u_i D_i \right) \mathbf{x}(t) + \mathbf{w}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

where

- $\mathbf{x}$ is the n-dimensional state vector
- $\mathbf{w}$ is an arbitrary n-dimensional external disturbance
- $\mathbf{y}$ is the p-dimensional output vector
- u_i, $i \in [1, M]$, are the scalar control inputs
- A is an $n \times n$ Metzler matrix

Positive Bilinear Systems

Consider a Positive Bilinear System described by

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^{M} u_i D_i\right)\mathbf{x}(t) + \mathbf{w}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

where

- $\mathbf{x}$ is the n-dimensional state vector
- $\mathbf{w}$ is an arbitrary n-dimensional external disturbance
- $\mathbf{y}$ is the p-dimensional output vector
- u_i, $i \in [1, M]$, are the scalar control inputs
- A is an $n \times n$ Metzler matrix
- D_i, $i \in [1, M]$, are $n \times n$ diagonal matrices

Positive Bilinear Systems

Consider a Positive Bilinear System described by

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^{M} u_i D_i\right)\mathbf{x}(t) + \mathbf{w}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

where

- $\mathbf{x}$ is the n-dimensional state vector
- $\mathbf{w}$ is an arbitrary n-dimensional external disturbance
- $\mathbf{y}$ is the p-dimensional output vector
- u_i, $i \in [1, M]$, are the scalar control inputs
- A is an $n \times n$ Metzler matrix
- D_i, $i \in [1, M]$, are $n \times n$ diagonal matrices
- C is a $p \times n$ nonnegative matrix

Optimal control of Positive Bilinear Systems

Consider a Positive Bilinear System described by

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^{M} u_i D_i\right)\mathbf{x}(t) + \mathbf{w}(t)$$

$$\mathbf{y}(t) = C\mathbf{x}(t)$$

Optimal control of Positive Bilinear Systems

Consider a Positive Bilinear System described by

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^{M} u_i D_i \right) \mathbf{x}(t) + \mathbf{w}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

Assume that the Metzler matrix A is non-Hurwitz.

Optimal control of Positive Bilinear Systems

Consider a Positive Bilinear System described by

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^{M} u_i D_i\right) \mathbf{x}(t) + \mathbf{w}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

Assume that the Metzler matrix A is non-Hurwitz.

Optimal robust control problem:

determine $u_i \in \mathbb{R}_+$, $i \in [1, M]$, with the following characteristics:

Optimal control of Positive Bilinear Systems

Consider a Positive Bilinear System described by

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^{M} u_i D_i \right) \mathbf{x}(t) + \mathbf{w}(t)$$

$$\mathbf{y}(t) = C\mathbf{x}(t)$$

Assume that the Metzler matrix A is non-Hurwitz.

Optimal robust control problem:

determine $u_i \in \mathbb{R}_+$, $i \in [1, M]$, with the following characteristics:

i) satisfy the constraint $\sum_{i=1}^{M} u_i < 1$

Optimal control of Positive Bilinear Systems

Consider a Positive Bilinear System described by

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^{M} u_i D_i\right)\mathbf{x}(t) + \mathbf{w}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

Assume that the Metzler matrix A is non-Hurwitz.

Optimal robust control problem:

determine $u_i \in \mathbb{R}_+$, $i \in [1, M]$, with the following characteristics:

i) satisfy the constraint $\sum_{i=1}^{M} u_i < 1$

ii) make the system asymptotically stable

Optimal control of Positive Bilinear Systems

Consider a Positive Bilinear System described by

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^{M} u_i D_i \right) \mathbf{x}(t) + \mathbf{w}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

Assume that the Metzler matrix A is non-Hurwitz.

Optimal robust control problem:

determine $u_i \in \mathbb{R}_+$, $i \in [1, M]$, with the following characteristics:

i) satisfy the constraint $\sum_{i=1}^{M} u_i < 1$

ii) make the system asymptotically stable

iii) maximize system robustness against the external disturbance

Optimal control of Positive Bilinear Systems

Consider a Positive Bilinear System described by

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^{M} u_i D_i\right)\mathbf{x}(t) + \mathbf{w}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

Assume that the Metzler matrix A is non-Hurwitz.

Optimal robust control problem:

determine $u_i \in \mathbb{R}_+$, $i \in [1, M]$, with the following characteristics:

i) satisfy the constraint $\sum_{i=1}^{M} u_i < 1$
ii) make the system asymptotically stable
iii) minimize either the $\mathcal{L}_1$-norm or the $\mathcal{H}_\infty$-norm of the system

Positive Bilinear Systems: application areas

Positive Bilinear Systems have been fruitfully employed in a variety of application areas:

- system biology

Positive Bilinear Systems: application areas

Positive Bilinear Systems have been fruitfully employed in a variety of application areas:

- system biology
  a Positive Bilinear System can be adopted to describe the behaviour of a group of self-replicating genotypes
  $\Rightarrow$ determing the optimal control means determing the best drugs concentration profile

Positive Bilinear Systems: application areas

Positive Bilinear Systems have been fruitfully employed in a variety of application areas:

- system biology
  a Positive Bilinear System can be adopted to describe the behaviour of a group of self-replicating genotypes
  $\Rightarrow$ determing the optimal control means determing the best drugs concentration profile
- network theory
  optimal control issues concern the proper selection of leaders in a directed graph

Compartmental Systems: definition

Compartmental Systems represent physical systems whose describing variables are intrinsically nonnegative and obey some conservation law (e.g., mass, energy, fluid).

Compartmental Systems: definition

Compartmental Systems represent physical systems whose describing variables are intrinsically nonnegative and obey some conservation law (e.g., mass, energy, fluid).

A *Positive System*

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

is a Compartmental System if the Metzler matrix A is such that the entries of each of its columns sum up to a nonpositive number

$$\mathbf{1}_n^\top A \leq 0,$$

i.e., A is a compartmental matrix.

Compartmental Systems: motivations

Compartmental Systems arise in the description of a good number of dynamical processes:

Compartmental Systems: motivations

Compartmental Systems arise in the description of a good number of dynamical processes:

- liquids flowing in a network of interconnected tanks

Compartmental Systems: motivations

Compartmental Systems arise in the description of a good number of dynamical processes:

- liquids flowing in a network of interconnected tanks
- time evolution of temperatures in adjacent rooms

Compartmental Systems: motivations

Compartmental Systems arise in the description of a good number of dynamical processes:

- liquids flowing in a network of interconnected tanks
- time evolution of temperatures in adjacent rooms
- kinetics of substances in tracer experiments

Compartmental Systems: motivations

Compartmental Systems arise in the description of a good number of dynamical processes:

- liquids flowing in a network of interconnected tanks
- time evolution of temperatures in adjacent rooms
- kinetics of substances in tracer experiments
- cell proliferation through mitosis

Compartmental Systems: motivations

Compartmental Systems arise in the description of a good number of dynamical processes:

- liquids flowing in a network of interconnected tanks
- time evolution of temperatures in adjacent rooms
- kinetics of substances in tracer experiments
- cell proliferation through mitosis
- physiological processes, e.g., insulin secretion, glucose absorption

Compartmental Switched Systems

Compartmental *Switched* Systems are adopted to describe the behaviour, under *different operating conditions*, of systems modeling the exchange of material between different compartments.

Compartmental Switched Systems

Compartmental *Switched* Systems are adopted to describe the behaviour, under *different operating conditions*, of systems modeling the exchange of material between different compartments.

A Compartmental Switched System is described by

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}u(t)$$

Compartmental Switched Systems

Compartmental *Switched* Systems are adopted to describe the behaviour, under *different operating conditions*, of systems modeling the exchange of material between different compartments.

A Compartmental Switched System is described by

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}u(t)$$

and consists of:

- a family of compartmental models (the subsystems) (A_i, B_i), $i \in [1, M]$

Compartmental Switched Systems

Compartmental *Switched* Systems are adopted to describe the behaviour, under *different operating conditions*, of systems modeling the exchange of material between different compartments.

A Compartmental Switched System is described by

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}u(t)$$

and consists of:

- a family of compartmental models (the subsystems) (A_i, B_i), $i \in [1, M]$
- a switching function describing which of the subsystems is active at every time instant $\sigma : \mathbb{R}_+ \to [1, M]$

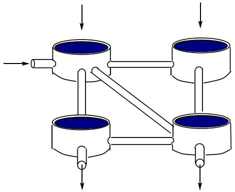
Compartmental Switched Systems: motivations

Compartmental Switched Systems can be employed to describe:

Compartmental Switched Systems: motivations

Compartmental Switched Systems can be employed to describe:

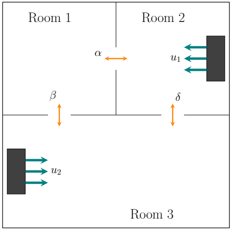
- a fluid network undergoing different open/closed configurations of the pipes connecting the tanks



Compartmental Switched Systems: motivations

Compartmental Switched Systems can be employed to describe:

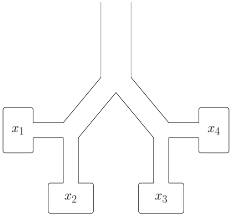
- a fluid network undergoing different open/closed configurations of the pipes connecting the tanks
- a thermal system whose heat transmission coefficients depend on the open/closed configurations of doors and windows



Compartmental Switched Systems: motivations

Compartmental Switched Systems can be employed to describe:

- a fluid network undergoing different open/closed configurations of the pipes connecting the tanks
- a thermal system whose heat transmission coefficients depend on the open/closed configurations of doors and windows
- the lung dynamics alternating between inspiration and expiration phases



Compartmental Switched Systems: stabilizability

Compartmental Switched System with *autonomous* subsystems

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad \sigma\colon \mathbb{R}_+ \to [1, M]$$

Compartmental Switched Systems: stabilizability

Compartmental Switched System with *autonomous* subsystems

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad \sigma\colon \mathbb{R}_+ \to [1, M]$$

Stabilizability:
under what conditions on the matrices A_i, $i \in [1, M]$, for every positive initial state $\mathbf{x}(0) > 0$, there exists a switching function σ that makes $\mathbf{x}(t)$ asymptotically converge to zero?

Compartmental Switched Systems: stabilizability

Compartmental Switched System with *autonomous* subsystems

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad \sigma : \mathbb{R}_+ \to [1, M]$$

Stabilizability:
under what conditions on the matrices A_i, $i \in [1, M]$, for every positive initial state $\mathbf{x}(0) > 0$, there exists a switching function σ that makes $\mathbf{x}(t)$ asymptotically converge to zero?

IFF there exists a Hurwitz convex combination of A_i, $i \in [1, M]$

Compartmental Switched Systems: stability

Compartmental Switched System with *autonomous* subsystems

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad \sigma\colon \mathbb{R}_+ \to [1, M]$$

Compartmental Switched Systems: stability

Compartmental Switched System with *autonomous* subsystems

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad \sigma\colon \mathbb{R}_+ \to [1, M]$$

Asymptotic stability under arbitrary switching:
under what conditions on the matrices A_i, $i \in [1, M]$, the state trajectory $\mathbf{x}(t)$ asymptotically converges to zero for every positive initial state $\mathbf{x}(0) > 0$ and for every switching function σ?

Compartmental Switched Systems: stability

Compartmental Switched System with *autonomous* subsystems

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad \sigma\colon \mathbb{R}_+ \to [1, M]$$

Asymptotic stability under arbitrary switching:
under what conditions on the matrices A_i, $i \in [1, M]$, the state trajectory $\mathbf{x}(t)$ asymptotically converges to zero for every positive initial state $\mathbf{x}(0) > 0$ and for every switching function σ?

IFF all matrices A_i, $i \in [1, M]$, are Hurwitz

The *non-autonomous* case

- Autonomous case: $\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t)$

Asympt. stable under arbitrary switching **IFF** A_i is Hurwitz for every $i \in [1, M]$

The *non-autonomous* case

- Autonomous case: $\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t)$

  Asympt. stable under arbitrary switching **IFF** A_i is Hurwitz for every $i \in [1, M]$

- *Non-autonomous* case: $\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}\mathbf{u}(t)$

The *non-autonomous* case

- Autonomous case: $\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t)$

  Asympt. stable under arbitrary switching **IFF** A_i is Hurwitz for every $i \in [1, M]$

- *Non-autonomous* case: $\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}\mathbf{u}(t)$

Feedback stabilization problem:
under what conditions on the pairs (A_i, B_i), $i \in [1, M]$, there exist feedback matrices K_i, $i \in [1, M]$, such that the control law $\mathbf{u}(t) = K_i\mathbf{x}(t)$ makes the closed-loop system compartmental and asymptotically stable under arbitrary switching?

The *non-autonomous* case

- Autonomous case: $\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t)$

  Asympt. stable under arbitrary switching **IFF** A_i is Hurwitz for every $i \in [1, M]$

- *Non-autonomous* case: $\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}\mathbf{u}(t)$

Feedback stabilization problem:
under what conditions on the pairs (A_i, B_i), $i \in [1, M]$, there exist feedback matrices K_i, $i \in [1, M]$, such that the control law $\mathbf{u}(t) = K_i\mathbf{x}(t)$ makes the closed-loop system compartmental and asymptotically stable under arbitrary switching?

IFF $\exists K_i$, $i \in [1, M]$, s.t. $A_i + B_iK_i$ is compartmental and Hurwitz

State-feedback stabilization problem

State-feedback stabilization problem

Consider a Compartmental System

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

State-feedback stabilization problem

Consider a Compartmental System

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

and assume that the compartmental matrix A is non-Hurwitz.

State-feedback stabilization problem

Consider a Compartmental System

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

and assume that the compartmental matrix A is non-Hurwitz.

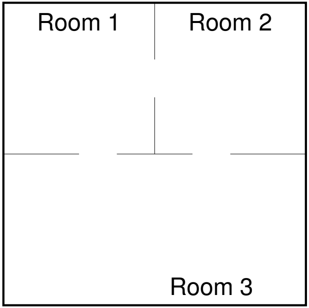
State-feedback stabilization problem:

determine $K \in \mathbb{R}^{m \times n}$ such that the state-feedback control law $\mathbf{u}(t) = K\mathbf{x}(t)$ makes the closed-loop system

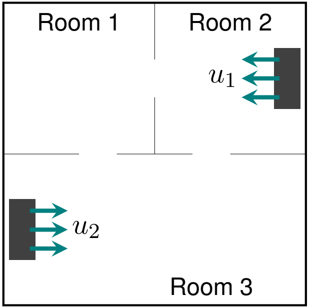
$$\dot{\mathbf{x}}(t) = (A + BK)\,\mathbf{x}(t)$$

compartmental and asymptotically stable.

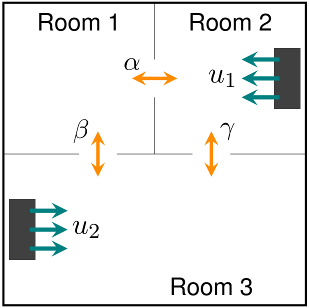
Example: room temperature regulation



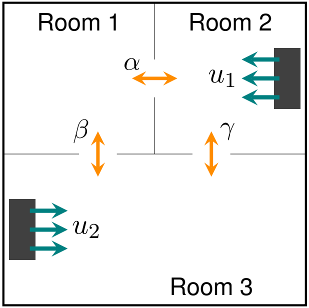
Example: room temperature regulation



Example: room temperature regulation

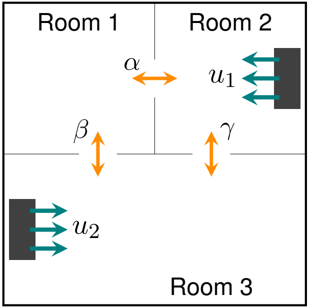


Example: room temperature regulation



Assume that the system is thermally isolated from the external environment.

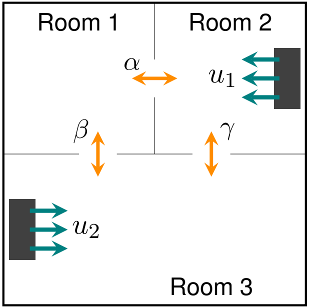
Example: room temperature regulation



Assume that the system is thermally isolated from the external environment.

$x_i = (i\text{th room temp} - \text{desired temp}) \geq 0$

Example: room temperature regulation



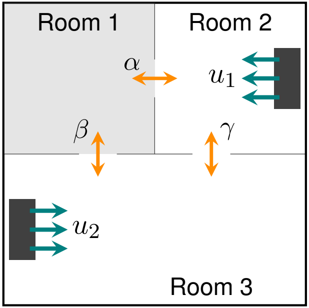
Assume that the system is thermally isolated from the external environment.

$x_i = (i\text{th room temp} - \text{desired temp}) \geq 0$

Compartmental model describing temperatures evolution:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t)\\ &= \begin{bmatrix} -(\alpha+\beta) & \alpha & \beta \\ \alpha & -(\alpha+\gamma) & \gamma \\ \beta & \gamma & -(\gamma+\beta) \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{u}(t)\end{aligned}$$

Example: room temperature regulation



Assume that the system is thermally isolated from the external environment.

$x_i = (i\text{th room temp} - \text{desired temp}) \geq 0$

Compartmental model describing temperatures evolution:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$
$$= \begin{bmatrix} -(\alpha+\beta) & \alpha & \beta \\ \alpha & -(\alpha+\gamma) & \gamma \\ \beta & \gamma & -(\gamma+\beta) \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t)$$

Room temperaturte regulation problem:
determine a state-feedback control law that regulates all temperatures by making use only of the temperature in Room 1.

Preliminary definitions

In the following we will denote by:

Preliminary definitions

In the following we will denote by:

- $\mathbf{e}_i$ the ith canonical vector

Preliminary definitions

In the following we will denote by:

- $\mathbf{e}_i$ the ith canonical vector
- $\mathbf{1}_n$ the n-dimensional vector with all entries equal to 1

Preliminary definitions

In the following we will denote by:

- $\mathbf{e}_i$ the ith canonical vector
- $\mathbf{1}_n$ the n-dimensional vector with all entries equal to 1
- $S_i \in \mathbb{R}^{(n-1)\times n}$ the selection matrix obtained by removing the ith row in the identity matrix I_n, i.e.,

$$S_i = \left[\begin{array}{c|c|c} I_{i-1} & 0 & 0 \\ \hline 0 & 0 & I_{n-i} \end{array}\right]$$

Preliminary definitions

In the following we will denote by:

- $\mathbf{e}_i$ the ith canonical vector
- $\mathbf{1}_n$ the n-dimensional vector with all entries equal to 1
- $S_i \in \mathbb{R}^{(n-1)\times n}$ the selection matrix obtained by removing the ith row in the identity matrix I_n, i.e.,

$$S_i = \left[\begin{array}{c|c|c} I_{i-1} & 0 & 0 \\ \hline 0 & 0 & I_{n-i} \end{array}\right]$$

For any vector $\mathbf{v} \in \mathbb{R}^n$, $S_i\mathbf{v}$ is the vector obtained from $\mathbf{v}$ by removing the ith entry.

Preliminary definitions

In the following we will denote by:

- $\mathbf{e}_i$ the ith canonical vector
- $\mathbf{1}_n$ the n-dimensional vector with all entries equal to 1
- $S_i \in \mathbb{R}^{(n-1)\times n}$ the selection matrix obtained by removing the ith row in the identity matrix I_n, i.e.,

$$S_i = \left[\begin{array}{c|c|c} I_{i-1} & 0 & 0 \\ \hline 0 & 0 & I_{n-i} \end{array}\right]$$

For any vector $\mathbf{v} \in \mathbb{R}^n$, $S_i\mathbf{v}$ is the vector obtained from $\mathbf{v}$ by removing the ith entry.
For any matrix $A \in \mathbb{R}^{n\times m}$, S_iA denotes the matrix obtained from A by removing the ith row.

Preliminary definitions (cont'd)

- A Metzler matrix A is reducible if there exists a permutation matrix Π such that

$$\Pi^\top A \Pi = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are square nonvacuous matrices, otherwise it is irreducible.

Frobenius normal form

For every Metzler matrix A a permutation matrix Π can be found such that

$$\Pi^\top A \Pi = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ 0 & A_{22} & \dots & A_{2s} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & A_{ss} \end{bmatrix} \tag{2}$$

where each diagonal block A_{ii}, of size $n_i \times n_i$, is either scalar ($n_i = 1$) or irreducible.

Frobenius normal form

For every Metzler matrix A a permutation matrix Π can be found such that

$$\Pi^\top A \Pi = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ 0 & A_{22} & \dots & A_{2s} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & A_{ss} \end{bmatrix} \qquad (2)$$

where each diagonal block A_{ii}, of size $n_i \times n_i$, is either scalar ($n_i = 1$) or irreducible. The block form (2) is usually known as Frobenius normal form of A.

Communication classes

Compartmental matrix $A \in \mathbb{R}^{n \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix}$$

Communication classes

Compartmental matrix $A \in \mathbb{R}^{n \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix}$$

Directed graph $\mathcal{G}(A) := \{\mathcal{V}, \mathcal{E}\}$

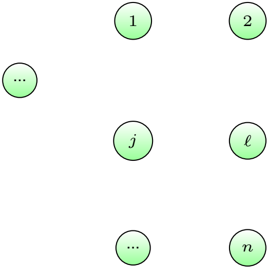
Communication classes

Compartmental matrix $A \in \mathbb{R}^{n\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix}$$

Directed graph $\mathcal{G}(A) := \{\mathcal{V}, \mathcal{E}\}$

set of vertices



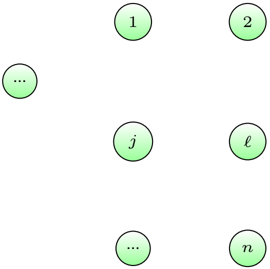
Communication classes

Compartmental matrix $A \in \mathbb{R}^{n \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix}$$

Directed graph $\mathcal{G}(A) := \{\mathcal{V}, \mathcal{E}\}$

set of edges



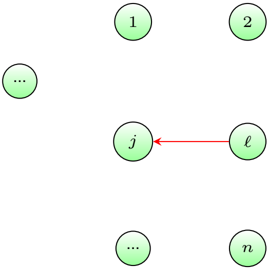
Communication classes

Compartmental matrix $A \in \mathbb{R}^{n \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & a_{j\ell} & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix}$$

Directed graph $\mathcal{G}(A) := \{\mathcal{V}, \mathcal{E}\}$

set of edges



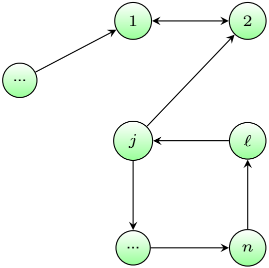
$a_{j\ell} > 0$

Communication classes

Compartmental matrix $A \in \mathbb{R}^{n \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix}$$

Directed graph $\mathcal{G}(A) := \{\mathcal{V}, \mathcal{E}\}$

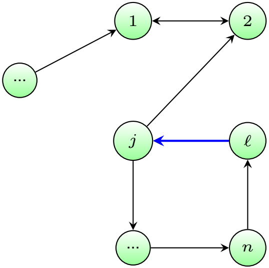


Communication classes

Compartmental matrix $A \in \mathbb{R}^{n \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix}$$

Directed graph $\mathcal{G}(A) := \{\mathcal{V}, \mathcal{E}\}$



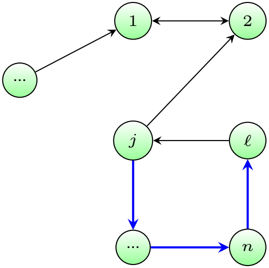
Vertex j and vertex ℓ can communicate

Communication classes

Compartmental matrix $A \in \mathbb{R}^{n\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix}$$

Directed graph $\mathcal{G}(A) := \{\mathcal{V}, \mathcal{E}\}$



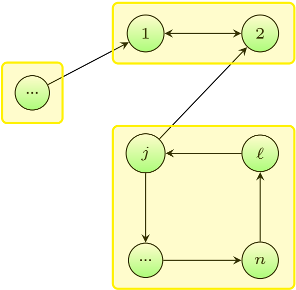
Vertex j and vertex ℓ can communicate

Communication classes

Compartmental matrix $A \in \mathbb{R}^{n \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix}$$

Directed graph $\mathcal{G}(A) := \{\mathcal{V}, \mathcal{E}\}$



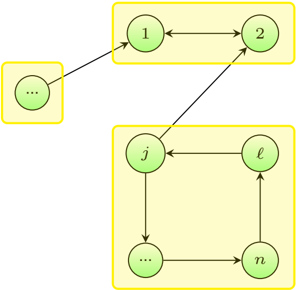
Communication class:
a set of vertices that communicate with every other vertex in the class and with no other vertex.

The corresponding subgraph is strongly connected.

Communication classes

Compartmental matrix $A \in \mathbb{R}^{n \times n}$

Directed graph $\mathcal{G}(A) := \{\mathcal{V}, \mathcal{E}\}$



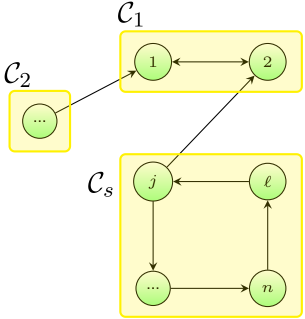
$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ 0 & A_{22} & \dots & A_{2s} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & A_{ss} \end{bmatrix}$$

Frobenius normal form of A

Communication classes

Compartmental matrix $A \in \mathbb{R}^{n \times n}$

Directed graph $\mathcal{G}(A) := \{\mathcal{V}, \mathcal{E}\}$



$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ 0 & A_{22} & \dots & A_{2s} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & A_{ss} \end{bmatrix}$$

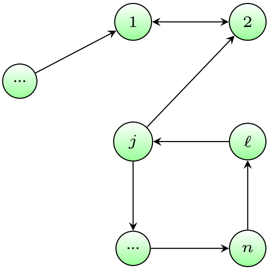
Frobenius normal form of A

Hurwitz stability

Compartmental matrix $A \in \mathbb{R}^{n\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix}$$

Directed graph $\mathcal{G}(A) := \{\mathcal{V}, \mathcal{E}\}$



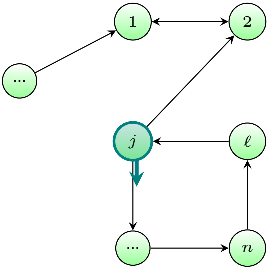
Hurwitz stability

Compartmental matrix $A \in \mathbb{R}^{n \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1j} & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & a_{nj} & a_{nn} \end{bmatrix}$$

Directed graph $\mathcal{G}(A) := \{\mathcal{V}, \mathcal{E}\}$

$$\mathbf{1}_n^\top A \mathbf{e}_j < 0$$



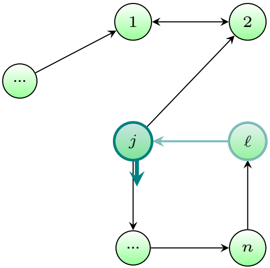
Vertex j has outflow to the environment

Hurwitz stability

Compartmental matrix $A \in \mathbb{R}^{n\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1j} & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & a_{nj} & a_{nn} \end{bmatrix}$$

Directed graph $\mathcal{G}(A) := \{\mathcal{V}, \mathcal{E}\}$



$$\mathbf{1}_n^\top A \mathbf{e}_j < 0$$

Vertex j has outflow to the environment

Vertex ℓ is outflow connected

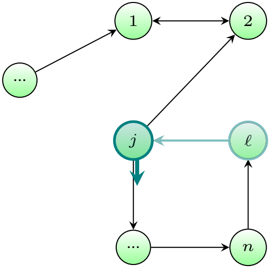
Hurwitz stability

Compartmental matrix $A \in \mathbb{R}^{n\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1j} & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & a_{nj} & a_{nn} \end{bmatrix}$$

$$\mathbf{1}_n^\top A \mathbf{e}_j < 0$$

Directed graph $\mathcal{G}(A) := \{\mathcal{V}, \mathcal{E}\}$



Lemma 1:
A compartmental matrix A is Hurwitz if and only if all compartments are outflow connected.

The case when A is irreducible

Proposition 1:

Consider a Compartmental System and assume that the state-space matrix A is *irreducible* and non-Hurwitz.

The case when A is irreducible

Proposition 1:

Consider a Compartmental System and assume that the state-space matrix A is *irreducible* and non-Hurwitz.
The state-feedback stabilization problem is solvable if and only if there exist $h \in [1, n]$ and $\mathbf{v} \in \mathbb{R}^m$ such that

$$S_h \left(A\mathbf{e}_h + B\mathbf{v} \right) \geq 0 \tag{3}$$

$$\mathbf{1}_n^\top B\mathbf{v} < 0 \tag{4}$$

The case when A is irreducible

Proposition 1:

Consider a Compartmental System and assume that the state-space matrix A is *irreducible* and non-Hurwitz.
The state-feedback stabilization problem is solvable if and only if there exist $h \in [1, n]$ and $\mathbf{v} \in \mathbb{R}^m$ such that

$$S_h \left(A\mathbf{e}_h + B\mathbf{v} \right) \geq 0 \qquad (3)$$

$$\mathbf{1}_n^\top B\mathbf{v} < 0 \qquad (4)$$

When so, any matrix $K = \varepsilon \mathbf{v}\mathbf{e}_h^\top$ with $\varepsilon \in (0, 1)$ is a possible solution.

The case when A is irreducible

Proposition 1:

Consider a Compartmental System and assume that the state-space matrix A is *irreducible* and non-Hurwitz.
The state-feedback stabilization problem is solvable if and only if there exist $h \in [1, n]$ and $\mathbf{v} \in \mathbb{R}^m$ such that

compartmental property

$$S_h \left(A\mathbf{e}_h + B\mathbf{v} \right) \geq 0 \quad (3)$$

$$\mathbf{1}_n^\top B\mathbf{v} < 0 \quad (4)$$

When so, any matrix $K = \varepsilon \mathbf{v}\mathbf{e}_h^\top$ with $\varepsilon \in (0, 1)$ is a possible solution.

The case when A is irreducible

Proposition 1:

Consider a Compartmental System and assume that the state-space matrix A is *irreducible* and non-Hurwitz.
The state-feedback stabilization problem is solvable if and only if there exist $h \in [1, n]$ and $\mathbf{v} \in \mathbb{R}^m$ such that

compartmental property
$$S_h \left(A\mathbf{e}_h + B\mathbf{v} \right) \geq 0 \qquad (3)$$

outflow to environment
$$\mathbf{1}_n^\top B\mathbf{v} < 0 \qquad (4)$$

When so, any matrix $K = \varepsilon \mathbf{v}\mathbf{e}_h^\top$ with $\varepsilon \in (0, 1)$ is a possible solution.

The case when A is irreducible

Proposition 1:

Consider a Compartmental System and assume that the state-space matrix A is *irreducible* and non-Hurwitz.
The state-feedback stabilization problem is solvable if and only if there exist $h \in [1, n]$ and $\mathbf{v} \in \mathbb{R}^m$ such that

compartmental property $$S_h \left(A\mathbf{e}_h + B\mathbf{v} \right) \geq 0 \qquad (3)$$

outflow to environment $$\mathbf{1}_n^\top B\mathbf{v} < 0 \qquad (4)$$

When so, any matrix $K = \varepsilon \mathbf{v} \mathbf{e}_h^\top$ with $\varepsilon \in (0, 1)$ is a possible solution.

graph (strong) connectedness

The case when A is irreducible

Proposition 1:

Consider a Compartmental System and assume that the state-space matrix A is *irreducible* and non-Hurwitz.
The state-feedback stabilization problem is solvable if and only if there exist $h \in [1, n]$ and $\mathbf{v} \in \mathbb{R}^m$ such that

$$S_h\left(A\mathbf{e}_h + B\mathbf{v}\right) \geq 0 \tag{3}$$

$$\mathbf{1}_n^\top B\mathbf{v} < 0 \tag{4}$$

When so, any matrix $K = \varepsilon \mathbf{v}\mathbf{e}_h^\top$ with $\varepsilon \in (0, 1)$ is a possible solution.

Every compartment is outflow connected!

Remarks

- When A is *irreducible* and the state-feedback stabilization problem is solvable, there always exists a solution K with a unique nonzero column.

Remarks

- When A is *irreducible* and the state-feedback stabilization problem is solvable, there always exists a solution K with a unique nonzero column. However, not all columns (i.e., not all indices $h \in [1, n]$) play an equivalent role.

Remarks

- When A is *irreducible* and the state-feedback stabilization problem is solvable, there always exists a solution K with a unique nonzero column. However, not all columns (i.e., not all indices $h \in [1, n]$) play an equivalent role.
- In the Single-Input case ($m = 1$), problem solvability does not depend on the specific values of the entries of A and B but only on their nonzero pattern.
  The nonzero pattern of a matrix $A \in \mathbb{R}^{n \times m}$ is the set $\overline{\mathrm{ZP}}(A) := \{(i, j) \in [1, n] \times [1, m] \colon [A_{ij}] \neq 0\}$.

The case when A is reducible

Proposition 2:

Consider a Compartmental System and assume that the state-space matrix A is *reducible* and non-Hurwitz.

The case when A is reducible

Proposition 2:

Consider a Compartmental System and assume that the state-space matrix A is *reducible* and non-Hurwitz.
Assume w.l.o.g. that A is in Frobenius normal form:

$$A = \begin{bmatrix} A_{11} & \dots & 0 & \dots & A_{1s} \\ 0 & \ddots & \vdots & & \\ \vdots & & A_{rr} & & A_{rs} \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & & 0 & A_{ss} \end{bmatrix}$$

The case when A is reducible

Proposition 2:

Consider a Compartmental System and assume that the state-space matrix A is *reducible* and non-Hurwitz.
Assume w.l.o.g. that A is in Frobenius normal form:

$$A = \begin{bmatrix} A_{11} & \dots & 0 & \dots & A_{1s} \\ 0 & \ddots & \vdots & & \\ \vdots & & A_{rr} & & A_{rs} \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & & 0 & A_{ss} \end{bmatrix}$$

A_{ii}, $i \in [1, r]$, irreducible non-Hurwitz

The case when A is reducible

Proposition 2:

Consider a Compartmental System and assume that the state-space matrix A is *reducible* and non-Hurwitz.
Assume w.l.o.g. that A is in Frobenius normal form:

$$A = \begin{bmatrix} A_{11} & \dots & 0 & \dots & A_{1s} \\ 0 & \ddots & \vdots & & \\ \vdots & & A_{rr} & & A_{rs} \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & & 0 & A_{ss} \end{bmatrix}$$

$A_{ii}, i \in [1, r]$, irreducible non-Hurwitz

$A_{ii}, i \in [r+1, s]$, irreducible Hurwitz

The case when A is reducible

Proposition 2:

Consider a Compartmental System and assume that the state-space matrix A is *reducible* and non-Hurwitz.
Assume w.l.o.g. that A is in Frobenius normal form:

$$A = \begin{bmatrix} A_{11} & \dots & 0 & \dots & A_{1s} \\ 0 & \ddots & \vdots & & \\ \vdots & & A_{rr} & & A_{rs} \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & & 0 & A_{ss} \end{bmatrix}$$

A_{ii}, $i \in [1, r]$, irreducible non-Hurwitz

A_{ii}, $i \in [r+1, s]$, irreducible Hurwitz

Let Ω_i, $i \in [1, s]$, be the set of all column indices corresponding to A_{ii}.

The case when A is reducible (cont'd)

If for every $i \in [1, r]$ there exist $\ell_i \in \Omega_i$ and $\mathbf{v}_i \in \mathbb{R}^m$ such that

$$S_{\ell_i}\left(A\mathbf{e}_{\ell_i} + B\mathbf{v}_i\right) \geq 0 \tag{5}$$

$$\mathbf{1}_n^\top B\mathbf{v}_i < 0 \tag{6}$$

then the state-feedback stabilization problem is solvable.

The case when A is reducible (cont'd)

If for every $i \in [1, r]$ there exist $\ell_i \in \Omega_i$ and $\mathbf{v}_i \in \mathbb{R}^m$ such that

$$S_{\ell_i}\left(A\mathbf{e}_{\ell_i} + B\mathbf{v}_i\right) \geq 0 \tag{5}$$

$$\mathbf{1}_n^\top B\mathbf{v}_i < 0 \tag{6}$$

then the state-feedback stabilization problem is solvable.
Moreover, any matrix

$$K := \varepsilon \sum_{i=1}^{r} \mathbf{v}_i \mathbf{e}_{\ell_i}^\top, \quad \varepsilon \in (0, 1)$$

is a possible solution.

The case when A is reducible (cont'd)

If for every $i \in [1, r]$ there exist $\ell_i \in \Omega_i$ and $\mathbf{v}_i \in \mathbb{R}^m$ such that

compartmental property

$$S_{\ell_i} \left(A\mathbf{e}_{\ell_i} + B\mathbf{v}_i \right) \geq 0 \quad (5)$$

$$\mathbf{1}_n^\top B\mathbf{v}_i < 0 \quad (6)$$

then the state-feedback stabilization problem is solvable.
Moreover, any matrix

$$K := \varepsilon \sum_{i=1}^{r} \mathbf{v}_i \mathbf{e}_{\ell_i}^\top, \quad \varepsilon \in (0, 1)$$

is a possible solution.

The case when A is reducible (cont'd)

If for every $i \in [1, r]$ there exist $\ell_i \in \Omega_i$ and $\mathbf{v}_i \in \mathbb{R}^m$ such that

compartmental property $$S_{\ell_i}\left(A\mathbf{e}_{\ell_i} + B\mathbf{v}_i\right) \geq 0 \quad (5)$$

outflow to environment $$\mathbf{1}_n^\top B\mathbf{v}_i < 0 \quad (6)$$

then the state-feedback stabilization problem is solvable.
Moreover, any matrix

$$K := \varepsilon \sum_{i=1}^{r} \mathbf{v}_i \mathbf{e}_{\ell_i}^\top, \quad \varepsilon \in (0, 1)$$

is a possible solution.

The case when A is reducible (cont'd)

If for every $i \in [1, r]$ there exist $\ell_i \in \Omega_i$ and $\mathbf{v}_i \in \mathbb{R}^m$ such that

compartmental property $$S_{\ell_i}\left(A\mathbf{e}_{\ell_i} + B\mathbf{v}_i\right) \geq 0 \quad (5)$$

outflow to environment $$\mathbf{1}_n^\top B\mathbf{v}_i < 0 \quad (6)$$

then the state-feedback stabilization problem is solvable.

Moreover, any matrix

$$K := \varepsilon \sum_{i=1}^{r} \mathbf{v}_i \mathbf{e}_{\ell_i}^\top, \quad \boxed{\varepsilon \in (0, 1)}$$

is a possible solution. subgraph (strong) connectedness

The case when A is reducible (cont'd)

If for every $i \in [1, r]$ there exist $\ell_i \in \Omega_i$ and $\mathbf{v}_i \in \mathbb{R}^m$ such that

$$S_{\ell_i}\left(A\mathbf{e}_{\ell_i} + B\mathbf{v}_i\right) \geq 0 \quad (5)$$

$$\mathbf{1}_n^\top B\mathbf{v}_i < 0 \quad (6)$$

then the state-feedback stabilization problem is solvable.
Moreover, any matrix

$$K := \varepsilon \sum_{i=1}^r \mathbf{v}_i \mathbf{e}_{\ell_i}^\top, \quad \varepsilon \in (0,1)$$

is a possible solution.

Every compartment is outflow connected!

Remarks

Let r be the number of non-Hurwitz communication classes.

Remarks

Let r be the number of non-Hurwitz communication classes.

- The previous proposition provides only a *sufficient* (but not necessary) condition for the problem solvability.

Remarks

Let r be the number of non-Hurwitz communication classes.

- The previous proposition provides only a *sufficient* (but not necessary) condition for the problem solvability. If the feedback stabilization problem is solvable, the minimum number of nonzero columns of any solution K is *at least* r but it might be greater.

Remarks

Let r be the number of non-Hurwitz communication classes.

- The previous proposition provides only a *sufficient* (but not necessary) condition for the problem solvability. If the feedback stabilization problem is solvable, the minimum number of nonzero columns of any solution K is *at least* r but it might be greater.
  → Algorithm that allows to assess problem solvability and provides a solution whenever it exists.

Remarks

Let r be the number of non-Hurwitz communication classes.

- The previous proposition provides only a *sufficient* (but not necessary) condition for the problem solvability. If the feedback stabilization problem is solvable, the minimum number of nonzero columns of any solution K is *at least* r but it might be greater.
  → Algorithm that allows to assess problem solvability and provides a solution whenever it exists.
- In the Single-Input case ($m = 1$) a necessary condition for the problem solvability is that $r = 1$.

Conclusions

- Positive Systems provide an effective mathematical description of a variety of real phenomena/processes.

Think positive, be positive!

Conclusions

- Positive Systems provide an effective mathematical description of a variety of real phenomena/processes.
- Positivity constraint makes it possible to tackle many control issues by resorting to ad hoc tools and techniques that in the general case cannot be employed.

Think positive, be positive!

Thank you!

Prof. Maria Elena Valcher

Prof. Anders Rantzer