Consensus on nonlinear spaces
and
Graph coloring

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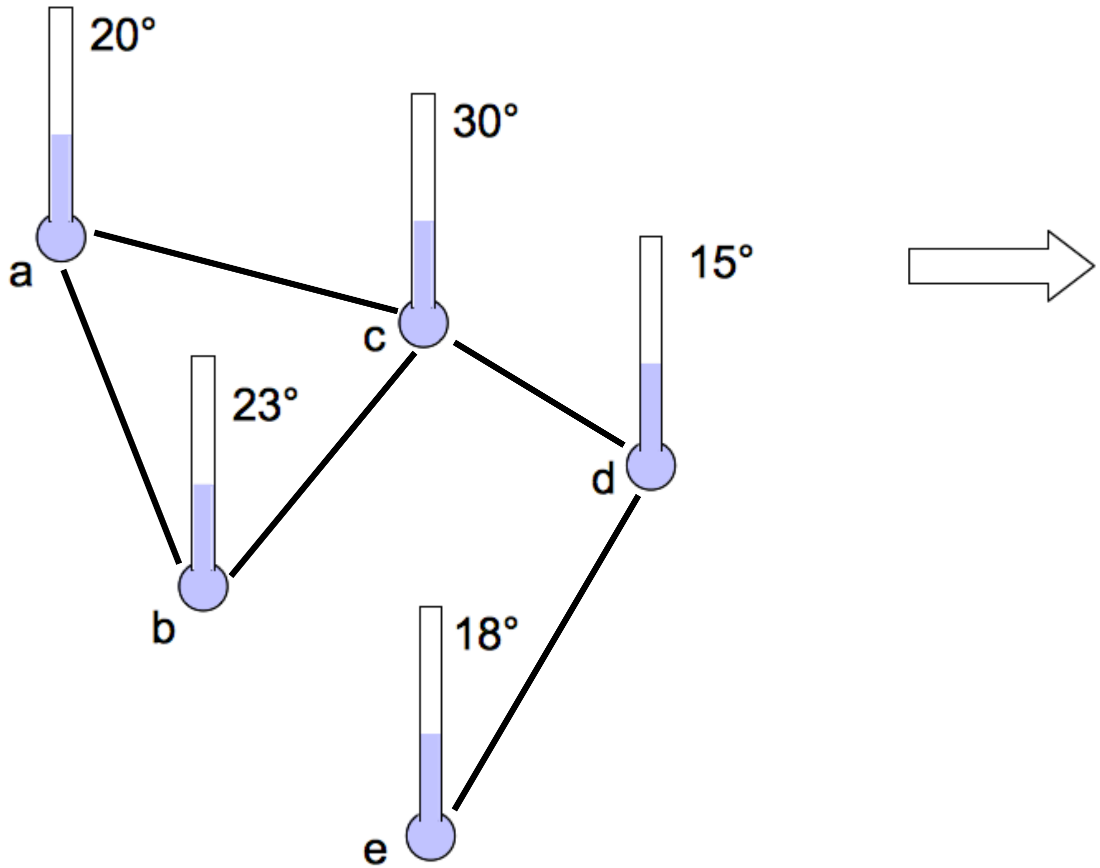
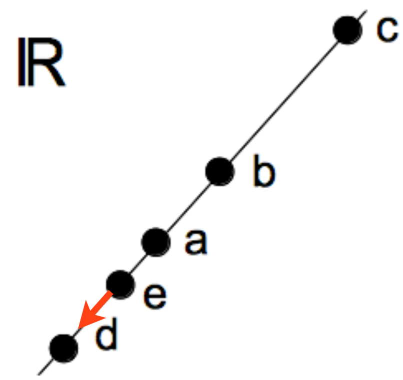
What temperature should we have in this room?

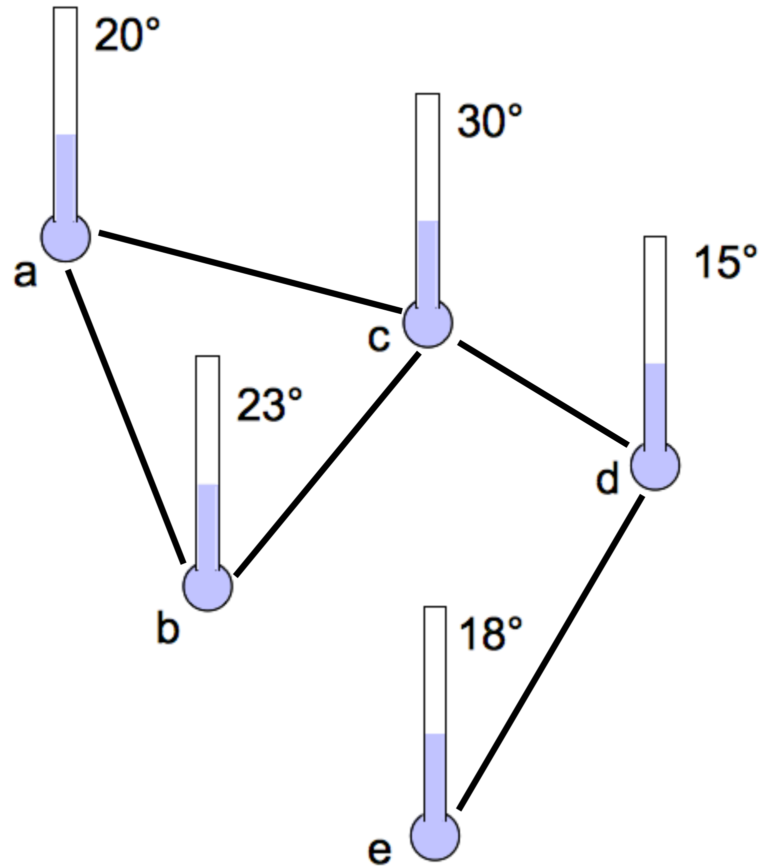


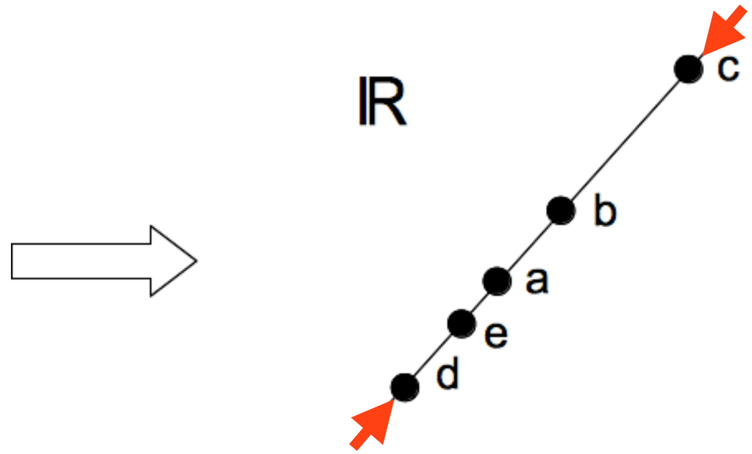


Synchronization = consensus
in a vector space

“move towards your neighbors”

What temperature should we have in this room?





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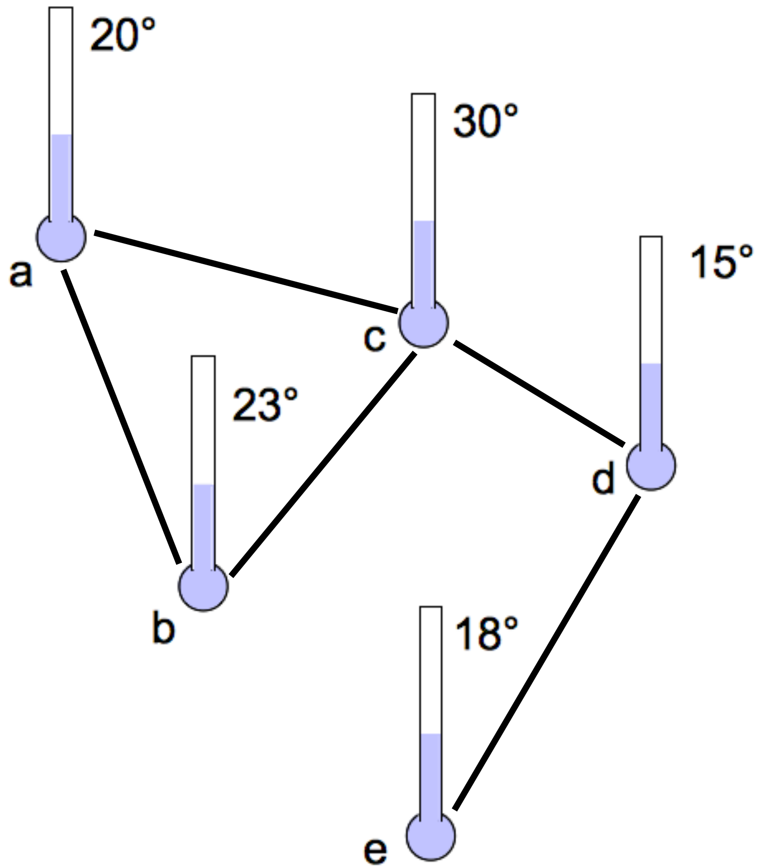

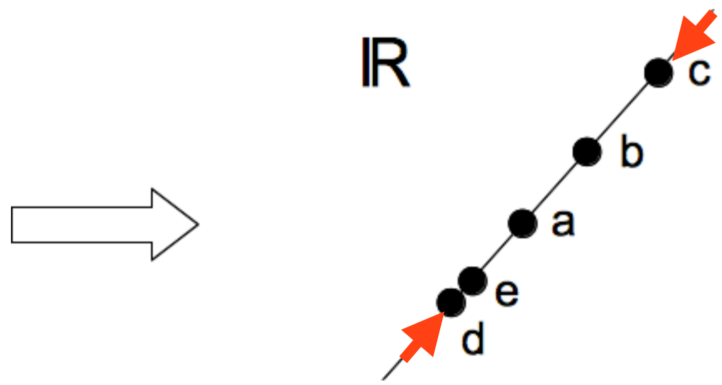

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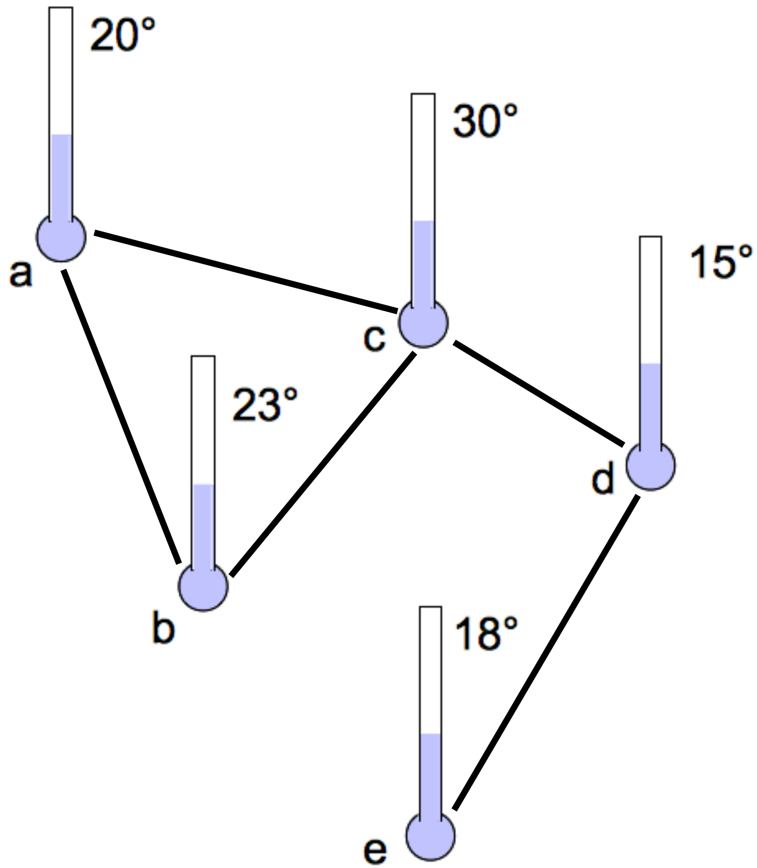

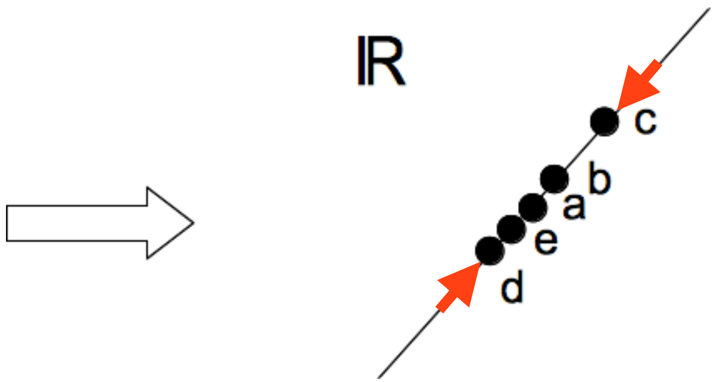

Synchronization = consensus
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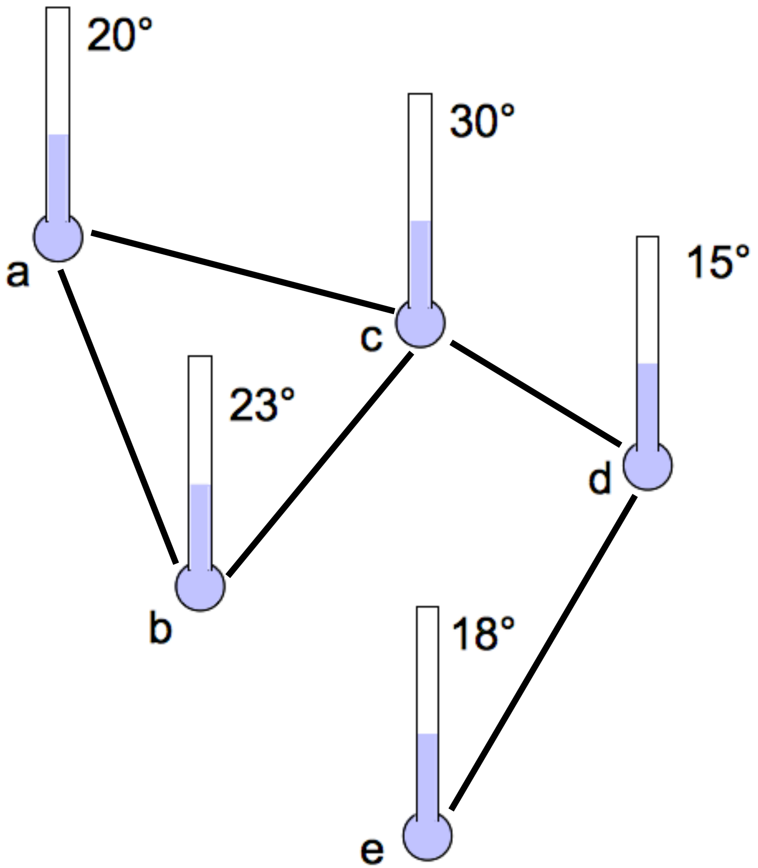
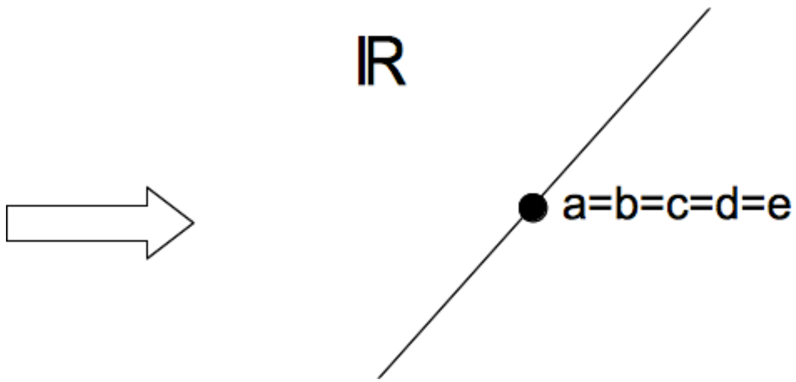
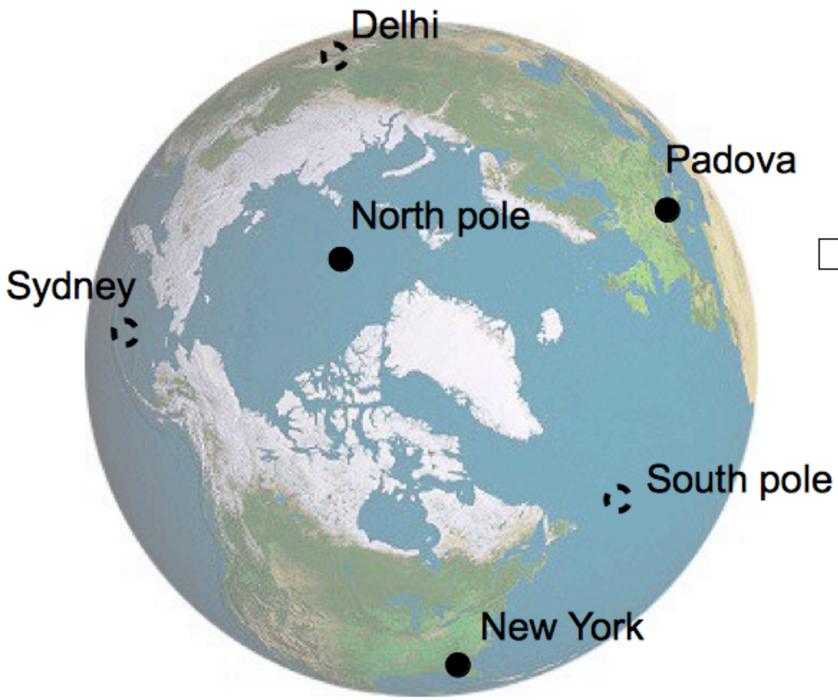


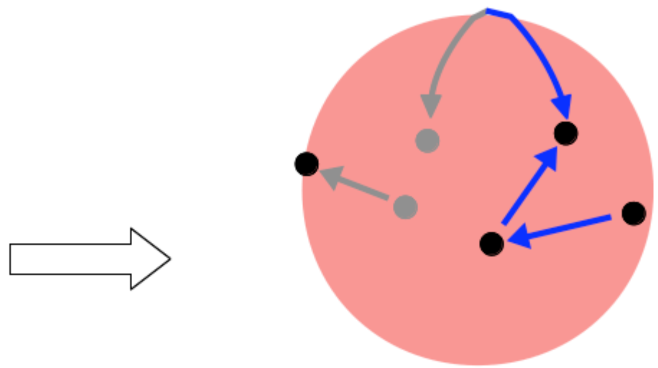


Synchronization = consensus in a vector space

“move towards your neighbors”

Where is the center of the world?

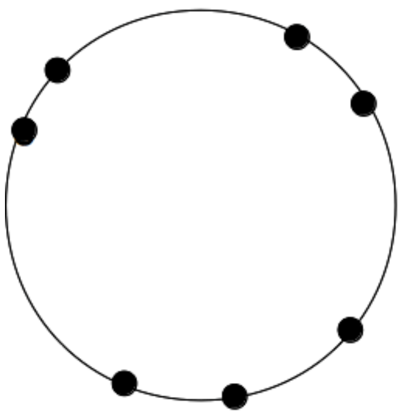




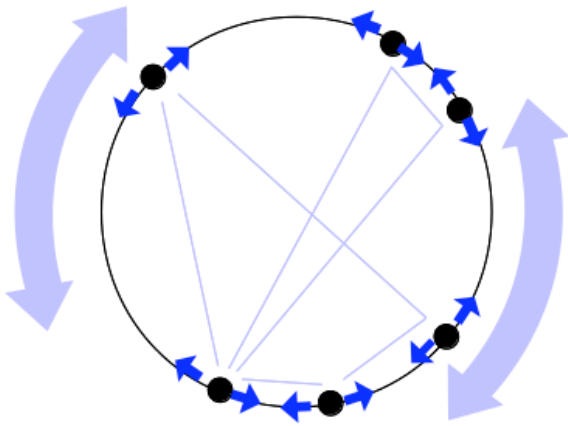
Synchronization on a sphere

Where is the center of the world?

Synchronization on a sphere



Where is the mean position?



How do agents move?

Consensus on nonlinear spaces & Graph coloring

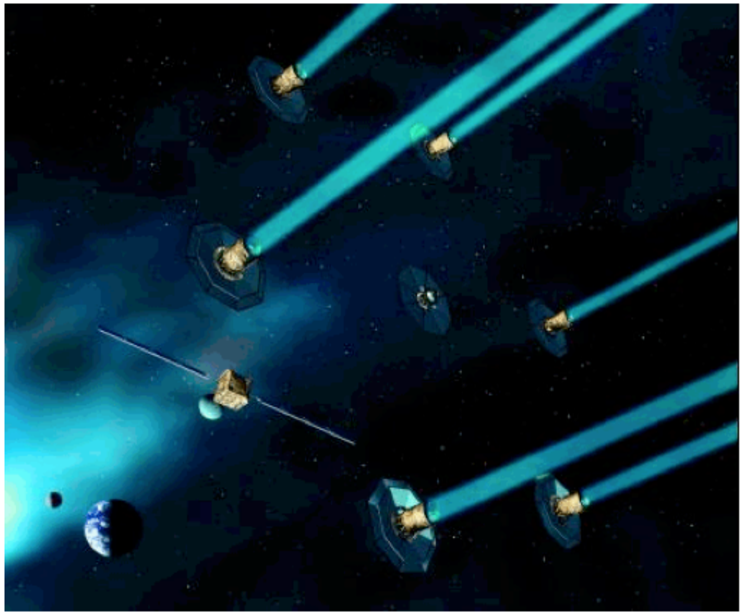
1. Some examples to motivate nonlinear consensus

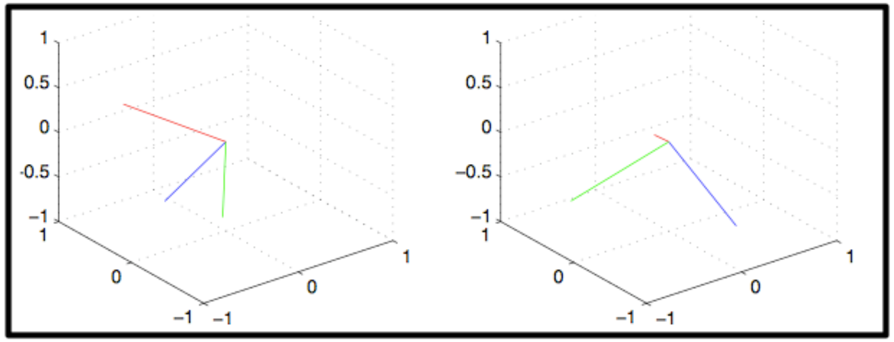
2. Formalizing consensus on nonlinear spaces

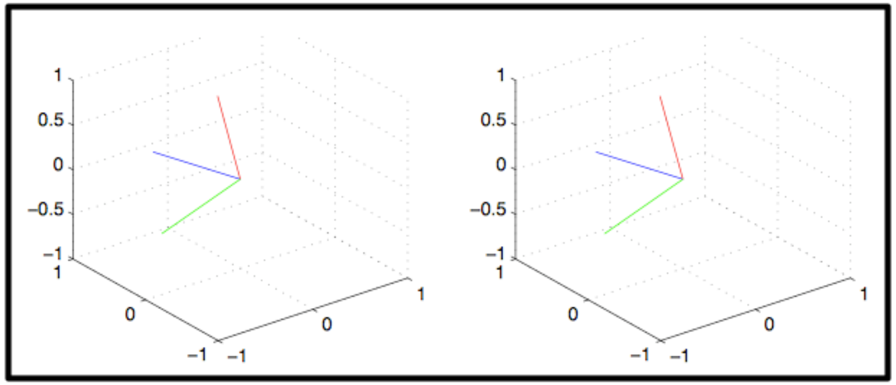
3. Link with graph coloring : (just) a complexity result

I. Orientation synchronization e.g. in formations of spacecraft

State space of orientations = manifold of rotation matrices SO(3)





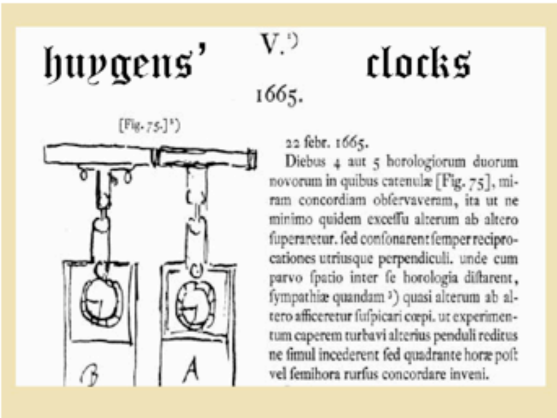


DARWIN interferometer

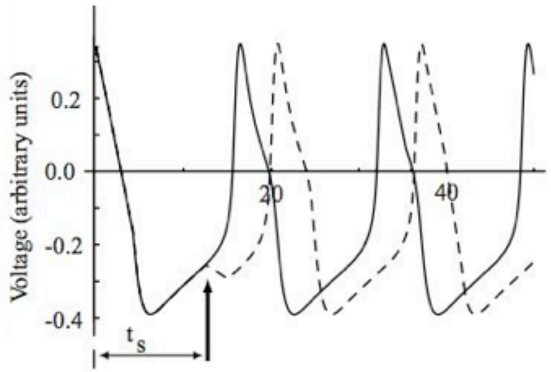
(NASA / ESA concept study)

II. Coordination on the circle appears in problems involving oscillator networks

Synchronized fireflies
Huygens' clocks



Laser tuning
Cell/neuron action



For $\theta_k \in S^1$, $k = 1,2,\ldots,N$

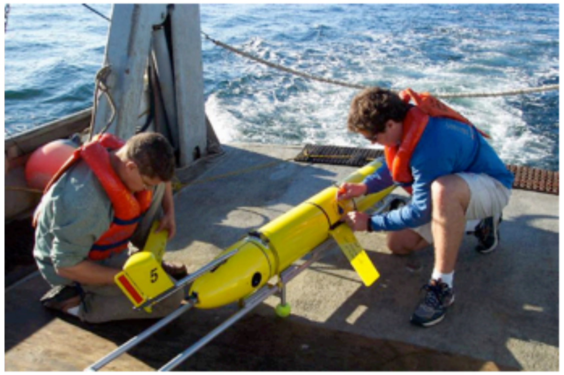
phase synchronization : $\theta_1 = \theta_2 = \ldots = \theta_N$

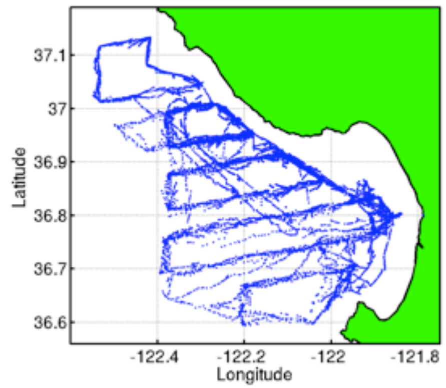
frequency synchronization : $\frac{d}{dt}\theta_1 = \frac{d}{dt}\theta_2 = \ldots = \frac{d}{dt}\theta_N$

III. Distributed sensor networks e.g. to collect ocean data (Naomi Leonard et al.)

Autonomous underwater vehicles, sparse communication

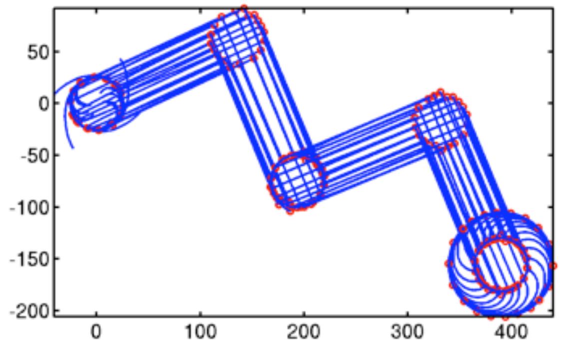
Buoyancy-driven at constant speed ~ 40 cm/s





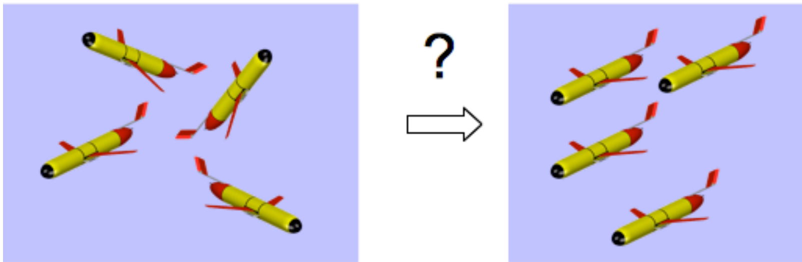
Goal : collective trajectory planning

on a simplified AUV model

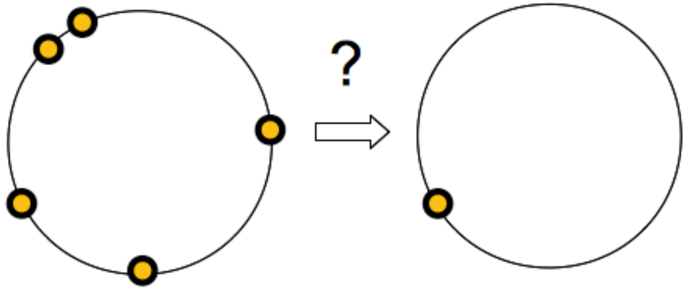


Agreement on collective motion involves nonlinear spaces

Decision on a direction of straight motion



Synchronization on S^1



General motion “in formation”

translations $\mathbb{R}^2$ }
rotations S^1 } non-trivial coupling: Lie group SE(2)

NB: In nonlinear spaces, coordinated motion differs (more difficult) from consensus

Coordinate motion in $\mathbb{R}^n$ = synchronize velocities in $\mathbb{R}^n$

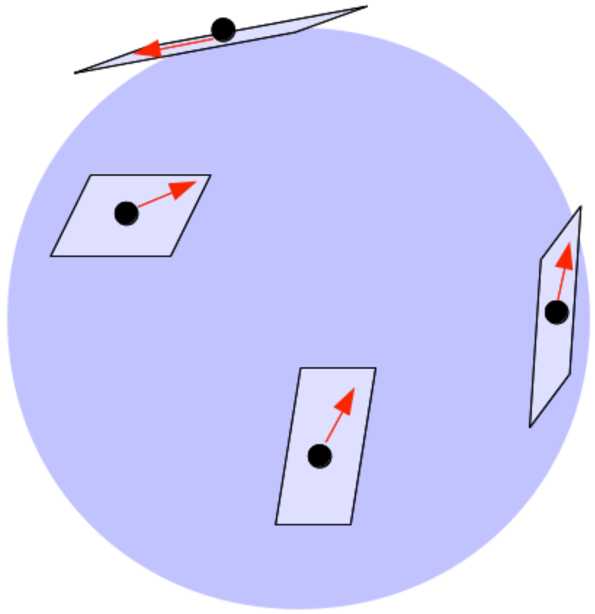
Motion “in formation”: relative positions of the agents are constant

=

Equal velocities for all the agents in $T\mathbb{R}^n = \mathbb{R}^n$

NB: In nonlinear spaces, coordinated motion differs (more difficult) from consensus

Coordinate motion on the sphere = ???



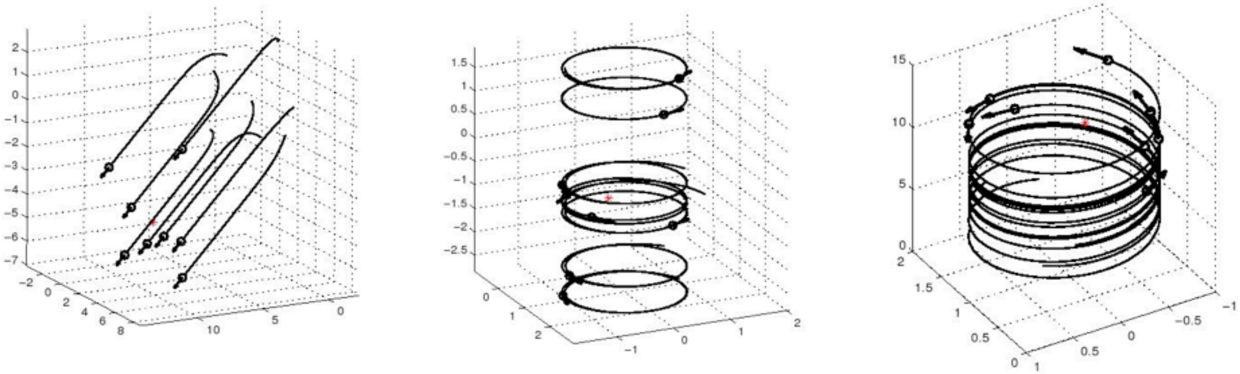
The velocities belong to different tangent spaces TS^n

The intersection of all tangent spaces is generically empty

NB: In nonlinear spaces, coordinated motion differs (more difficult) from consensus

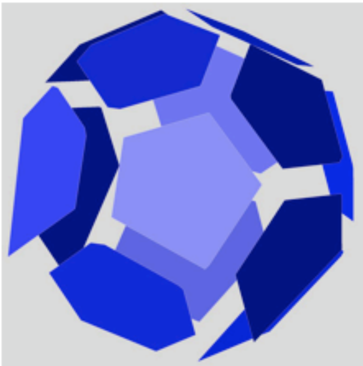
Algorithms for coordinated motion on Lie groups, see:

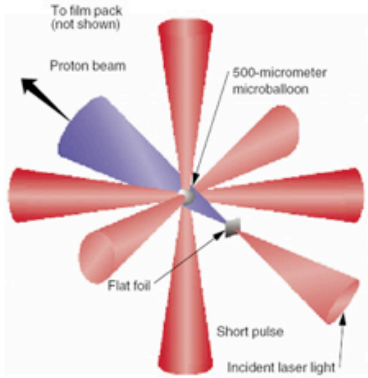
"Coordinated motion design on Lie groups"
A. Sarlette, S. Bonnabel and R. Sepulchre,
IEEE Trans. Automatic Control, vol. 55 nr. 5, pp. 1047-1058 (2010)



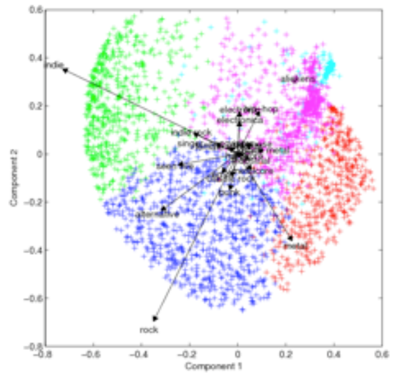
IV. Coordination on nonlinear spaces is linked to algorithmic applications

Packing





Clustering



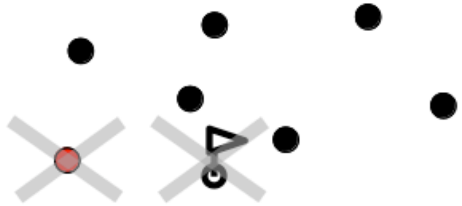
- points on a sphere
- lines or subspaces of $\mathbb{R}^n$ (Grassmann manifolds)

Applications: optimal coding, numerical integration, learning of structure in data, optimal placement of converging laser beams / representative planar projections,...

Setting

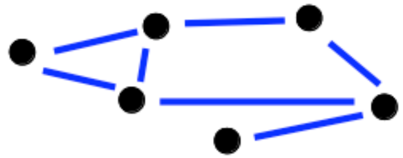
Identical autonomous agents

same control law for each agent
no “leader” , no external supervisor



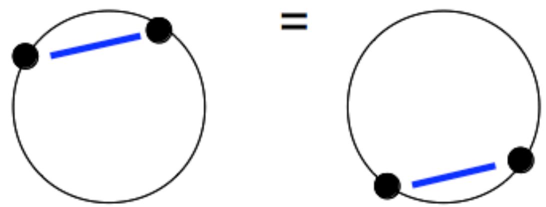
Limited interconnection links between agents

agent k has access only to some agents j
interconnection graph G (directed, varying)



Invariance with respect to absolute position

the agents' behavior only depends
on their relative positions



Consensus on nonlinear spaces & Graph coloring

1. Some examples to motivate nonlinear consensus

2. Formalizing consensus on nonlinear spaces

    Synchronization: from vector spaces to the circle

    Formalization on compact homogeneous manifolds

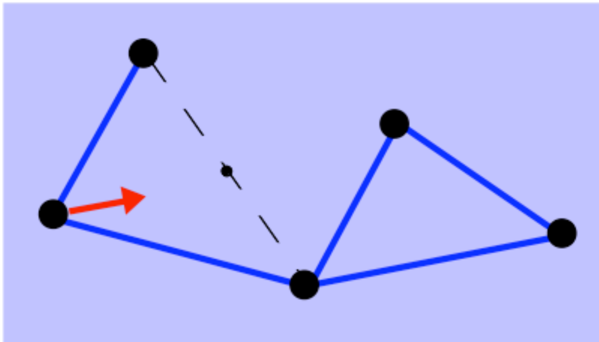
    Global synchronization properties

3. Link with graph coloring : (just) a complexity result

A linear algorithm achieves global exponential synchronization on vector spaces

$$\frac{d}{dt}x_k = \sum_{j \rightsquigarrow k} (x_j - x_k) = d(m_k - x_k)$$

$$\text{with} \quad \begin{cases} d = \sum_{j \rightsquigarrow k} 1 \\ m_k = \frac{1}{d} \sum_{j \rightsquigarrow k} x_j \end{cases}$$

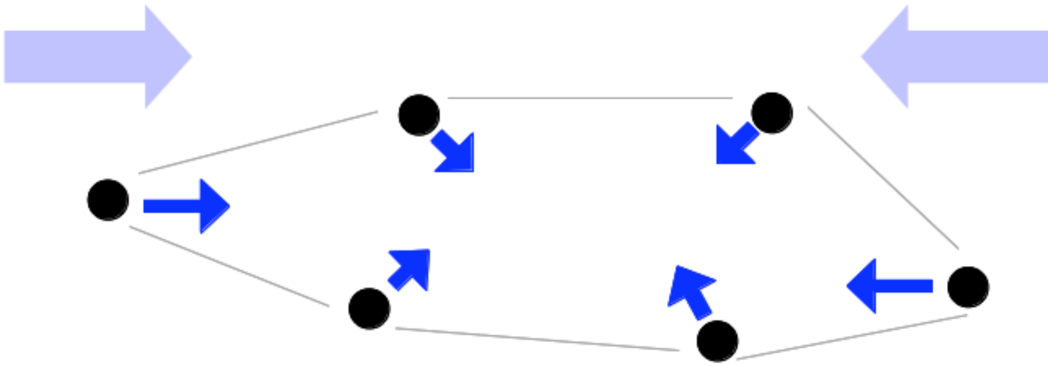


For graph G fixed undirected : gradient of $\frac{1}{2} \sum_{k} \sum_{j \rightsquigarrow k} \|x_j - x_k\|^2$

A linear algorithm achieves global exponential synchronization on vector spaces

Exponential synchronization is ensured for any initial condition iff G is uniformly connected, i.e. ∃ T such that the union of links during [t, t+T] is connected for all t.

Stability of multi-agent systems with time-dependent communication links,
L.Moreau, IEEE Trans. Automatic Control vol. 50(2), 2005



For G undirected: final state = arithmetic mean of the $x_k(0)$

This result has two fundamental limitations

The convergence result involves a condition on G.
But often interconnections depend on the states of the agents.

What about state-dependent graphs ?

⇒ under investigation see Bullo et al., Aeyels/De Smet, Blondel/Hendrickx

The global convergence argument does not extend to nonconvex spaces like the circle, sphere,...

How do synchronization algorithms behave globally on manifolds ?

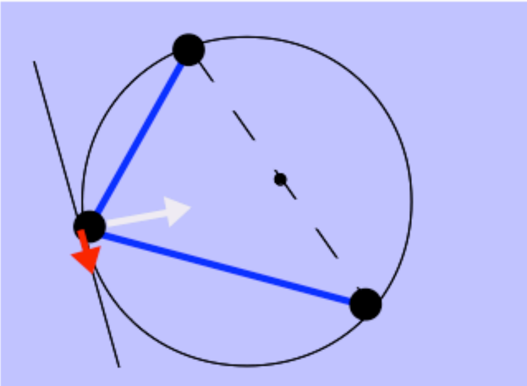
⇒ topic of this talk

An algorithm with the same local behavior can be designed on the circle

$$\frac{d}{dt}\theta_k = \sum_{j \rightsquigarrow k} \sin(\theta_j - \theta_k) \quad = d \ \mathrm{Proj}_{TS^1(\theta_k)} \left(M_k - e^{i\theta_k} \right)$$

$$\text{with} \quad M_k = \textstyle\sum_{j \rightsquigarrow k} e^{i\theta_j}$$

Similar to Kuramoto and Vicsek models describing natural behavior



For graph G fixed undirected : gradient of $\frac{1}{2}\sum_k \sum_{j \rightsquigarrow k} \|e^{i\theta_j} - e^{i\theta_k}\|^2$

In the following we will extend this to other “perfectly symmetric” nonlinear spaces

= compact homogeneous manifolds (CCH)

Formally : quotient manifold of a Lie group by a subgroup

Intuitively: “all points are identical”

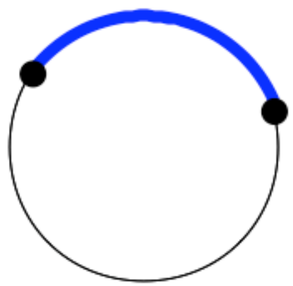
Examples: sphere S^n
rotation matrices SO(n) (and all other compact groups)
Grassmann manifolds (see last part)

In this talk: compact homogeneous manifolds H embedded in $\mathbb{R}^n$
such that $\|x\| = r$ constant for $x \in H$

An alternative distance measure yields convenient properties

Geodesic distance

$d_g(\theta_k, \theta_j) = |\theta_k - \theta_j|$ on S^1

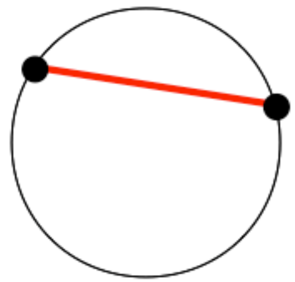


Not obvious on general manifolds

d_g^2 not smooth everywhere

Chordal distance

$d_c(\theta_k, \theta_j) = 2\sin\left|\frac{\theta_k - \theta_j}{2}\right|$ on S^1



On CCH manifolds: consider $d_c(x_k, x_j) = \|x_k - x_j\|$

d_c^2 smooth everywhere

An alternative distance measure yields convenient properties

Geodesic distance

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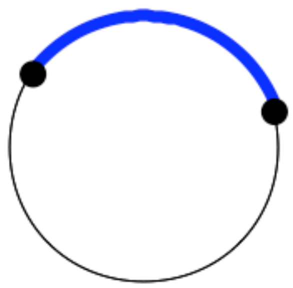

Not obvious on general manifolds

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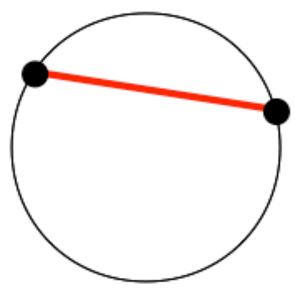
Chordal distance

$d_c(\theta_k, \theta_j) = \|e^{i\theta_k} - e^{i\theta_j}\|$ on S^1



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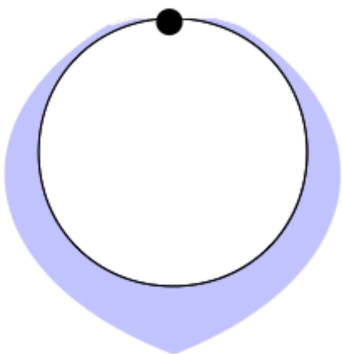

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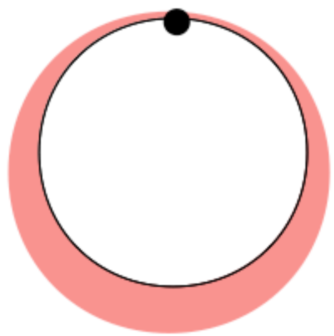


On CCH manifolds: consider $d_c(x_k, x_j) = \|x_k - x_j\|$

d_c^2 smooth everywhere

The “induced arithmetic mean” of the chordal distance is easily computable

Induced arithmetic mean

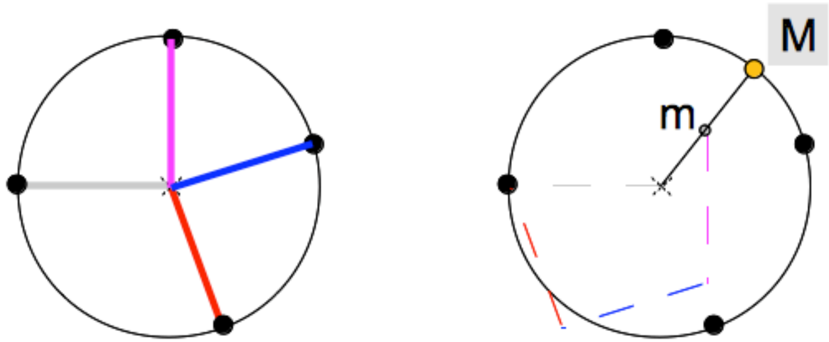
$$M = \min_{x \in H} \left(\sum_k d_c(x, x_k)^2 \right) = \mathrm{Proj}_H \left(m = \frac{1}{N} \sum_k x_k \right)$$

≠ traditional Karcher (or Fréchet) mean $= \min_{x \in H} \left(\sum_k d_g(x, x_k)^2 \right)$

Anti-M $= \max_{x \in H} \left(\sum_k d_c(x, x_k)^2 \right) = \mathrm{Proj}_H \left(-m \right)$

The “induced arithmetic mean” of the chordal distance is easily computable

On S^1: $M = \arg\left(\sum_k e^{i\theta_k}\right)$



On SO(n): M = polar decomposition of m

On the Grassmann manifold, representing an element of Gr(p,n) by the orthogonal projection matrix Π_k on the corresponding subspace:

M = p-dimensional principal eigenspace of m = $\sum \Pi_k$

The induced arithmetic mean allows to define several specific configuration types

Synchronization	$x_j = x_k$ for all j,k
Consensus	each agent k moves as close as possible to its fixed neighbors, such that $\forall\, k\,, \quad x_k \in M(\{x_j : j \rightsquigarrow k\})$
Anti-Consensus	each agent k moves as far as possible to its fixed neighbors, such that $\forall\, k\,, \quad x_k \in \text{Anti-}M(\{x_j : j \rightsquigarrow k\})$
Balancing	each point on the manifold is equally close to the agents, i.e. $M(\{x_k\}) = H$

The gradient of V_G yields consensus algorithms

$$\frac{d}{dt} x_k = -\alpha \; \mathrm{grad}_{H,k}(V_\Gamma) \qquad \text{for} \quad k = 1, 2, \ldots, N$$

with $\alpha > 0$ for consensus, $\alpha < 0$ for anti-consensus

$$\Rightarrow \quad \frac{d}{dt} x_k = \alpha \; \mathrm{Proj}_{TH(x_k)} \left(\sum_{\{j : j \rightsquigarrow k \text{ or } k \rightsquigarrow j\}} (x_j - x_k) \right)$$

OK only for undirected G

Final algorithm (not gradient for directed, varying graphs)

$$\frac{d}{dt} x_k = \alpha \; \mathrm{Proj}_{TH(x_k)} \left(\sum_{j \rightsquigarrow k} (x_j - x_k) \right)$$

explicit forms on SO(n), Grassmann,...

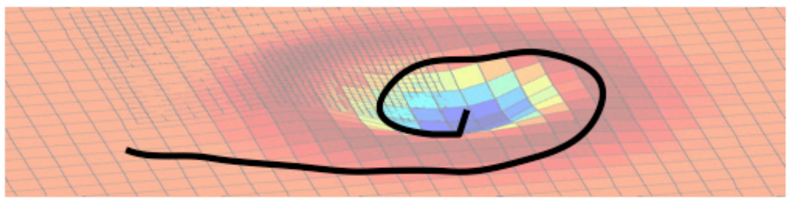
These developments can be adapted to more complex agent dynamics

"Cascade" approach

use the result of the consensus algorithm as desired velocity, function of the relative positions of the agents, at the input of a tracking algorithm

"Energy shaping" approach

for a mechanical system, use V_Γ as artificial potential combined with appropriate artificial dissipation



Consensus on nonlinear spaces & Graph coloring

1. Some examples to motivate nonlinear consensus

2. Formalizing consensus on nonlinear spaces

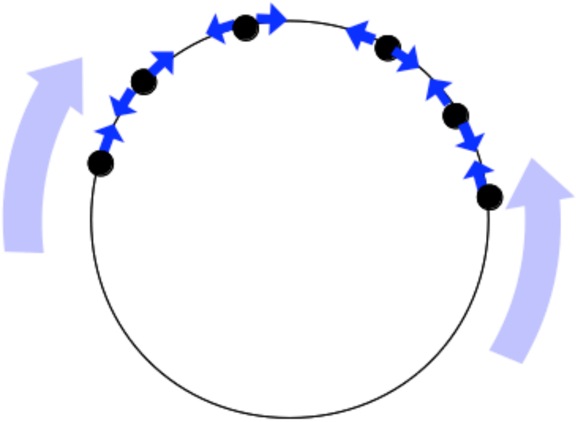
    Synchronization: from vector spaces to the circle

    Formalization on compact homogeneous manifolds

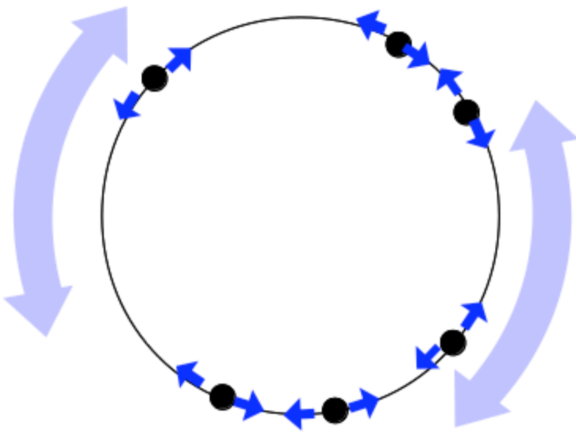
    Global synchronization properties

3. Link with graph coloring : (just) a complexity result

Synchronization is ensured locally.
The global behavior is a priori unclear.

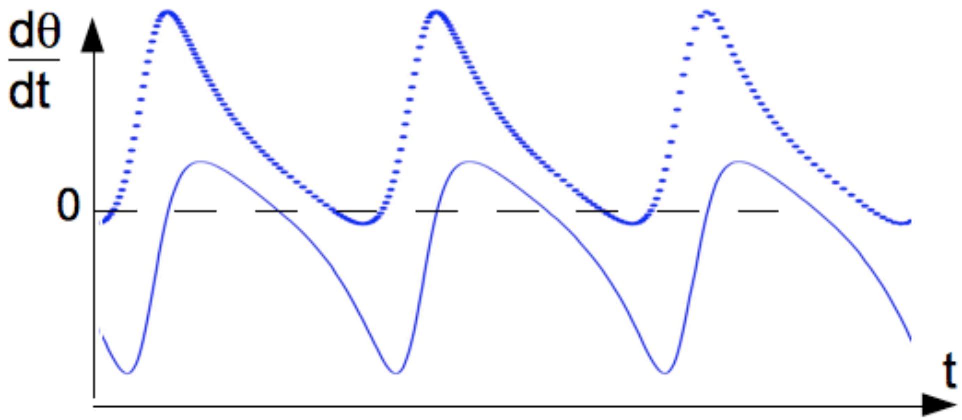


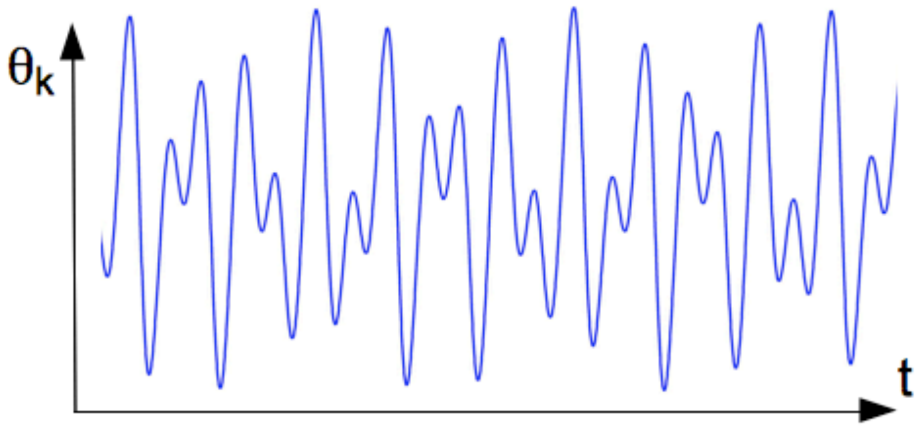
Contraction arguments hold if
all agents are in a semicircle



Convergence ?
What is the mean of $\theta_k(0)$?

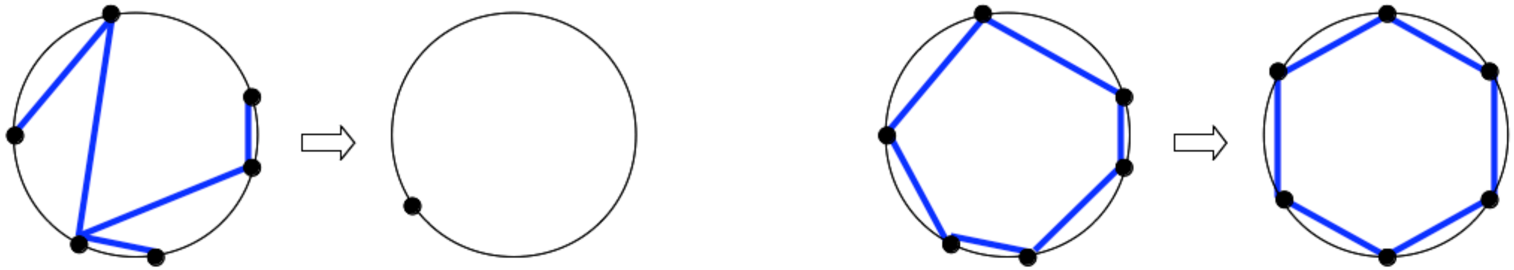
Fixed but directed graphs can lead to limit cycles, quasi-periodic behavior,...





Undirected graphs ensure convergence to an equilibrium set, but which one ?

Some graphs feature stable local attraction equilibria ≠ synchronization



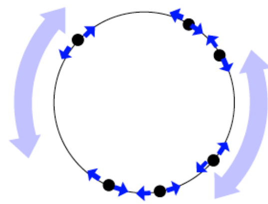
What about repulsive agents?

$$\frac{d}{dt}x_k = -\sum_{j \rightsquigarrow k} (x_j - x_k) \quad \text{on } R^n$$

Agents drive away to infinity

$$\frac{d}{dt}\theta_k = -\sum_{j \rightsquigarrow k} \sin(\theta_j - \theta_k) \quad \text{on circle}$$

Stable equilibria are not trivial

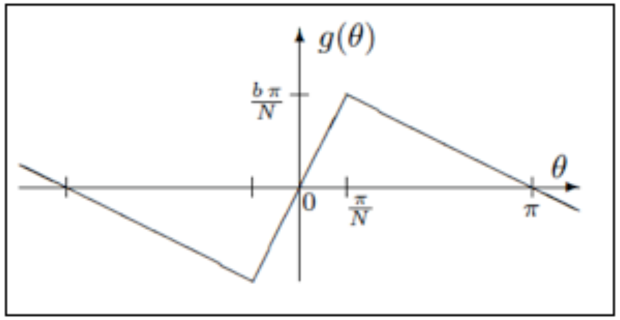


The existence of local equilibria is sensitive to the attraction profile between connected agents

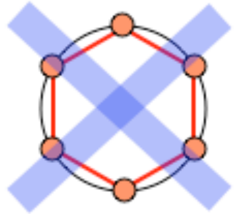
Circle : $\frac{d}{dt}\theta_k = \sum_{j \rightsquigarrow k} g(\theta_j - \theta_k)$

IAM gradient: $g(\theta) = \sin(\theta)$

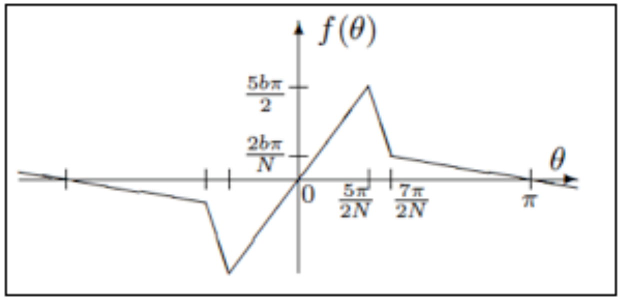
Variation 1



Synchronization is only stable equilibirum for any fixed undirected graph



Variation 2



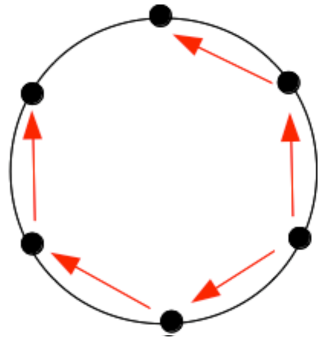
Stable equilibrium different from synchronization even for all-to-all graph

Alternative algorithms can overcome spurious local equilibria of standard consensus motion

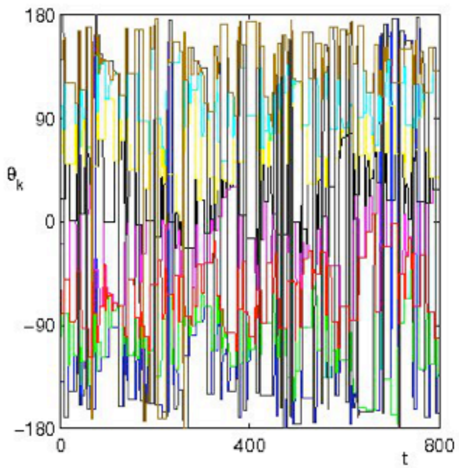
Gossip algorithm = forced asynchrony

At each time, select a single link, and only its 2 agents move towards each other

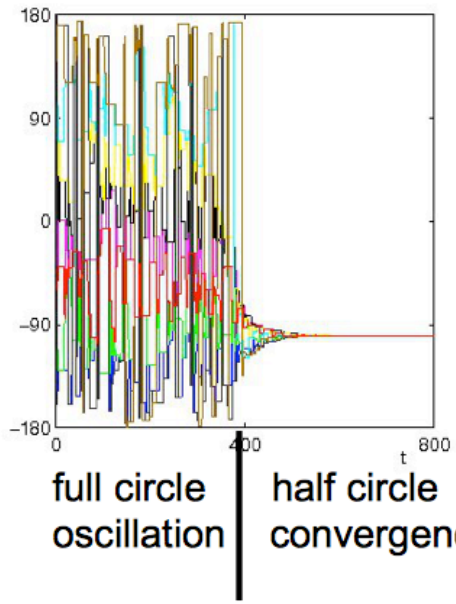
<u>Thm:</u> If G is uniformly connected, synchronizes with probability 1 also on the circle, sphere,...



Simulations on S^1



75% of random runs : nothing at 800 iterations



full circle oscillation | half circle convergence

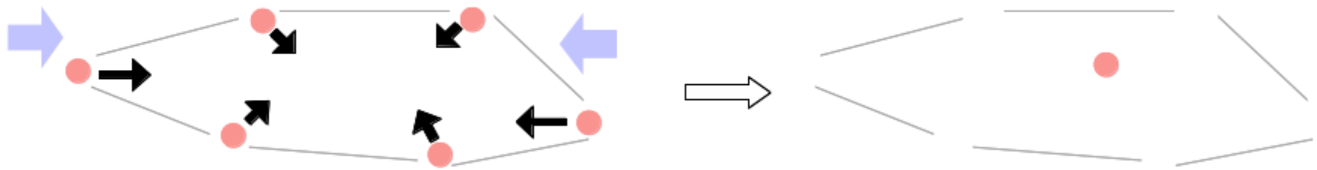
Alternative algorithms can overcome spurious local equilibria of standard consensus motion

Auxiliary variables (can be written with agent-based coordinates)

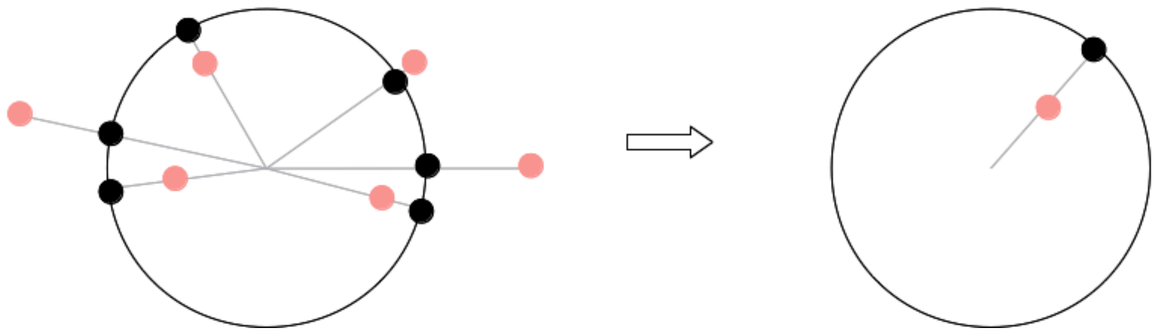
Embed the manifold in vector space $\mathbb{R}^n$

Assign an auxiliary variable $y_k \in \mathbb{R}^n$ to each agent

The y_k reach agreement by consensus in $\mathbb{R}^n$



Positions $x_k \in H$ are made to follow the projection of y_k on H



Consensus on nonlinear spaces & Graph coloring

1. Some examples to motivate nonlinear consensus

2. Formalizing consensus on nonlinear spaces

    Synchronization: from vector spaces to the circle

    Formalization on compact homogeneous manifolds

    Global synchronization properties

3. Link with graph coloring : (just) a complexity result

Consensus algorithms seem much harder to analyze on nonlinear spaces

Attractive agents, fixed undirected interaction graph
⇨ seems difficult to say if synchronization is the only stable equilibrium

How hard can equilibrium characterization be ?

"Consensus on nonlinear spaces and graph coloring"
A. Sarlette, CDC Orlando, pp. 4885-4890 (2011)

Idea: interacting agents setting ⇄ graph coloring

Graph theory

many complexity results

Analog computation:

continuous dynamics solve computational problem

Graph parametrizes interacting agents in continuous dynamics

Result: equilibrium characterization on projective space is NP-hard

graph k-coloring

NP-hard for k>2

(robust) repulsion on

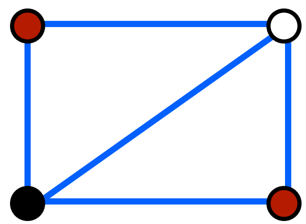
projective space $P^{k-1}R$

Graph coloring is a classical computational problem

- Given graph $G(V,E)$ and integer k,

  find $\varrho : V \longrightarrow \{1,2,\ldots,k\}$ (colors)

  s.t. $\varrho(a) \neq \varrho(b)$ for all $(a,b) \in E$



☺

Ex. country maps, Sudoku,...

Graph coloring is a classical computational problem

≙ Given graph G(V,E) and integer k,

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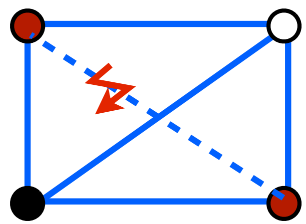


Ex. country maps, Sudoku,...

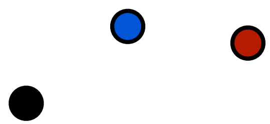
Complexity

For k=2: G is 2-colorable $\Leftrightarrow$ G is bipartite (polynomial)

For k>2: NP-hard (in #V) to determine if G(V,E) is k-colorable

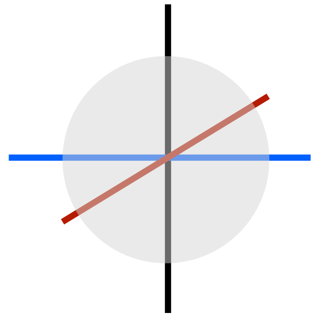
Graph k-coloring & Directions in R^k

k different equivalent colors
{1,2,...,k}



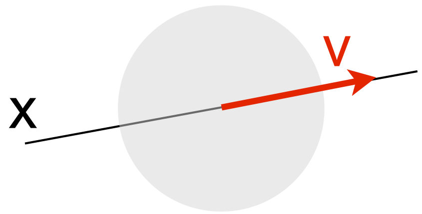
$\longleftrightarrow$

k orthogonal lines of R^k



Lines of R^k define the projective space $P^{k-1}R$

$x \in P^{k-1}R$ represents a line of R^k



Handy representation:

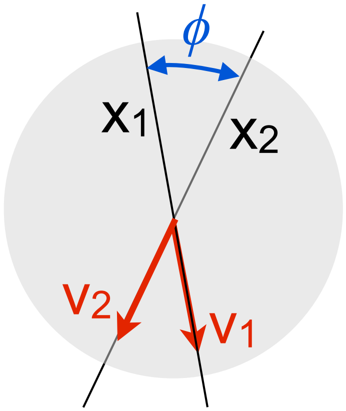
orthonormal projection Π onto line x

$\Pi \in R^{k \times k}$, $\text{rank}(\Pi)=1$, $\text{trace}(\Pi)=1$

$\Pi = v\, v^T \;/\; (v^T v)$

Chordal distance on $P^{k-1}R$

$$d_c(\Pi_1, \Pi_2) \quad := \quad \|\Pi_1 - \Pi_2\|_F$$
$$= \sqrt{2 - 2(v_1^T v_2)^2}$$
$$= \sqrt{2 \sin^2(\phi)}$$



Repulsive agents try to maximize their mutual distance

Cost function

$$W = \sum_{(a,b)\in E} g(d_c(\Pi_a, \Pi_b)^2)$$

graph dependence

with $g(\mathrm{x})$ a strictly monotonically increasing function on $[0,2]$

Gradient dynamics

$$\begin{aligned}\frac{d}{dt}\Pi_a &= \mathrm{grad}_{\Pi_a} W \\ &= -\sum g'(d_c(\Pi_a, \Pi_b)^2)\,(\Pi_a\Pi_b\Pi_a^\perp + \Pi_a^\perp\Pi_b\Pi_a)\end{aligned}$$

= anti-consensus motion on projective space

Goal: relate the stable equilibria to graph coloring solutions

Stable equilibria = local maxima of W

↳ result about complexity of characterizing stable equilibrium set
(as complex as deciding graph coloring)

↳ possibility to solve graph coloring by swarm optimization?
(continuous evolution of the swarm converges to solution
= distributed analog computation)

Two particular sets in $P^{k-1}R$

$S_o = \{(\Pi_1, \Pi_2, ..., \Pi_N) \in (\mathbb{P}^{k-1}\mathbb{R})^N : \Pi_a \Pi_b = \Pi_b \Pi_a \quad \forall\, a, b \}$

all states belong to a discrete set of k orthogonal lines = "colors"

$S_p(G) = \{(\Pi_1, \Pi_2, ..., \Pi_N) \in (\mathbb{P}^{k-1}\mathbb{R})^N : \Pi_a \Pi_b = 0 \;\; \forall\, (a, b) \in E \}$

every edge is stretched to the maximum distance

Properties :

$S_p(G)$ can be empty depending on G

$S_p(G)$ global maxima of W if $\neq \varnothing$ (*)

$S_o \cap S_p(G) \neq \varnothing$ if and only if G is k-colorable (**)

The complexity result

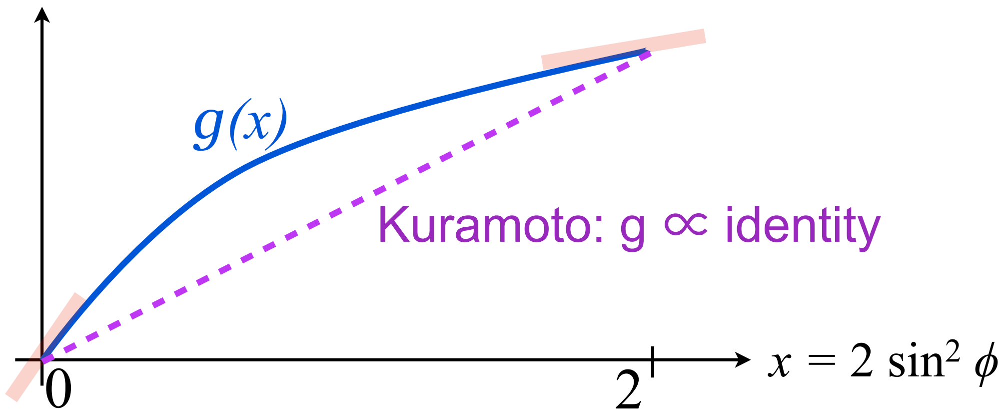
Question : Given G(V,E) and $P^{k-1}R$, is any point in S_o a stable equilibrium for the repulsive agents ?

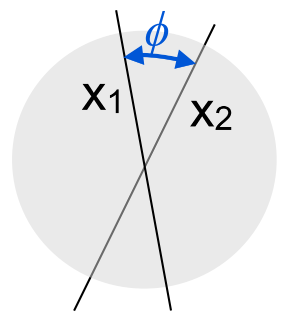
`yes/no` question (typical decision problem)
about specific property of equilibrium set

Theorem: This question is as difficult as graph coloring
-- that is NP-hard for k>2 --
if g(x) satisfies $g'(0)/g'(2) > \lfloor \frac{N}{k} \rfloor / (\lceil \frac{N}{k} \rceil - 1)$

The condition on coupling function $g(x)$ is not too restrictive ...

Condition $g'(0)/g'(2) > \lfloor \frac{N}{k} \rfloor / (\lceil \frac{N}{k} \rceil - 1)$

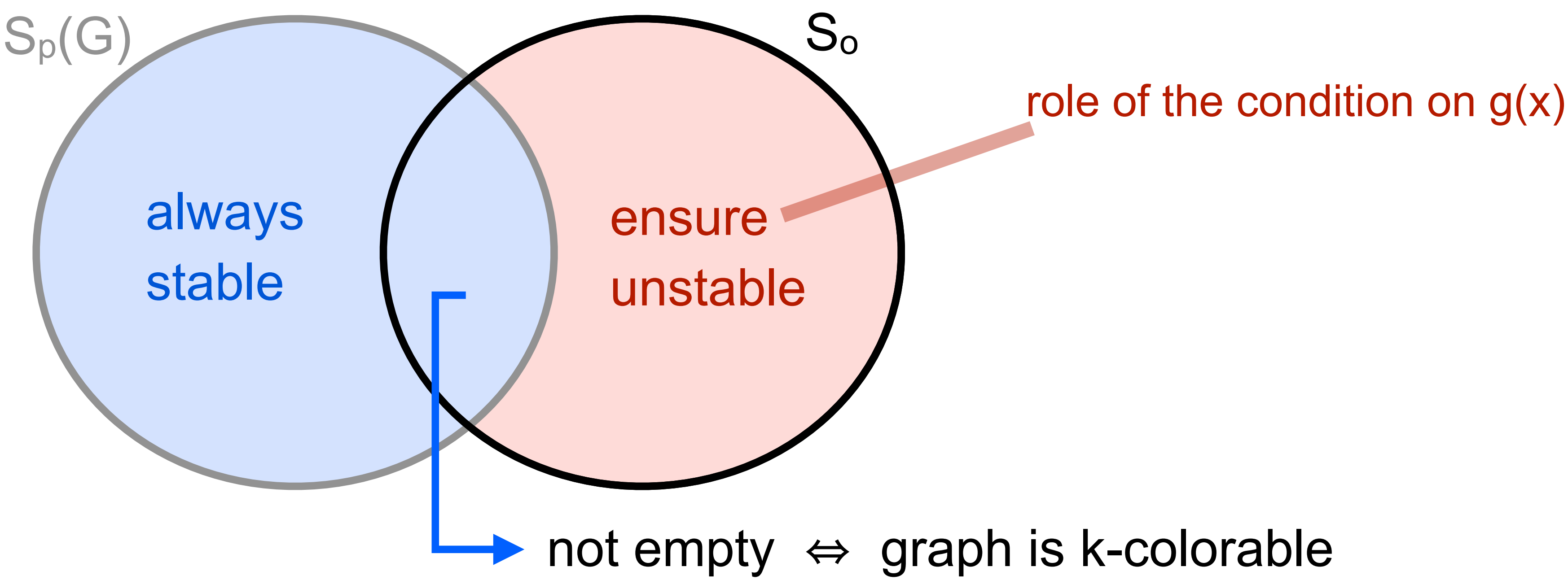




Large class of $g(x)$ coupling functions

Allows $g(x) \propto \text{identity}$ (canonical consensus) for $N/k \longrightarrow \infty$

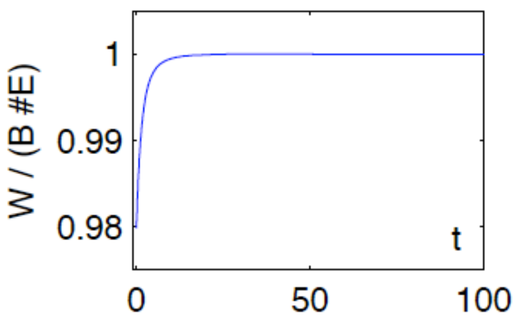
Proof idea

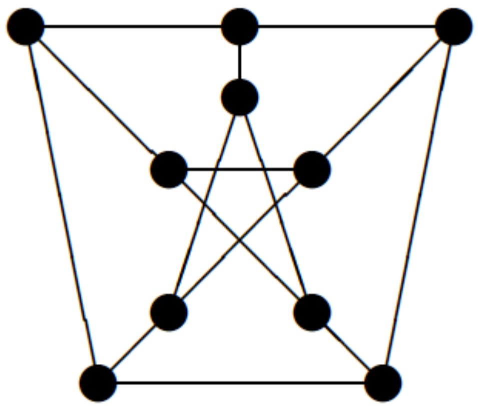


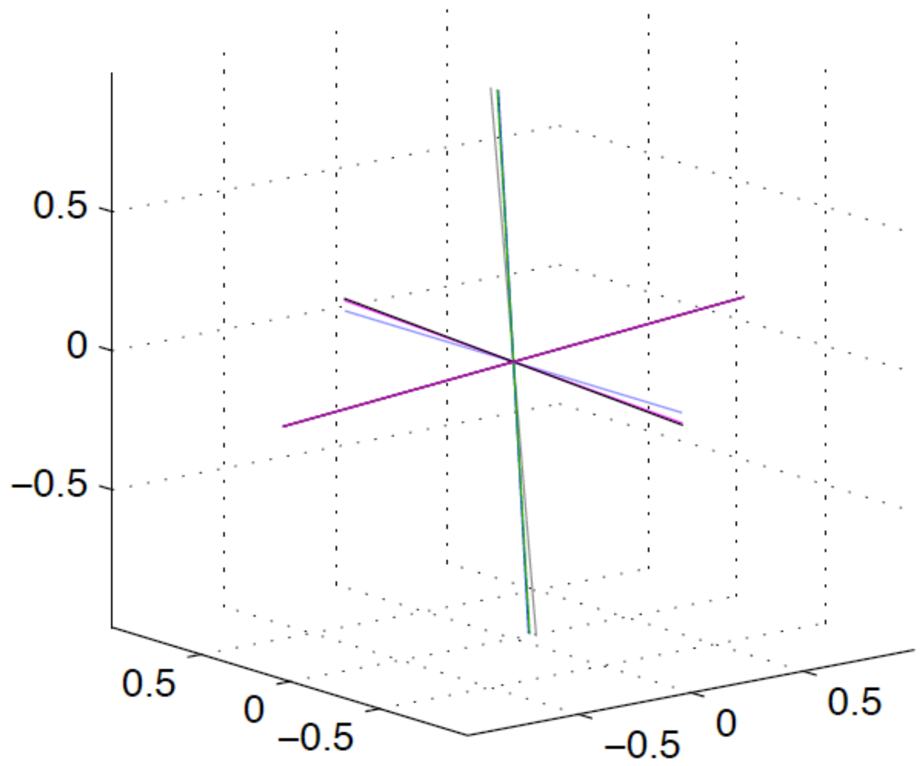
⇨ stable eq. in S_o ⇔ graph k-colorable

Simulations with $g(x) = \mathrm{atan}(x/2)$ for k=3

Petersen graph, 3-colorable

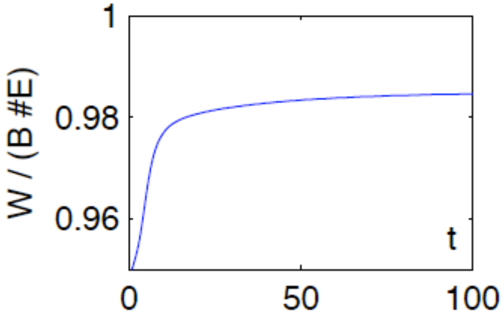


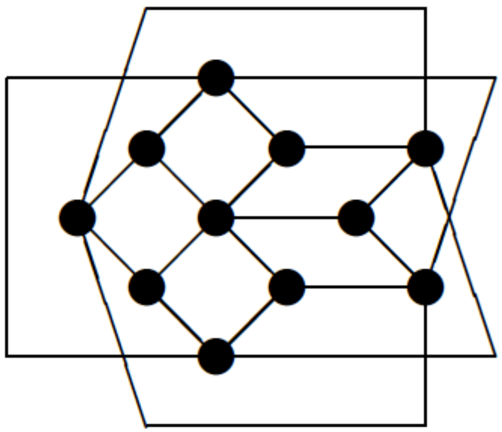


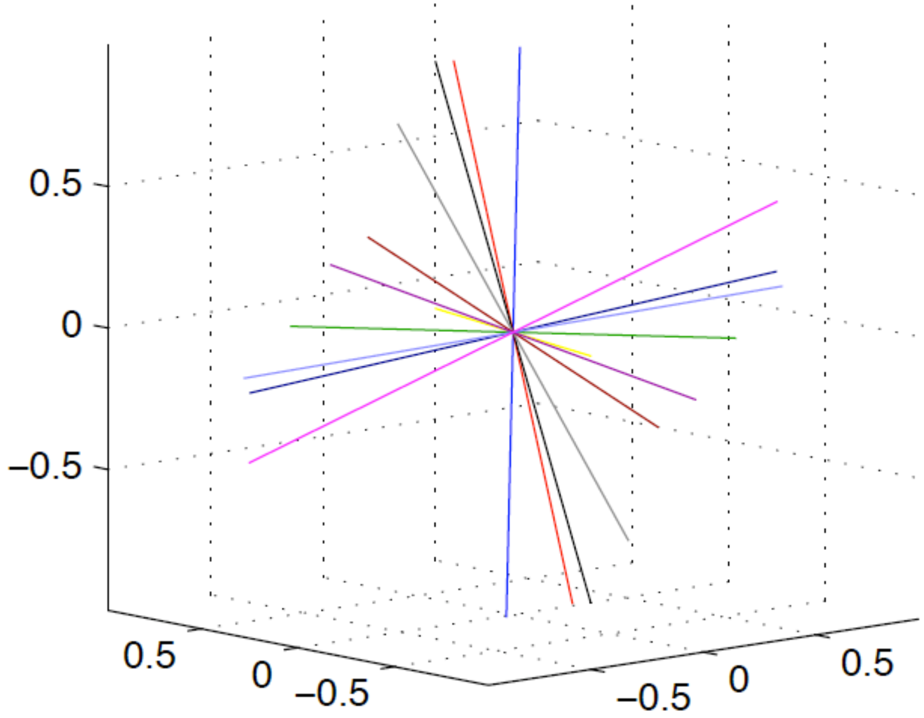


Simulations with $g(x) = \mathrm{atan}(x/2)$ for k=3

Grötzsch graph, not 3-colorable







Can we use the distributed dynamical system to solve graph coloring ?

Stable equilibria = local maxima of W

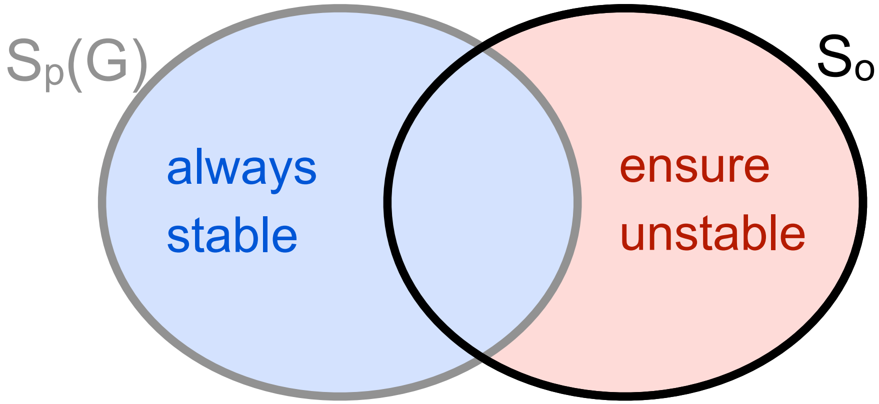
↳ result about complexity of characterizing stable equilibrium set (as complex as deciding graph coloring) OK

↳ possibility to solve graph coloring by swarm optimization? (continuous evolution of the swarm converges to solution = distributed analog computation) ??

NO

The multi-agent system on $P^{k-1}R$ does not solve graph-coloring

Global maxima of W in $S_o \cap S_p(G)$ $\longleftrightarrow$ graph k-coloring



Multi-agent system for **colorable G converges to** **$S_p(G)$** $\neq$ $S_o \cap S_p(G)$

[Kochen-Specker Theorem]
There exist non-colorable G for k=3 **with $S_p(G) \neq \emptyset$**

⇨ A system that converges to a point in $S_p(G) \setminus S_o$ can correspond to colorable or non-colorable G...

The Kochen-Specker theorem discusses fundamentals of quantum measurement

element of $P^{k-1}R$

$\equiv$ possible result of projective quantum measurement on R^k

Kochen-Specker:

For $k \geq 3$, there <u>does not</u> exist a function f from the set of possible measurement projectors $P_i \in P^{k-1}R$ to associated measurement results in {0,1} such that for every $\{P_i\}$ that form a physical observable (i.e. that commute and $\sum P_i = I$) we have $\sum f(P_i) = 1$

Use: show a contradiction with classical re-interpretations of quantum laws

The Kochen-Specker theorem discusses fundamentals of quantum measurement

Proof:

Constructs an example of N elements of $P^{k-1}R$, where mutually orthogonal lines are connected in a graph. Then assigning

f(color 1)=1 , f(other colors)=0 would solve the task if colorable

They have a counterexample with N=31 agents for k=3

⇨ They construct a situation where all pairs of connected agents are orthogonal in R^3, but the graph is NOT 3-colorable

Conclusion

General geometric interpretation of consensus
allows extension to nonlinear spaces

Consensus motion yields more complex global behavior than on R^n

- possible limit cycles, quasi-periodicity,... for directed graphs

- multiple equilibria for undirected graphs
  depending on precise coupling function & interaction graph

Conclusion

Graph-coloring ⟶ complexity of consensus on projective space

For a class of repulsion functions (robustly difficult)

Link not bi-directional: provides no solution for graph coloring

Equilibrium stability as key feature to characterize

NP-hard for k>2

⇨ leaves open the case k=2 correspoding to the circle
(which seems not trivial, but further unclear how hard)

Consensus on nonlinear spaces
and
Graph coloring

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