Condizioni per la dicotomia esponenziale e sue applicazioni al tracking esatto per i sistemi non lineari a fase non minima

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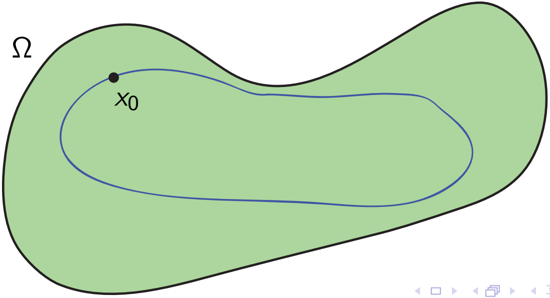
Problem

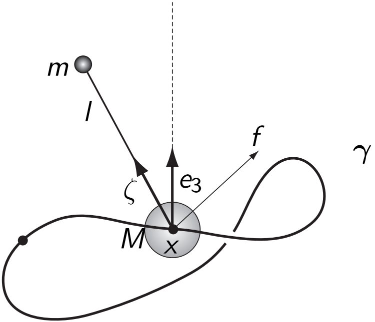
Find sufficient conditions for the existence of a T-periodic solution for system

$$\begin{aligned} \dot{x} &= f(x,t) \\ x(0) &= x_0 , \end{aligned} \tag{1}$$

on a compact set Ω and find a bound on its norm.

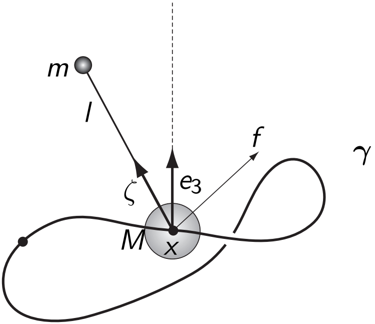


An example: the spherical pendulum



- Spherical inverted pendulum of mass m linked to a moving base of mass M through a massless rod of length l
- force $f \in \mathbb{R}^3$ is applied on the center of mass x of M.

An example: the spherical pendulum



Problem

Given a T-periodic curve $\gamma \in C^3(\mathbb{R}, \mathbb{R}^3)$, *find a control force* $f \in \mathcal{C}(\mathbb{R}, \mathbb{R}^3)$, *applied to the point* x, *such that if* $x(0) = \gamma(0)$, *then* $x(t) = \gamma(t)$, $\forall t \geq 0$ *and* $\|\zeta - e_3\|$ *is sufficiently small, where* $e_3 = (0, 0, 1)^T$.

Model derivation

- $q = (x, \zeta) \in \mathbb{R}^3 \times S^2$ is the vector of generalized coordinates, x is the position of the center of mass, ζ the orientation versor of the rod.
- The Lagrangian is $L = T - U$ with

$$T = 1/2(m + M)\|\dot{x}\|^2 + 1/2ml^2\|\dot{\zeta}\| + ml\langle\dot{\zeta}\,,\,\dot{x}\rangle$$

$$U = g\langle(M + m)x + lm\zeta\,,\,e_3\rangle\ .$$

- Through the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = f$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\zeta}} - \frac{\partial L}{\partial \zeta} = 0\ .$$

- write $\dot{\zeta} = \zeta \times \omega$, with $\omega \in \mathbb{R}^3$ and $\langle\zeta\,,\,\omega\rangle = 0$.

Model equations

- The resulting dynamical system is

$$\begin{cases} (m+M)\ddot{x} + ml\ddot{\zeta} + (m+M)ge_3 = f \\ I\dot{\omega} = \zeta \times \ddot{x} + g(\zeta \times e_3) \\ \dot{\zeta} = \zeta \times \omega \end{cases} \tag{2}$$

Model equations

- The resulting dynamical system is

$$\left\{ \begin{array}{l} (m+M)\ddot{x} + ml\ddot{\zeta} + (m+M)ge_3 = f \\ I\dot{\omega} = \zeta \times \ddot{x} + g(\zeta \times e_3) \\ \dot{\zeta} = \zeta \times \omega \end{array} \right. \quad (2)$$

Exact tracking:

$$\begin{array}{l} x(t) = \gamma(t) \\ \dot{x}(t) = \dot{\gamma}(t) \\ \ddot{x}(t) = \ddot{\gamma}(t) \rightarrow f = (m+M)\ddot{x} + ml\ddot{\zeta} + (m+M)ge_3 \end{array}$$

Model equations

- The resulting dynamical system is

$$\begin{cases} (m+M)\ddot{x} + ml\ddot{\zeta} + (m+M)ge_3 = f \\ I\dot{\omega} = \zeta \times \ddot{x} + g(\zeta \times e_3) \\ \dot{\zeta} = \zeta \times \omega \end{cases} \qquad (2)$$

  Resulting internal dynamics

$$\begin{cases} I\dot{\omega} = \zeta \times \ddot{\gamma} + g(\zeta \times e_3) \\ \dot{\zeta} = \zeta \times \omega \end{cases}$$

Problem statement

Problem

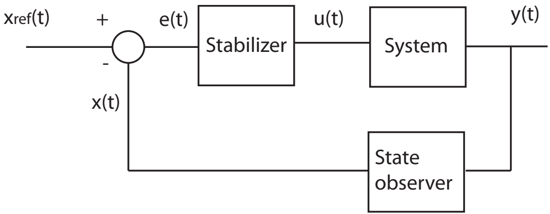
Find $\zeta_0 \in S^2_+$ *and* $\omega_0 \in \mathbb{R}^3$ *such that* $\langle \zeta_0 \, , \, \omega_0 \rangle = 0$ *and the following system*

$$\begin{cases} \dot{\zeta} = \zeta \times \omega \\ I\dot{\omega} = \zeta \times \ddot{\gamma} + g(\zeta \times e_3) \\ \omega(0) = \omega_0 \\ \zeta(0) = \zeta_0 \, . \end{cases} \qquad (3)$$

has a T*-periodic solution, with the property*

$$\zeta(t) \in S^2_+, \; \forall t \in \mathbb{R} \, ,$$

Tracking scheme



Exact tracking

Consider system

$$\begin{aligned} \dot{x} &= F(x) + G(x)u(t), \quad \text{with } F(0) = 0 \\ y(t) &= H(x) \ . \end{aligned} \tag{4}$$

Given a sufficiently regular curve γ we want that

$$y(t) = \gamma(t)$$

Normal form

$$\begin{aligned} \dot{\xi}_{1,i} &= \xi_{2,i} \\ &\vdots \\ \dot{\xi}_{r_i,i} &= \alpha_i(\xi,\eta) + \beta_i(\xi,\eta)u(t) \end{aligned} \tag{5}$$

for $i = 1, \ldots, m$, where $\xi = (\xi_{j,i}) = y_j^{(i)}$, $j = 1, \ldots, m$, $i = 1, \ldots, r_j$ and

$$\dot{\eta} = \gamma(\eta,\xi) + \delta(\eta,\xi)u(t) \ . \tag{6}$$

Exact tracking

Setting $y(t) = \gamma(t)$, then

$$\xi_{j,i} = \gamma_j^{(i)}, j = 1, \dots, m, i = 1, \dots, r_j \ .$$

we obtain the *internal dynamics system*

$$\dot{\eta}(t) = f(\eta(t), \Gamma(t)), \ \forall t \in [0, T] \ ,$$

- the problem is challenging for hyperbolic systems in which the internal dynamic are unstable both forward and backward for generic initial conditions

Different choices of boundary conditions

$$\dot{\eta}(t) = f(\eta(t), \Gamma(t)),\ \forall t \in [0, T]\ ,$$

- if Γ has limited length

$$\lim_{t \to \pm\infty} \eta(t) = 0$$

- if Γ is T-periodic

$$\eta(T) = \eta(0)\ .$$

References

Some literature on dynamic inversion for nonminimum phase systems

- S. Devasia, D. Chen, and B. Paden (1996) (Fixed Point approach with a Picard iteration)
- Notarstefano, Hauser, Frezza (2005) (vtol)
- Hauser, Saccon, Frezza (2005) (driven pendulum)
- Pavlov, Pettersen (2007) (Convergent systems approach)
- Zou (2009) (preview based inversion)

Picard Iterations

Idea of the fixed point approach (Devasia-Chen-Paden, Hauser-Saccon-Frezza).

$$\dot{x} = f(x,t) + B(t)u \, .$$

- Consider the linearized system on the origin

$$\dot{x} = A(t)x + w(t) \, ,$$

  where $A(t) = \frac{df}{dx}|_{x=0}$.
- Find a bounded operator $\Phi[w]$, such that $\Phi[w]$ is a solution of the system
- Rewrite system as

$$\dot{x} = A(t)x + (f(x,t) + B(t)u(t) - A(t)x)$$

- Consider the fixed point iteration

$$x_{i+1} = \Phi[f(x_i,t) + Bu(t) - A(t)x_i]$$

Picard Iterations

- If operator

$$\Psi[x_i] : x_i \to \Phi[f(x_i,t) + Bu(t) - A(t)x_i]$$

is a contraction, then the iteration converges to a point which is a solution of the system.

- this holds when

$$\begin{aligned}\|\Psi[x_2] - \Psi[x_1]\| &= \|\Phi[f(x_2,t) - A(t)x_2 - (f(x_1,t) - A(t)x_1)]\| \\ &= \|\Phi[g(x_2,t) - g(x_1,t)] \leq \|x_1 - x_2\| \ .\end{aligned}$$

- where $g(x,t) = f(x,t) - A(t)x$, this holds if

$$\|\Phi\|\|g(\cdot,t)\| \leq 1 \ .$$

Homotopy approach

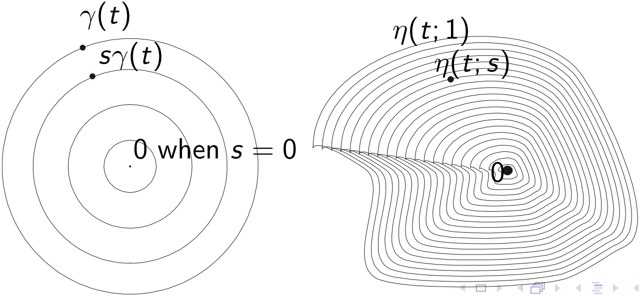
To find a T-periodic solution for

$$\dot{\eta} = f(\eta, \Gamma(t)) \ ,$$

introduce the family of differential systems

$$\dot{\eta}(t; s) = f(\eta(t), s\Gamma(t)) = F(t, \eta; s) \ ,$$

- for $s = 0$ there is a trivial bounded solution (the equilibrium)
- if it is possible to reach $s = 1$ we have solved our problem.



Some literature on this method

- Starting point: Degree theory of Leray/Schauder ('30)
- Capietto, Mawhin, Zanolin (1992)
- Mawhin (1997)
- Kamenskii, Makarenkov, Nistri (2008)

Topological conditions. Basic idea.

- Consider the fixed points of the Poincarè map

$$x_0 \rightsquigarrow x(T)\ .$$

- Given a subset $\Omega \in \mathbb{R}^n$, if there is a periodic solution for $s = 0$, there is still a periodic solution for $s = 1$ unless the solution *exits* from the boundary of Ω.
- Difficult to check

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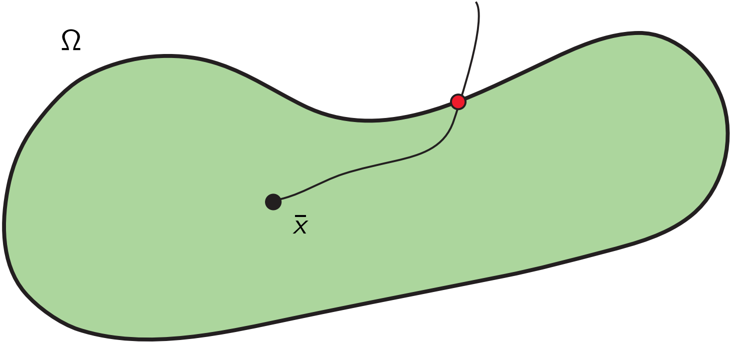
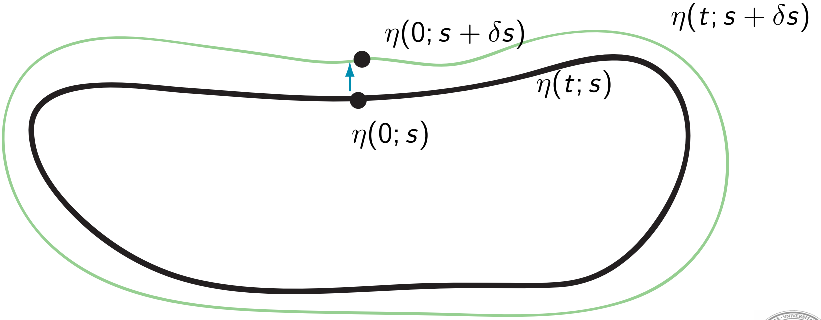


Periodic perturbations

Let $\eta(t;s)$ be a family of T-periodic solutions

$$\begin{cases} \dot{\eta}(t;s) = F(t,\eta;s) \\ \eta(T;s) = \eta(0;s) \ , \end{cases}$$



Consider system

$$\begin{cases} \dot{\eta}(t; s) = F(t, \eta; s) \\ \eta(T; s) = \eta(0; s) \ , \end{cases}$$

derive with respect to s

$$\begin{cases} \frac{\partial}{\partial s}\dot{\eta}(t; s) = \partial_\eta F(t, \eta(t; s); s)\partial_s \eta(t; s) + \partial_s F(t, \eta(t; s); s) \\ \frac{d}{ds}\eta(T; s) = \frac{d}{ds}\eta(0; s) \end{cases}$$

Set $A(t) = \partial_\eta F(t, \eta(t, ; s); s)$, $B(t) = \partial_s F(t\eta(t; s); s)$, $x = \frac{\partial}{\partial s}\dot{\eta}(t; s)$

Periodic perturbations

The system becomes

$$\begin{cases} \dot{x} = A(t)x + B(t) \\ x(T) = x(0) \end{cases}$$

the solution is

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t)\Phi(\tau)^{-1}B(t)ds \ ,$$

where $\Phi(t)$ is the solution of

$$\begin{cases} \dot{\Phi}(t) = A(t)\Phi(t) \\ \Phi(0) = I \end{cases}$$

Periodic perturbations

By periodicity $x(0) = x(T)$, we obtain

$$x(0) = (I - \Phi(T))^{-1} \int_0^T \Phi(t)\Phi(\tau)^{-1} \partial_s F(t, s, \eta_s) \, .$$

Matrix

$$(I - \Phi(T))$$

must be invertible

that is $\Phi(T)$ must not have 1 as eigenvalue.

Discussion

- The differential step for applying the homotopy method consists in finding periodic (and therefore bounded) solution of the linear non autonomous system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)\\ x(0) &= x(T)\end{aligned}$$

- This problem is strongly connected to the existence of an exponential "dichotomy" of system

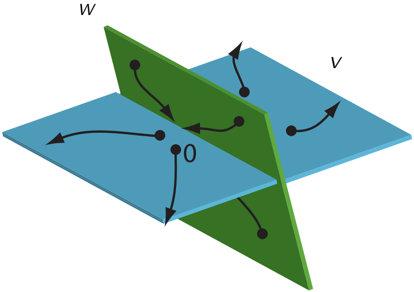
$$\dot{x}(t) = A(t)x(t)\ .$$

Exponential dichotomy

Given system

$$\begin{cases} \dot{x} = A(t)x \\ x(0) = x_0 \ , \end{cases}$$

- there exists subspaces V, W with $V \oplus W = \mathbb{R}^n$,



- solutions with $x_0 \in V$ decrease exponentially for $t \rightarrow -\infty$,
- solutions with $x_0 \in W$ decrease exponentially for $t \rightarrow +\infty$,

$A(t)$ is constant $\rightarrow$ no eigenvalues on the imaginary axis.

Some references on dichotomy

- Daleckii, Kerin (1974)
- Coppel (1978)
- Muldowney (1984)

Concept used in control theary, in particular in the context of dynamic inversion of non minimum phase systems by

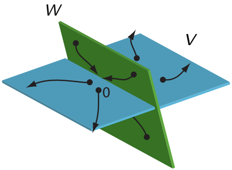
- Devasia, Paden (1998)
- Hauser, Saccon, Frezza (2005).

Dichotomy + T-invariance

Set

$$\begin{cases} \dot{\Phi} = A(t)\Phi \\ \Phi(0) = I \,, \end{cases}$$

find subspaces V, W with $V \oplus W = \mathbb{R}^n$ such that



- solutions with $x_0 \in V$ decrease exponentially for $t \to -\infty$,
- solutions with $x_0 \in W$ decrease exponentially for $t \to +\infty$,
- $\Phi(T)V = V$, $\Phi(T)W = W$.

if this happens

$$1 \notin \sigma(\Phi(T)) = \sigma(\Phi(T)|_V) \cup \sigma(\Phi(T)|_W)$$

Some notations

- $\| \cdot \|$ is any induced norm
- $\mu(\cdot)$ is the associated logarithmic norm

Logarithmic norm: defined as

$$\mu(A) = \lim_{h \to 0^+} \frac{\|I + hA\| - 1}{h} ,$$

property if $\dot{X} = A(t)X$ then

$$\frac{d}{dt}\|X(t)\| \leq \mu(A(t))\|X(t)\| .$$

For example if $\| \cdot \| = \| \cdot \|_2$, then $\mu(A) = \sigma_M(\frac{A+A^T}{2})$.

Flow of planes

Given system $\dot{x} = A(t)x$, separate the $x \in \mathbb{R}^n$ in $x_1 \in \mathbb{R}^k$ and $x_2 \in \mathbb{R}^{n-k}$ and

$$A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix}, \ \forall t \in [0, T],$$

The two subsystems

- subsystem $\dot{x}_1(t) = -A_{11}(t)x_1$
- subsystem $\dot{x}_2(t) = A_{22}(t)x_2$

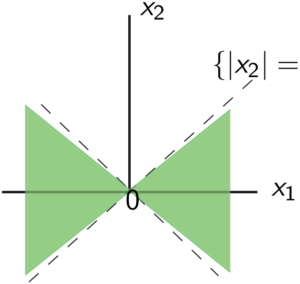
are asymptotically stable.

Flow of planes

Set $X \in \mathbb{R}^{(n-k)\times k} : \|X\| < 1$, then

$$\mathrm{Im}\begin{pmatrix} I \\ X \end{pmatrix},$$

- is supplementary to $\mathrm{Im}\begin{pmatrix} 0 \\ I \end{pmatrix}$
- for each vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathrm{Im}\begin{pmatrix} I \\ X \end{pmatrix}$, $\|x_2\| \le \|x_1\|$



$$\mathrm{Im}\begin{pmatrix} 1 \\ x \end{pmatrix}$$

Flow of planes

$$A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix}, \ \forall t \in [0, T] \,,$$

Represent a plane with $\mathrm{Im} \begin{pmatrix} I \\ X(t) \end{pmatrix}$.

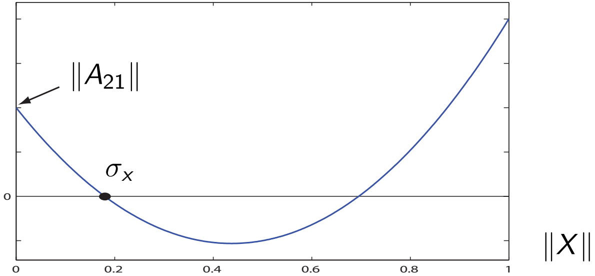
The flow of X satisfies the equation

$$\dot{X}(t) = A_{21}(t) + A_{22}(t)X(t) - X(t)A_{11}(t) - X(t)A_{12}(t)X(t),$$

write $X(t) = \Phi(t)X(0)$ then

$$\frac{d}{dt}\|X\| \leq (\mu(A_{22}(t)) + \mu(-A_{11}(t)))\|X\| + \|A_{21}(t)\| + \|A_{12}(t)\|\|X\|^2 \,.$$

$$\frac{d}{dt}\|X\| \leq \underline{(\mu(A_{22}(t)) + \mu(-A_{11}(t)))}\|X\| + \overline{\|A_{21}(t)\|} + \overline{\|A_{12}(t)\|}\|X\|^2 .$$



If

$$\Delta = \underline{(\mu(A_{22}(t)) + \mu(-A_{11})(t))^2} - 4\overline{\|A_{21}(t)\|\|A_{12}(t)\|} < 0$$

Then the set $\|X\| < \sigma_x$ is forward invariant.

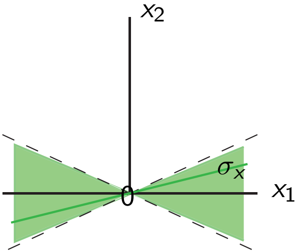
Forward invariant plane

The set of all planes

$$\mathrm{Im}\begin{pmatrix} I \\ X \end{pmatrix},$$

with $X : \|X\| < \sigma_x$, is forward invariant for the system,
by Brouwer's Fixed Point Theorem there exists $\hat{X}$ for which

$$\mathrm{Im}\begin{pmatrix} I \\ \hat{X} \end{pmatrix} = \mathrm{Im}\,\Phi(T)(\begin{pmatrix} I \\ \hat{X} \end{pmatrix}).$$



Forward invariant plane

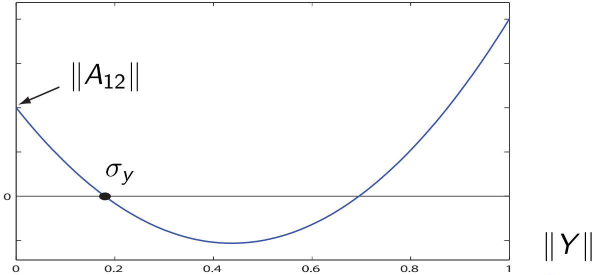
Represent a plane with $\mathrm{Im}\begin{pmatrix} Y(t) \\ I \end{pmatrix}$.

The flow of Y satisfies the equation

$$\dot{Y}(t) = A_{12}(t) + A_{11}(t)Y(t) - Y(t)A_{22}(t) - Y(t)A_{21}(t)Y(t),$$

write $Y(t) = \Phi(t)Y(0)$ then

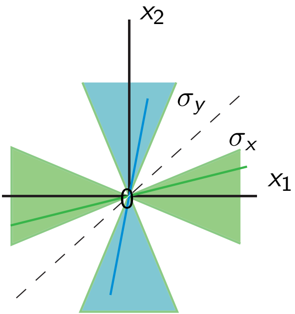
$$-\frac{d}{dt}\|Y\| \leq (\mu(A_{22}(t)) + \mu(-A_{11}(t)))\|X\| + \|A_{12}(t)\| + \|A_{21}(t)\|\|X\|^2 .$$



Forward invariant plane

Again, by Brouwer's fixed point theorem there exists $\hat{Y}$ such that

$$\mathrm{Im}\begin{pmatrix}\hat{Y}\\ I\end{pmatrix} = \mathrm{Im}\,\Phi(-T)\left(\begin{pmatrix} I\\ \hat{Y}\end{pmatrix}\right)\ .$$



Subspaces $\mathrm{Im}\begin{pmatrix} I \\ \hat{X} \end{pmatrix}$, $\mathrm{Im}\begin{pmatrix} \hat{Y} \\ I \end{pmatrix}$ are T-invariant and such that

$$\begin{pmatrix} I \\ \hat{X} \end{pmatrix} \oplus \begin{pmatrix} \hat{Y} \\ I \end{pmatrix} = \mathbb{R}^n .$$

Moreover if $x_0 \in \mathrm{Im}\begin{pmatrix} I \\ \hat{X} \end{pmatrix}$, $\forall t < 0$

$$\|x(t)\| \leq e^{(\underline{\mu(A_{11}(t))} - \overline{\|A_{12}(t)\|}\sigma_x)t}$$

if $x_0 \in \begin{pmatrix} \hat{Y} \\ I \end{pmatrix}$, $\forall t > 0$

$$\|x(t)\| \leq e^{(\underline{\mu(A_{22}(t))} + \overline{\|A_{21}(t)\|}\sigma_y)t} .$$

Existence of periodic solution

Problem

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t), \ \forall t \in [0, T] \\ x(0) = x(T) \ , \end{cases}$$

has solution if

- $\Delta = \underline{\mu(A_{22}) + \mu(-A_{11}))^2} - 4\overline{\|A_{21}\|\|A_{12}\|} < 0,$
- $\alpha_x = \mu(A_{11}) + \|A_{12}\|\sigma_x < 0,$
- $\alpha_y = \mu(A_{22}) - \|A_{21}\|\sigma_y > 0$

if $\sigma = \max\{\sigma_x, \sigma_y\}$

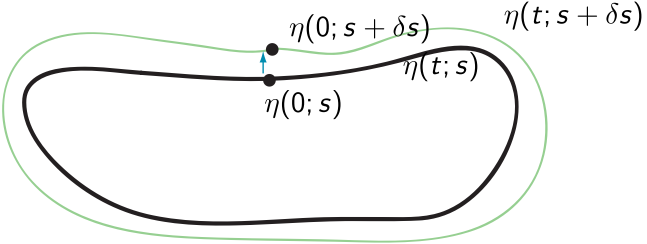
$$\|x\|_\infty \leq \frac{1+\sigma}{1-\sigma} \, \frac{\|B\|_\infty}{\min\{\alpha_x, \ |\alpha_y|\}} \ .$$

Periodic solutions

- It is possible to continue T-periodic solution of

$$\dot{\eta}(t;s) = F(t,\eta;s)$$

when the linear variation system satisfies the previous hypotheses.



- The periodic solution growth is bounded by

$$\|\eta(t;s+\delta s)-\eta(t;s)\| \leq \frac{1+\sigma}{1-\sigma}\,\frac{\|B\|_\infty}{\min\{\alpha_1,\ |\alpha_2|\}}\delta s\ .$$

Region of local continutation

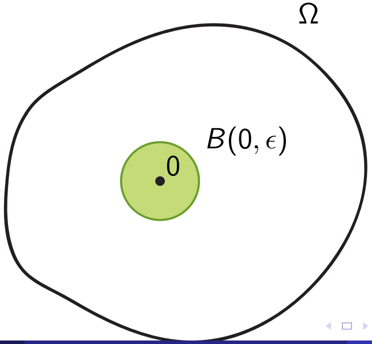
The internal dynamics are

$$\dot{\eta} = f(\eta, \Gamma(t)) ,$$

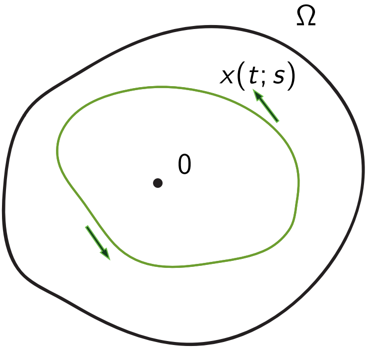
the homotopy is

$$\dot{\eta}(t; s) = f(\eta(t), s\Gamma(t)) = F(t, \eta; s) ,$$

if $A = \partial_\eta f(0,0)$ has not eigenvalues on the imaginary axis, the system $\dot{x} = Ax$ has a T-periodic solution

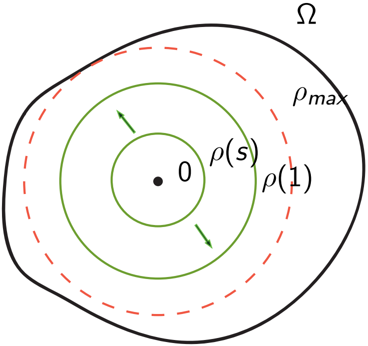


Region of local continutation



- All T-periodic solutions contained in Ω satisfy the sufficient conditions for continuation

Region of local continutation



- The growth of solutions is bounded by expression

$$\dot{\rho} = \frac{1+\sigma(\rho)}{1-\sigma(\rho)} \frac{\|B(\rho)\|_\infty}{\alpha_1(\rho) \wedge |\alpha_2(\rho)|}$$

.

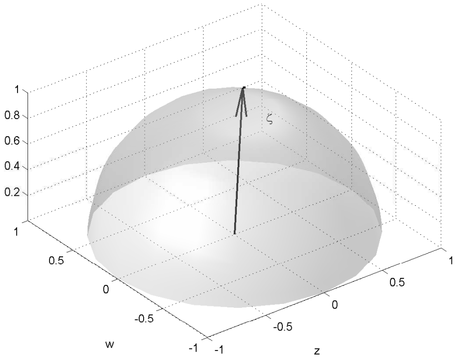
- If $\rho(1) < \rho_{max}$ then the system admits a periodic solution for $s = 1$ by homotopy.

Spherical pendulum

Set $\mathcal{B} = \{(z,w) \in \mathbb{R}^2 | \|(z,w)\| < 1\}$, the map

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & S^2_+ \\ (z,w) & \rightsquigarrow & (z, w, \sqrt{1 - z^2 - w^2}) \end{array}$$

is a diffeomorphism.



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is a diffeomorphism.

Problem

Find $(z_0, w_0, \dot{z}_0, \dot{w}_0) \in \mathcal{B} \times \mathbb{R}^2$ *such that the system*

$$\begin{cases} \begin{pmatrix} \ddot{z} \\ \ddot{w} \end{pmatrix} = -\begin{pmatrix} z \\ w \end{pmatrix} (\dot{z}^2 + \dot{w}^2 + \frac{(\dot{z}z+\dot{w}w)^2}{1-z^2-w^2}) + \\ +l^{-1}\begin{pmatrix} z \\ w \end{pmatrix} \langle \begin{pmatrix} z \\ w \\ \sqrt{1-z^2-w^2} \end{pmatrix}, \ddot{\gamma} + \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix} \rangle - l^{-1}\begin{pmatrix} \ddot{\gamma}_1 \\ \ddot{\gamma}_2 \end{pmatrix} \\ z(0) = z_0,\ w(0) = w_0 \\ \dot{z}(0) = \dot{z}_0,\ \dot{w}(0) = \dot{w}_0\ , \end{cases} \tag{4}$$

has a T*-periodic solution.*

Homotopy for the pendulum

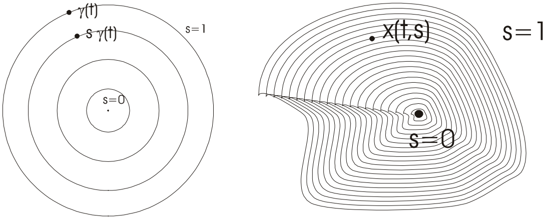
- Consider the family of ordinary differential systems depending on $s \in \mathbb{R}$

$$\begin{pmatrix} \ddot{z} \\ \ddot{w} \end{pmatrix} = -\begin{pmatrix} z \\ w \end{pmatrix}(\dot{z}^2 + \dot{w}^2 + \frac{(\dot{z}z+\dot{w}w)^2}{1-z^2-w^2}) +$$
$$+ l^{-1}\begin{pmatrix} z \\ w \end{pmatrix}\langle \begin{pmatrix} z \\ w \\ \sqrt{1-z^2-w^2} \end{pmatrix}, s\ddot{\gamma} + \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\rangle - sl^{-1}\begin{pmatrix} \ddot{\gamma}_1 \\ \ddot{\gamma}_2 \end{pmatrix}. \tag{5}$$

- for $s = 1$ this is exactly the spherical pendulum
- for $s = 0$, system (5) has an obvious T-periodic solution: the one identically zero *the pendulum is kept in the vertical unstable equilibrium in the point* $(0, 0)$.

- Consider the family of ordinary differential systems depending on $s \in \mathbb{R}$

$$\begin{pmatrix} \ddot{z} \\ \ddot{w} \end{pmatrix} = -\begin{pmatrix} z \\ w \end{pmatrix}\left(\dot{z}^2 + \dot{w}^2 + \frac{(\dot{z}z+\dot{w}w)^2}{1-z^2-w^2}\right) + \\ + l^{-1}\begin{pmatrix} z \\ w \end{pmatrix}\left\langle \begin{pmatrix} z \\ w \\ \sqrt{1-z^2-w^2} \end{pmatrix}, s\ddot{\gamma} + \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix} \right\rangle - s l^{-1}\begin{pmatrix} \ddot{\gamma}_1 \\ \ddot{\gamma}_2 \end{pmatrix}. \quad (5)$$



Application to the spherical pendulum

Proposition

There exists a strictly decreasing function $k \in \mathcal{C}(]0,+\infty[,\mathbb{R}^+)$ *and for every* $\sqrt{\frac{g}{l}} \in \mathbb{R}^+$ *there exists a strictly increasing function* $r_{\sqrt{\frac{g}{l}}} \in \mathcal{C}([0,k(d)[,\mathbb{R}])$ *such that for any* T*-periodic curve* $\gamma \in \mathcal{C}^2(\mathbb{R},\mathbb{R}^3)$*, with*

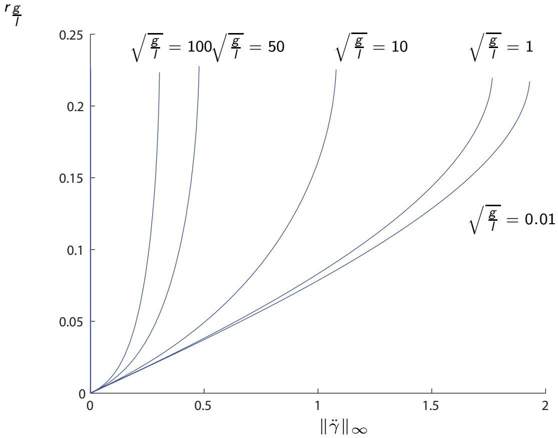
$$\|\ddot{\gamma}\|_\infty \leq k(\sqrt{\frac{g}{l}}) \ ,$$

there exists an initial condition $(z_0, w_0, \dot{z}_0, \dot{w}_0)$ *such that the solution of (5) is periodic and satisfies the following bounds*

$$\|(z,w)\|_\infty \leq \sqrt{2} \quad r_{\sqrt{\frac{g}{l}}}(\|\ddot{\gamma}_\infty\|) \ ,$$

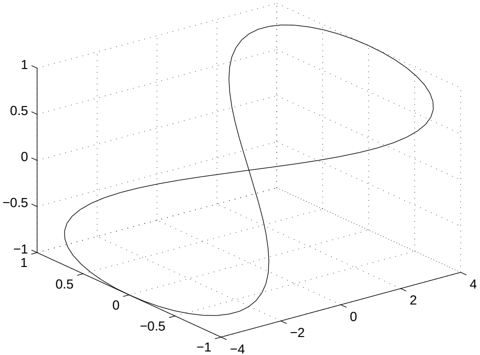
$$\|(\dot{z},\dot{w})\|_\infty \leq \sqrt{2}\sqrt{\frac{g}{l}} \quad r_{\sqrt{\frac{g}{l}}}(\|\ddot{\gamma}_\infty\|) \ .$$

The value $k(\sqrt{\frac{g}{l}})$ represents the maximum acceleration of γ for which this method guarantees the existence of a T-periodic solution for the pendulum internal dynamics.



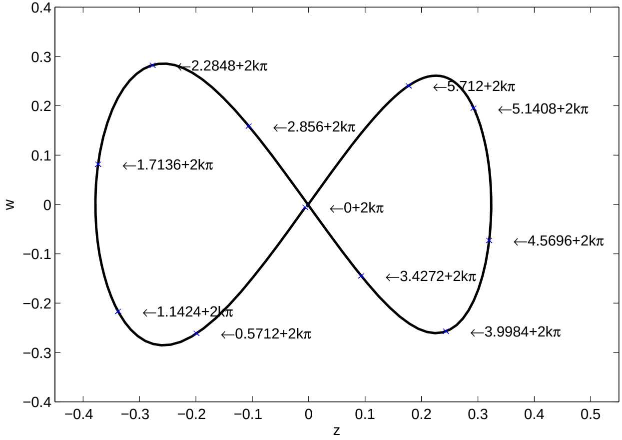
Set the length of the pendulum rod $l = 1$, consider the eight-shaped 2π-periodic trajectory in $\mathbb{R}^3$

$$\gamma(t) = \begin{pmatrix} 4 \sin t \\ \sin(2t) \\ \sin t \end{pmatrix} .$$

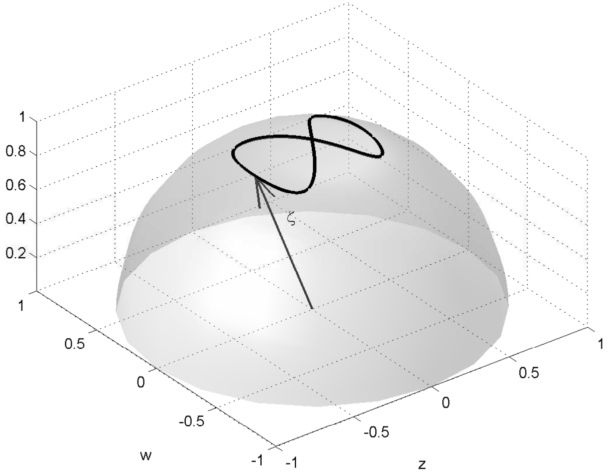


Simulation results

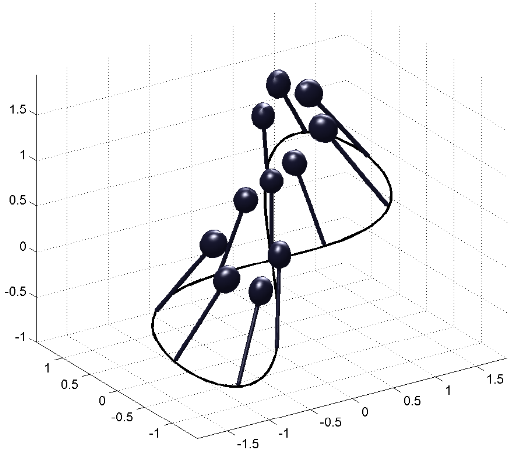
Resulting 2π-periodic trajectories for internal dynamics projected on plane (z, w)



The attitude versor ζ.



The pendulum motion along the curve.



Simulation results

Publications

L. Consolini and M. Tosques, "On the vtol aircraft exact tracking with bounded internal dynamics via a poincarè map approach," *IEEE Trans. On Automatic Control*, vol. 52, no. 9, pp. 1757–1762, 2007.

——, "On the existence of small periodic solutions for the 2-dimensional inverted pendulum on a cart," *SIAM Journal on Applied Mathematics*, vol. 68, no. 2, pp. 486–502, 2007.

——, "A homotopy method for exact output tracking of some non-minimum phase nonlinear control systems," *Int. J. Robust Nonlinear Control*, 2009.

——, "On the exact tracking of the spherical inverted pendulum via an homotopy method," *Systems & Control Letters*, vol. 58, no. 1, pp. 1 – 6, 2009.