A variational integrators approach to second order modeling and identification of linear mechanical systems

Mattia Bruschetta
Department of Information Engineering
Università degli Studi di Padova

Advisor : **Giorgio Picci**

Ph.D. thesis defense

Introduction

Linear mechanical systems

Linear mechanical systems are typically modeled by a second-order matrix differential equation

$$M\ddot{q} + D\dot{q} + Kq = Bu\,,$$

where u is the external forcing and $M, D, K \in \mathbb{R}^{n\times n}$ satisfy

$$\begin{aligned} M &= M^T > 0 && \text{Inertia} \\ D &= D^T \geq 0 && \text{Damping} \\ K &= K^T > 0\,. && \text{Stiffness} \end{aligned}$$

Today: $B = I$ (fully actuated system) and $y = q$ (fully observed system) Can treat more general situations.

Problem statement

The identification of second order models from *sampled* force/displacement measurements can be a challenging task. Reasons:

- Want accurate estimates of the *continuous-time* model parameters (M, D, K) from sampled data.
- State of the art identification software is *discrete-time:* leads to discrete time models

$$
\begin{aligned}
x_{k+1} &= F x_k + G u_k \, , \\
q_k &= H x_k + J u_k \, .
\end{aligned}
$$

- How to get reliable estimates of (M, D, K) from identified (H, F, G, J) ?
  Non-trivial problem.

Main difficulties

This is a well known challenging task, in fact we must deal with:

- **deteriorating** estimation for very high sampling frequency.

Main difficulties

This is a well known challenging task, in fact we must deal with:

- **deteriorating** estimation for very high sampling frequency.
- **huge number of parameters** to be estimated (high dimensional MIMO systems);
  - In continuous time we have Hamiltonian structure and a series of properties (**passivity**);
  - What for discrete domain? We would like a “discrete Hamiltonian structure”.

Main difficulties

This is a well known challenging task, in fact we must deal with:

- **deteriorating** estimation for very high sampling frequency.
- **huge number of parameters** to be estimated (high dimensional MIMO systems);
  - In continuous time we have Hamiltonian structure and a series of properties (**passivity**);
  - What for discrete domain? We would like a “discrete Hamiltonian structure”.
- **Ill conditioned** transformation from discrete to continuous (d2c) time model (matricial $\log$);

Main difficulties: Ill conditioning

Let h be the sampling period ($t = kh,\ k = \ldots, -1, 0, 1, 2\ldots$)

- Discrete-to-Continuous conversion. Imagine ZOH: recover matrices A and B of a continuous time model, from estimates of (F, G) of a discrete time model. Invert (MATLAB d2c)

$$F = \exp Ah, \qquad G = A^{-1}(I - \exp Ah)\, B$$

Main difficulties: Ill conditioning

Let h be the sampling period ($t = kh,\ k = \ldots, -1, 0, 1, 2 \ldots$)

- Discrete-to-Continuous conversion. Imagine ZOH: recover matrices A and B of a continuous time model, from estimates of (F, G) of a discrete time model. Invert (MATLAB d2c)

$$F = \exp Ah, \qquad G = A^{-1}(I - \exp Ah)\, B$$

- Ill-conditioning. No matter of how accurate the discrete-time estimates, the cont. time estimates can be very bad. Ex: Scalar case,

$$A + \delta A = 1/h \log(F + \delta F)\,,$$
$$\frac{\delta A}{A} = \frac{1}{logF} \frac{\delta F}{F}$$

For $h \to 0$, $F \to I$ and $\frac{1}{logF} \to \infty$. Errors are amplified!

Key point: Discretization

Want: discrete models of linear mechanical systems for which the discrete-to-continuous reconstruction map

$$\varphi_h : (H,\, F,\, G,\, J) \;\to (M,\, D,\, K)$$

- guarantees Hamiltonian (second order) structure
- is well conditioned (no dramatic error amplification)

Keep in mind that the map φ_h is always *approximate*. Ex Euler discretization

$$F = I + Ah \qquad G = Bh$$

has well-conditioned inverse but is generally inaccurate.

Discrete Mechanics (Variational Integrators)

Discrete Mechanics

General idea: Instead of discretizing the equations of motion, derive discrete equations from a *discrete variational principle*.
In continuous time, Lagrange-D'Alembert principle (virtual works) for a system with external forces $f(q, \dot{q}, t)$ is

$$\delta \int_{t_0}^{t_f} L(q, \dot{q})\, dt + \int_{t_0}^{t_f} f(q, q, t)\, \delta q(t)\, dt = 0\,.$$

for arbitrary variations with *fixed* end points.
This leads to forced Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = f(q, \dot{q}).$$

Exact discrete Lagrangian

Let $q_k = q(kh)\,,\ \ k \in [0, N]$ sampled trajectory solution of EL eqns.

$$L_d^E(q_k, q_{k+1}, h) := \int_{kh}^{(k+1)h} L(q(t), \dot{q}(t))dt\,,$$

$$f_d^{E-}(q_k, q_{k+1}, h) := \int_{kh}^{(k+1)h} f(q(t), \dot{q}(t))\frac{\partial q}{\partial q_k}(t)dt\,,$$

$$f_d^{E+}(q_k, q_{k+1}, h) := \int_{kh}^{(k+1)h} f(q(t), \dot{q}(t))\frac{\partial q}{\partial q_{k+1}}(t)dt\,.$$

where $q(\cdot)$ is the solution of the forced Euler-Lagrange equations such that $q(kh) = q_k$ and $q((k+1)h) = q_{k+1}$.

The Discrete Euler-Lagrange Equations

Discrete Lagrange-D'Alembert principle *A continuous variation* $\delta q(t)$ *is replaced by arbitrary discrete variations* $\{\delta q_k\}_{k=0,\ldots,N}$.

$$\delta \sum_{k=0}^{N-1} L_d^E(q_k, q_{k+1}, h) + \sum_{k=1}^{N-1} \left(f^{E+}(q_{k-1}, q_k, h) + f^{E-}(q_k, q_{k+1}, h) \right) \delta q_k = 0\,. \quad (1)$$

The Discrete Euler-Lagrange Equations

Discrete Lagrange-D'Alembert principle *A continuous variation* $\delta q(t)$ *is replaced by arbitrary discrete variations* $\{\delta q_k\}_{k=0,\ldots,N}$.

$$\delta \sum_{k=0}^{N-1} L_d^E(q_k, q_{k+1}, h) + \sum_{k=1}^{N-1} \left(f^{E+}(q_{k-1}, q_k, h) + f^{E-}(q_k, q_{k+1}, h) \right) \delta q_k = 0 \,. \quad (1)$$

Leads to **Discrete Euler-Lagrange (DEL) equations**

$$\mathbf{D}_2 L_d^E(q_{k-1}, q_k, h) + \mathbf{D}_1 L_d^E(q_k, q_{k+1}, h) + f^{E+}(q_{k-1}, q_k, h) + f^{E+}(q_k, q_{k+1}, h) = 0 \,, \quad (2)$$

which can be interpreted as an algorithm (variational integrator) $\mathrm{DEL} : (q_k, q_{k+1}) \mapsto (q_{k+1}, q_{k+2})$.

Approximate *discrete Lagrangian*

Exact discrete Lagrangian + *Exact* discrete forces impossible to compute in general.

$$L_d(q_k, q_{k+1}, h) \approx \int_{kh}^{(k+1)h} L(q(t), \dot{q}(t))d\,,$$

$$f_d^-(q_k, q_{k+1}, h) \approx \int_{kh}^{(k+1)h} f(q(t), \dot{q}(t))\frac{\partial q}{\partial q_k}(t)dt\,,$$

$$f_d^+(q_k, q_{k+1}, h) \approx \int_{kh}^{(k+1)h} f(q(t), \dot{q}(t))\frac{\partial q}{\partial q_{k+1}}(t)dt\,.$$

Approximate DEL equations

$$\mathbf{D}_2 L_d(q_{k-1}, q_k, h) + \mathbf{D}_1 L_d(q_k, q_{k+1}, h) + f^+(q_{k-1}, q_k, h) + f^-(q_k, q_{k+1}, h) = 0$$

Q: How good can we make the approximation?

Simplest Approximation: The Midpoint Rule

The *midpoint rule*: approximate integrals by trapezoidal rule

$$L_d(q_k, q_{k+1}, hk, h(k+1)) := hL\left(\frac{q_{k+1}+q_k}{2}, \frac{q_{k+1}-q_k}{h}\right)$$

$$f^-(q_k, q_{k+1}, h\,k, h\,(k+1)) := \frac{h}{2} f\left(\frac{q_{k+1}+q_k}{2}, \frac{q_{k+1}-q_k}{h}, \frac{h\,k+h\,(k+1)}{2}\right)$$

$$f^+(q_k, q_{k+1}, h\,k, h\,(k+1)) := \frac{h}{2} f\left(\frac{q_{k+1}+q_k}{2}, \frac{q_{k+1}-q_k}{h}, \frac{h\,k+h\,(k+1)}{2}\right)$$

Error on trajectory is $O(h^2)$. Can do much better by more complicated schemes.

Midpoint rule variational integrator for linear systems

Linear mechanical system with Lagrangian

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^T M \dot{q} - \frac{1}{2} q^T K q$$

and external force $f(q,\dot{q},t) = -D\dot{q} + u(t)$, $D = D^T \geq 0$,
Midpoint rule approximate DEL equations,

$$M_d\, q_{k+2} + D_d\, q_{k+1} + K_d\, q_k = f_d(k+2)$$

(Discrete Newton Law !) where

$$M_d := \left(\frac{M}{h} + \frac{hK}{4} + \frac{D}{2}\right), \quad D_d := \left(\frac{hK}{2} - \frac{2M}{h}\right), \quad K_d := \left(\frac{M}{h} + \frac{hK}{4} - \frac{D}{2}\right)$$

and

$$f_d(k+2) := h\,\frac{u(h(k+2)) + 2u(h(k+1)) + u(hk)}{4}\,. \quad \text{Discrete force}$$

Discrete-to-Continuous

The continuous time parameters (M, D, K) can be recovered by **linear relations**

$$
\begin{aligned}
M &:= \frac{h}{4}[M_d + K_d - D_d]\,, \\
D &:= M_d - K_d\ , \\
K &:= \frac{1}{h}[M_d + K_d + D_d]\,.
\end{aligned}
$$

Theoretical procedure:

1. Identify Variational integrator (Discrete Newton Law). Get estimates of $(M_d. D_d,\ K_d)$
2. Recover Continuous parameter using the above linear relations

This is precisely what we wanted to achieve.

Identification

Identification

The data set is $\{u_k, y_k\}$.Assume white measurement noise

$$y_k = q_k + w_k.$$

Want to identify

$$y_k = A_1 y_{k-1} + A_2 y_{k-2} + B f_k + e_k\,, \qquad (f \equiv f_d)$$

where

$$A_1 := -M_d^{-1} D_d\,, \qquad A_1 := -M_d^{-1} K_d\,, \qquad B := M_d^{-1}\,,$$

with $e_k = w_k - A_1 w_{k-1} - A_2 w_{k-2}$ colored noise. A vector *Output Error* model.

Nonlinear problem. Constrained PEM (`idgrey`)

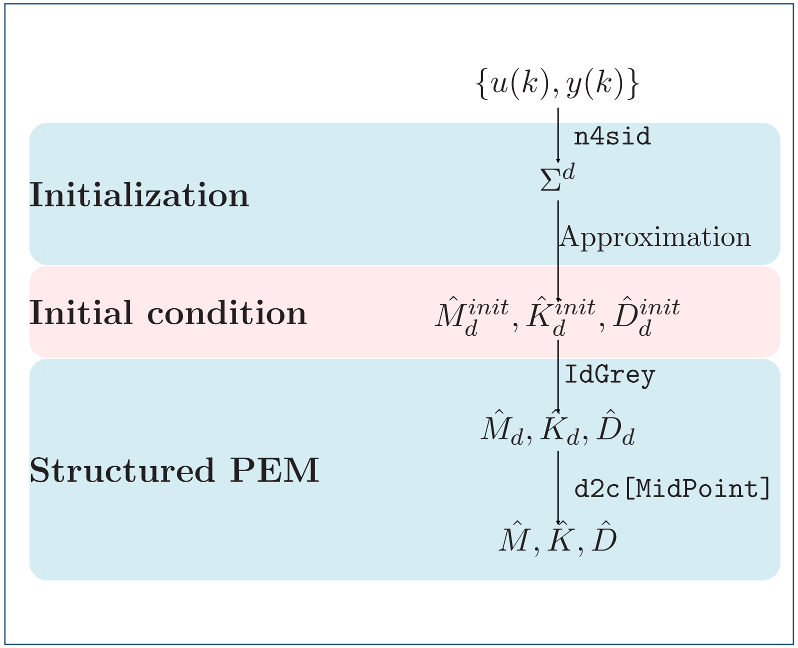
Identification method

- Identify the *output error* model via a nonlinear PEM (`idgrey`), imposing symmetry on the parameters (M_d, D_d, K_d) and Hamiltonian structure.
- Good initial condition on parameter estimates are absolutely necessary! Otherwise *local minima*!
- Initialize by running a subspace id method (`n4sid`)
- Must compute good initial estimates of (M_d, D_d, K_d) from identified discrete state-space model....(?)

Identification method: algorithm

1. Firstly we perform a subspace identification

$$\Sigma_d = \begin{cases} x_{k+1} = F x_k + G u_k \, , \\ q_k = H x_k + J u_k \, . \end{cases}$$



2. Keeping constant the system gain, with algebraic manipulation we get the initial condition

$$q_k = (\hat{M}_d^{init})^{-1} \hat{D}_d^{init} q_{k-1} + (\hat{M}_d^{init})^{-1} \hat{K}_d^{init} q_{k-2} + (\hat{M}_d^{init})^{-1} f_k$$

3. With the initial condition just computed the PEM optimization is performed;
4. Finally it is computed the conversion to continuous time domain using the just defined transformation.

Compare with `n4sid+d2c`

Compare with state of the art continuous-time parameters identification based on the 'd2c' conversion (matrix logarithm).

- The d2c conversion is applied to a discrete-time state space model identified via a subspace method (n4sid).
- Then, using a change of coordinate and projection method of Lus *et al.* (2003), one gets

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & I \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ B_{12} \end{bmatrix} u$$

$$y = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix}.$$

$$A_{21} := -M^{-1}K\,, \qquad A_{22} := -M^{-1}D\,, \qquad B_{12} := M^{-1}\,.$$

Numerical results: 3x3

$$M = \begin{bmatrix} 0.8 & 0.0 & 0.0 \\ 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 1.2 \end{bmatrix}$$

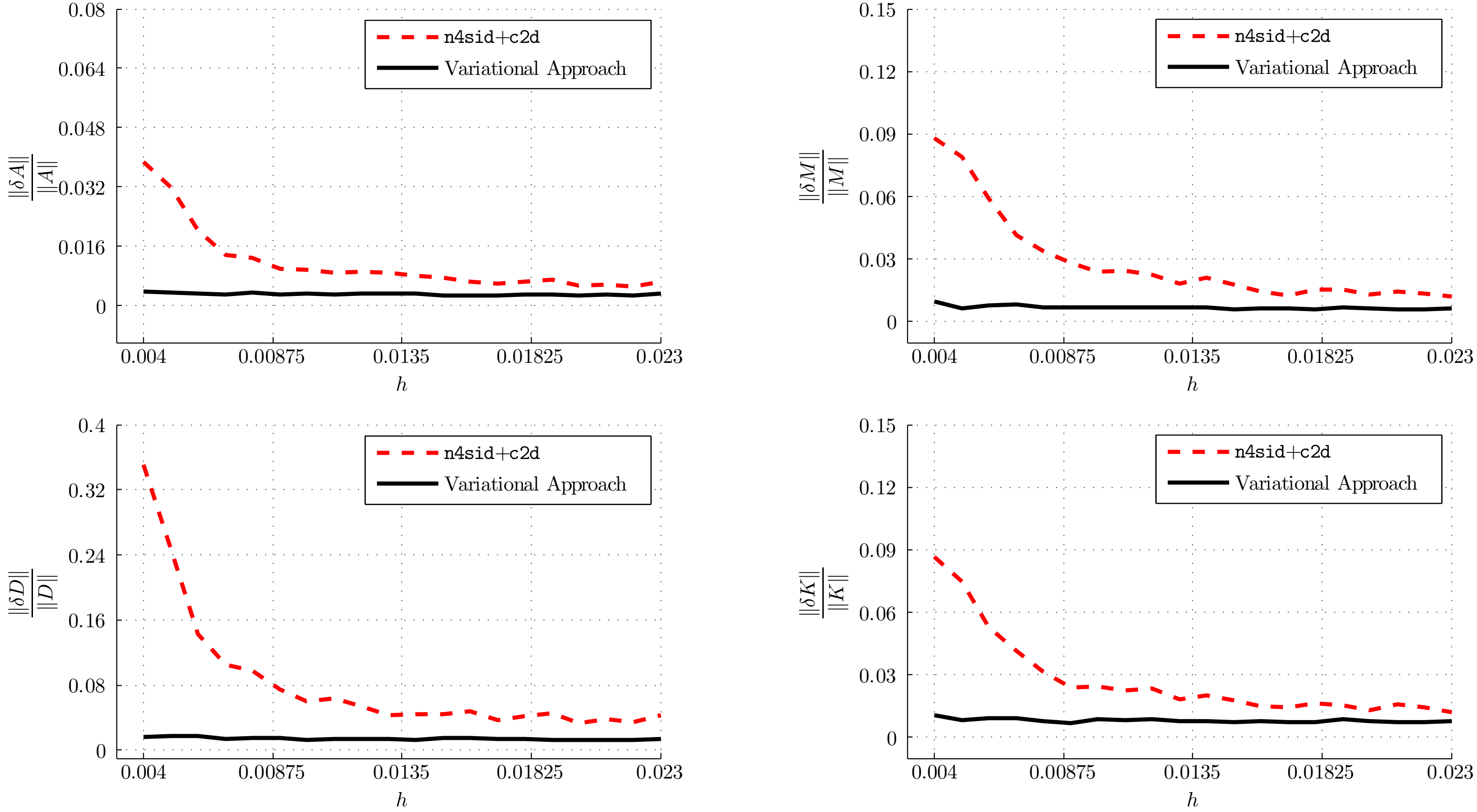
$$D = \begin{bmatrix} 0.4 & -0.1 & -0.1 \\ -0.1 & 0.4 & -0.1 \\ -0.1 & -0.1 & 0.4 \end{bmatrix}$$

$$K = \begin{bmatrix} 4.0 & -1.0 & -1.0 \\ -1.0 & 4.0 & -1.0 \\ 1.0 & 1.0 & 4.0 \end{bmatrix}$$

- 3 external forces (pseudo-random binary sequence),
- 3 sensors measuring the three positions.

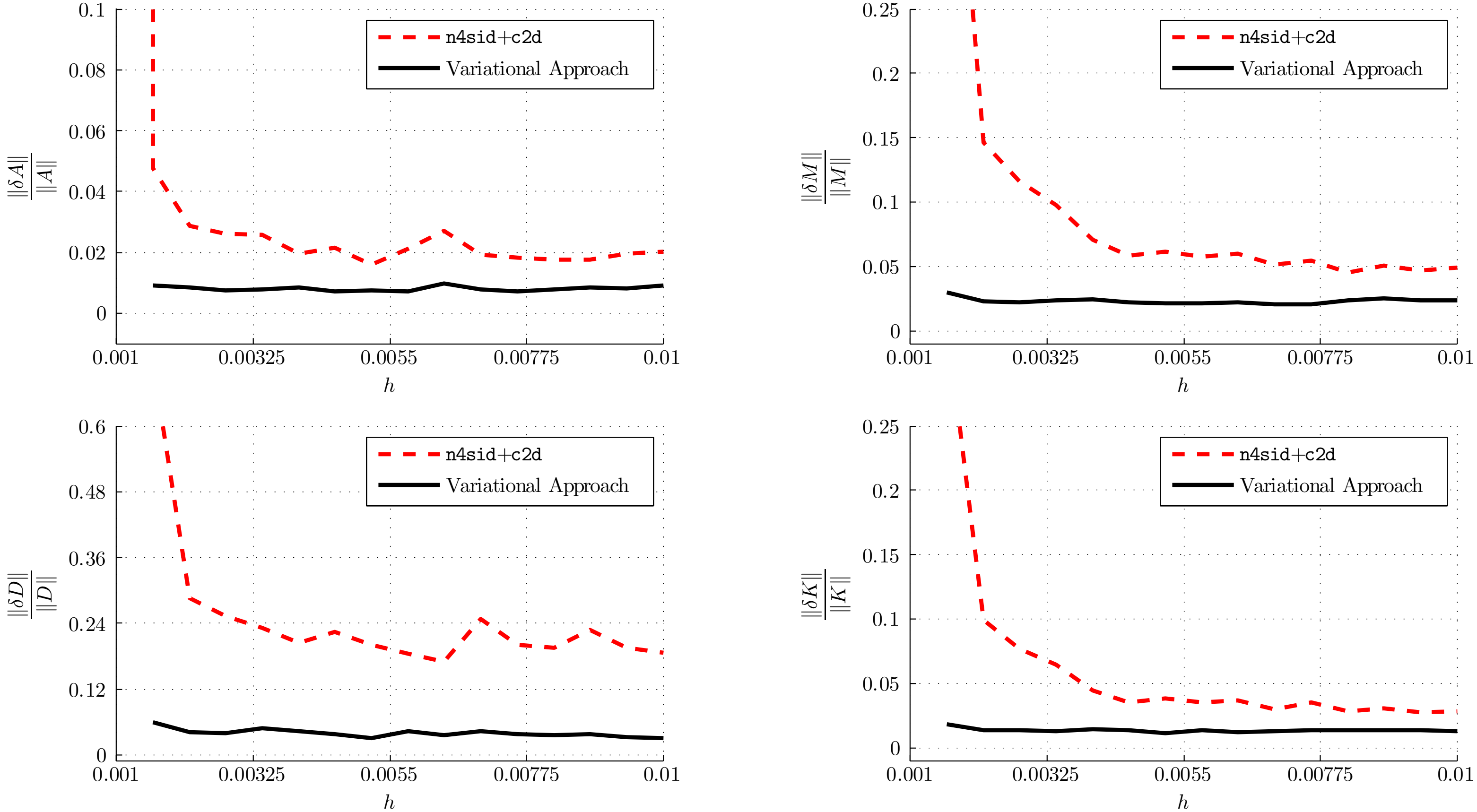
Numerical results: 3x3

Comparison of the `n4sid+d2c` procedure and the Variation approach for 3×3 system. SNR $= 15$.



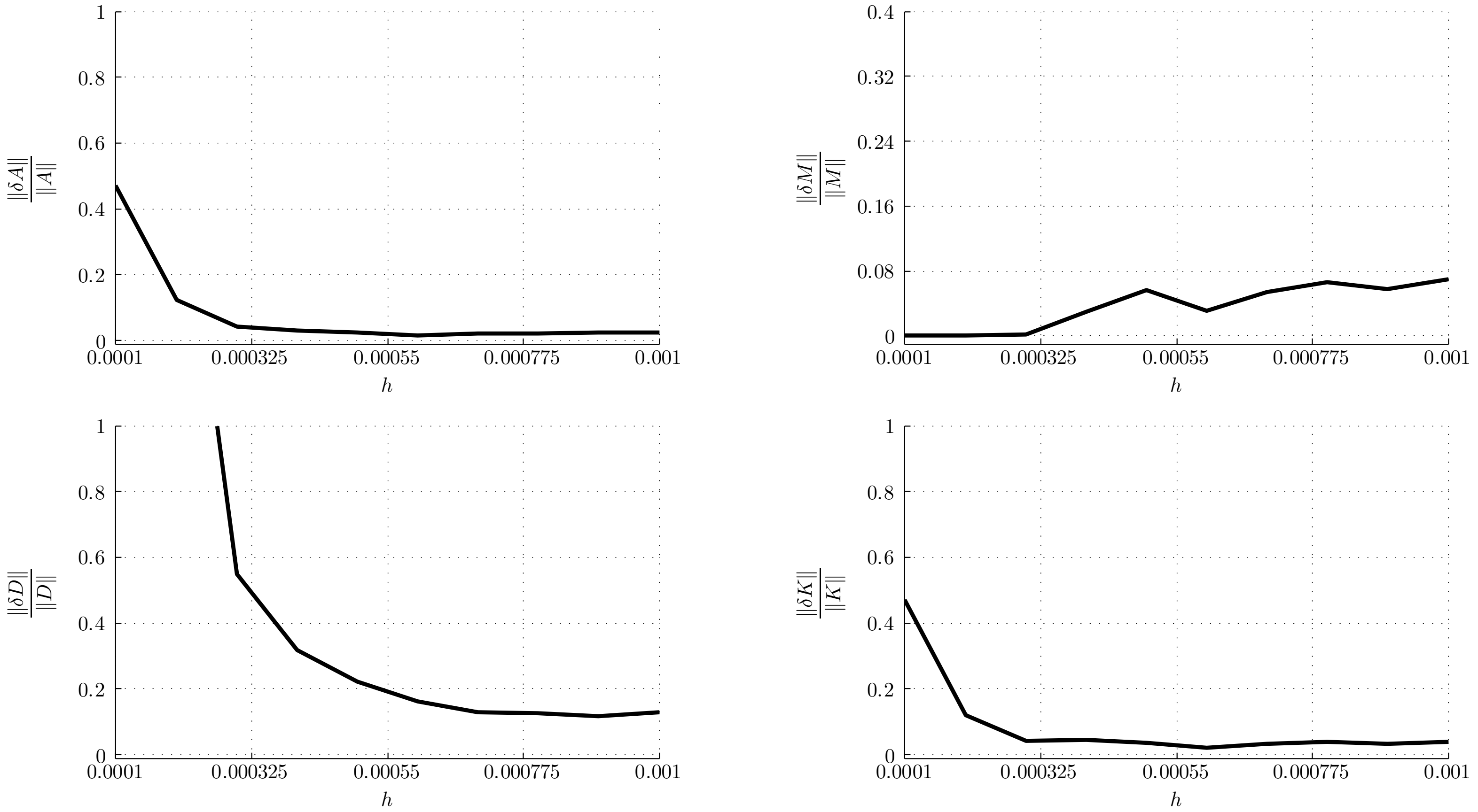
Numerical results: 8x8

Comparison of the `n4sid+d2c` procedure and the Variation approach for 8×8 system. SNR $= 25$.



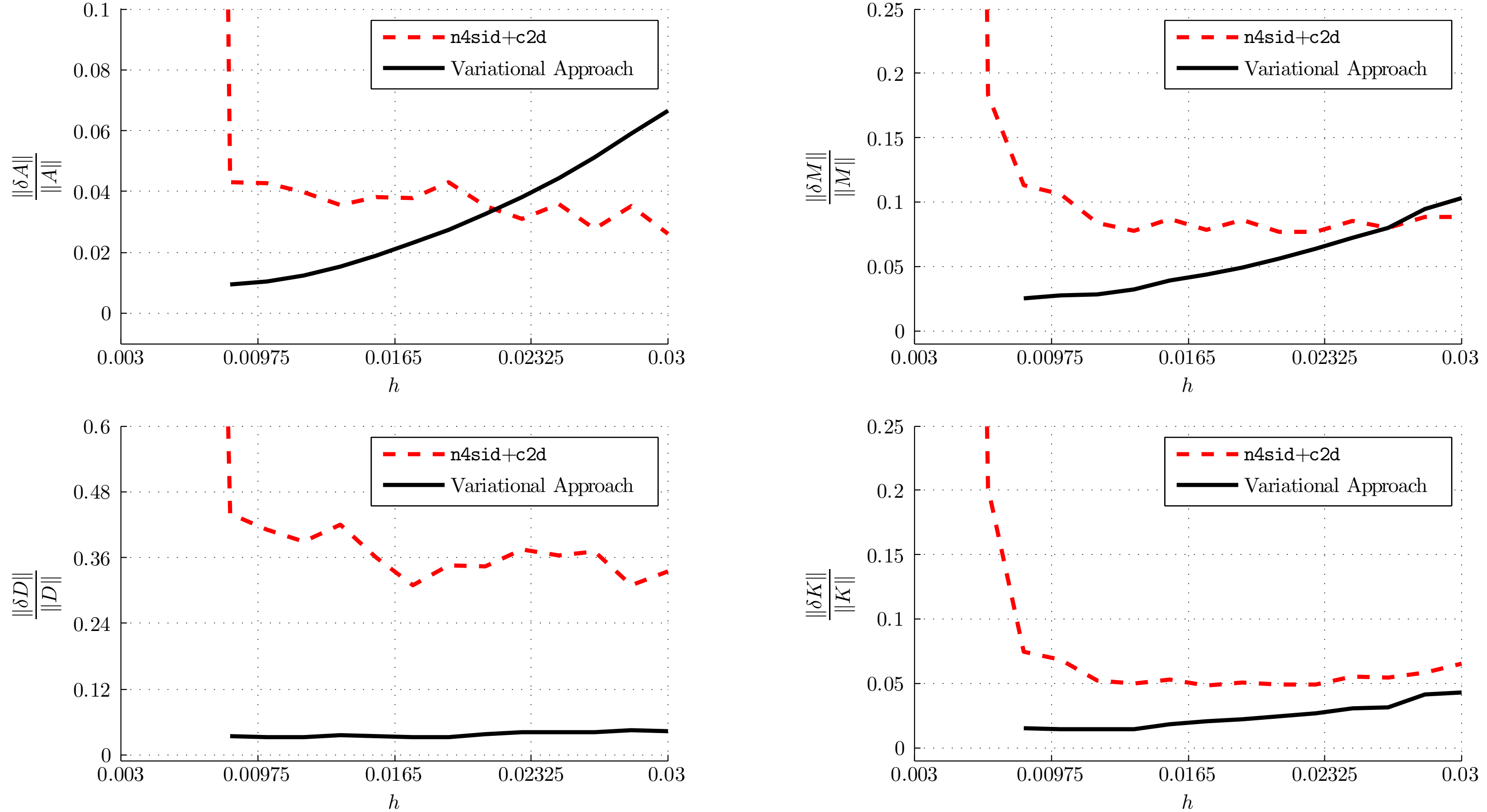
Numerical results: Ill-conditioning

Ill conditioning of the proposed approach for small values of h with the 8×8 system.



Numerical results: Approximation error

Comparison of the approximation error for large values of h with the 8×8 system.



Conclusions

Conclusions

- A new discretization procedure for second order Lagrangian equations of a linear mechanical systems has been proposed which leads to a much better recovery of the continuous time mechanical parameters than the usual discretizations.
- The proposed algorithm can deal with non fully-actuated, non fully-sensed mechanical systems: $y = Jq$, $f = J^T u$, under appropriate rank conditions.
- Variational discretization preserves symplectic structure, preserves conserved physical quantities, passivity etc.

Conclusions

- A new discretization procedure for second order Lagrangian equations of a linear mechanical systems has been proposed which leads to a much better recovery of the continuous time mechanical parameters than the usual discretizations.
- The proposed algorithm can deal with non fully-actuated, non fully-sensed mechanical systems: $y = Jq$, $f = J^T u$, under appropriate rank conditions.
- Variational discretization preserves symplectic structure, preserves conserved physical quantities, passivity etc.

[1] M. Bruschetta, G. Picci, A. Saccon. *Discrete Mechanical Systems: Second Order Modelling and Identification*. in Proceedings of SISID 2009 (Saint-Malo, France)

[2] M. Bruschetta, G. Picci, A. Saccon. *A Variational Integrators Approach to Second Order Modeling and Identification of Linear Mechanical Systems*. Submitted to Automatica.

End

Thank you for your attention!