# Large Population Consensus in an Adversarial Environment

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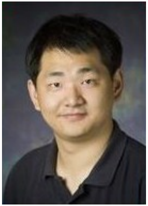


University of Padua, 14 July 2011.

# Outline

# Advection

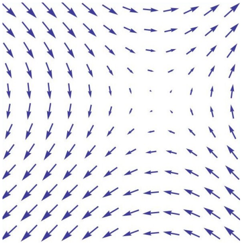
- $N \to \infty$ homogeneous agents with dynamics

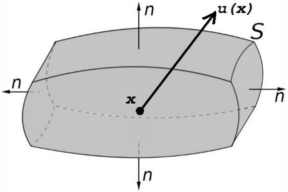
$$\dot{x}(t) = u(x(t))$$

- $\dot{x}(t)$ defines vector field

- density $m(x,t)$ in $x$ evolves according to advection equation

$$\partial_t m + div(m \cdot u(x)) = 0$$





# Mean field games

- Agents wish to minimize $J =$
$\int_0^T \Big[ \underbrace{\frac{1}{2}|u(t)|^2}_{\text{penalty on control}} + \underbrace{g(x(t), m(\cdot, t))}_{\text{...on state \& distribution}} \Big] dt + \underbrace{G(x(T), m(\cdot, T))}_{\text{...on final state}}$
- optimal control $u(t) = -\nabla_x J$
- coupled partial differential equations

$$-\partial_t J + \tfrac{1}{2}|\nabla_x J|^2 = g(x, m) \qquad \text{(HJB) - backward}$$

$$\partial_t m + div(m \cdot u(x)) = 0 \qquad \text{(advection) - forward}$$

- boundary conditions
$m(\cdot, 0) = m_0, \quad J(x, T) = G(x, m(\cdot, T))$

# Hamilton Jacobi Bellman

- From Bellman

$$\underbrace{J(x_0,t_0)}_{\text{today's cost}} = \min_u \underbrace{[\frac{1}{2}|u|^2 + g(x,m)]dt}_{\text{stage cost}} + \underbrace{J(x_0+dx,t_0+dt)}_{\text{future cost}}$$

- Taylor expanding future cost

$$J(x_0+dx,t_0+dt) = J(x_0,t_0) + \partial_t J dt + \nabla_x J \dot{x} dt$$

- $\min_u \underbrace{[\frac{1}{2}|u|^2 + g(x,m) + \partial_t J + \nabla_x J \overbrace{\dot{x}}^{u}]}_{\text{Hamiltonian}} = 0$
- optimal control $u(t) = -\nabla_x J$ yields

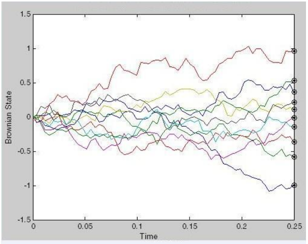
$$-\partial_t J + \frac{1}{2}|\nabla_x J|^2 = g(x,m) \quad \text{HJB}$$

# Stochastic differential game

- stochastic dynamics is

$$dx = udt + \sigma dB_t$$

- $dB_t$ infinitesimal Brownian motion




- Mean field games ($\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ Laplacian)

$$-\partial_t J + \frac{1}{2}|\nabla_x J|^2 - \frac{\sigma^2}{2}\Delta J = g(x,m) \qquad \text{(HJB)-backward}$$

$$u \downarrow \qquad \uparrow m$$

$$\partial_t m + div(m \cdot u(x)) - \frac{\sigma^2}{2}\Delta m = 0 \qquad \text{(Kolmogorov)-forward}$$

# Hamilton Jacobi Bellman (with $dx = udt + \sigma dB_t$)

- From Bellman

$$\underbrace{J(x_0,t_0)}_{\text{today's cost}} = \min_u \underbrace{[\frac{1}{2}|u|^2 + g(x,m)]dt}_{\text{stage cost}} + \underbrace{\mathbf{E}J(x_0+dx,t_0+dt)}_{\text{exp. future cost}}$$

- Taylor expanding future cost ($\mathbf{E}dB_t = 0$, $\mathbf{E}dB_t^2 \to dt$)

$$J(x_0+dx,t_0+dt) = J(x_0,t_0)+\partial_t J dt+\underbrace{\mathbf{E}\nabla_x J dx}_{\nabla_x J u dt} + \underbrace{\mathbf{E}\frac{1}{2}dx'\nabla_x^2 J dx}_{\frac{\sigma^2}{2}\Delta J \mathbf{E}dB_t^2}$$

- $\min_u \underbrace{[\frac{1}{2}|u|^2 + g(x,m) + \partial_t J + \nabla_x J u + \frac{\sigma^2}{2}\Delta J]}_{\text{Hamiltonian}} = 0$
- optimal control $u(t) = -\nabla_x J$ yields

$$-\partial_t J + \frac{1}{2}|\nabla_x J|^2 - \frac{\sigma^2}{2}\Delta J = g(x,m) \quad \text{HJB}$$

# Average cost

- $J = E \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left[\frac{1}{2}|u(t)|^2 + g(x(t), m(\cdot,t))\right] dt$
- Mean field games ($\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ Laplacian)

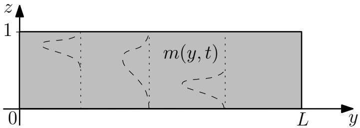
$$\begin{tikzcd}
\bar{\lambda} + \frac{1}{2}|\nabla_x \bar{J}|^2 - \frac{\sigma^2}{2}\Delta \bar{J} = g(x, \bar{m}) \arrow[d, bend right, "u"'] & \text{(HJB)} \\
div(\bar{m}\cdot u(x)) - \frac{\sigma^2}{2}\Delta \bar{m} = 0 \arrow[u, bend right, "\bar{m}"'] & \text{(Kolmogorov)}
\end{tikzcd}$$

# Discounted cost

- $J = E\int_0^\infty e^{-\rho t}\Big[\frac{1}{2}|u(t)|^2 + g(x(t), m(\cdot,t))\Big]dt$
- Mean field games ($\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ Laplacian)

$$
\begin{array}{lr}
-\partial_t J + \frac{1}{2}|\nabla_x J|^2 - \frac{\sigma^2}{2}\Delta J + \rho J = g(x,m) & \text{(HJB)} \\
\quad u \downarrow \qquad\qquad\qquad\qquad\qquad\qquad \uparrow m & \\
\partial_t m + div(m \cdot u(x)) - \frac{\sigma^2}{2}\Delta m = 0 & \text{(Kolmogorov)}
\end{array}
$$

# Mexican wave (mimicry & fashion)



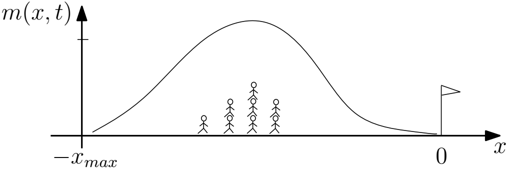

- state $x = [y, z]$, $y \in [0, L)$ coordinate, $z$ position:

$$z = \begin{cases} 1 & \text{standing} \\ 0 & \text{seated} \end{cases}, \quad z \in (0,1) \quad \text{intermediate}$$

- dynamics $dz = udt$ ($u$ control)
- penalty on state and distribution $g(x, m) =$

$$\underbrace{Kz^{\alpha}(1-z)^{\beta}}_{\text{comfort}} + \underbrace{\frac{1}{\epsilon^2}\int (z-\tilde{z})^2 m(\tilde{y}; t, \tilde{z}) \frac{1}{\epsilon} s(\frac{y-\tilde{y}}{\epsilon}) d\tilde{z} d\tilde{y}}_{\text{mimicry}}$$

# Meeting starting time (coordination with externality)




- dynamics $dx_i = u_i dt + \sigma dB_t$
- $\tilde{\tau}_i = min_s(x_i(s) = 0)$ arrival time, $t_s$ scheduled time, $\bar{t}$ actual starting time
- penalty on final state and distribution
  $$G(x(\tilde{\tau}_i), m(\cdot, \tau_i)) = \underbrace{c_1[\tilde{\tau}_i - t_s]_+}_{\text{reputation}} + \underbrace{c_2[\tilde{\tau}_i - \bar{t}]_+}_{\text{inconvenience}} + \underbrace{c_3[\bar{t} - \tilde{\tau}]_+}_{\text{waiting}}$$
- people arrived up to time $s$: $F(s) = -\int_0^s \partial_x m(0, v) dv$
- starting time $\bar{t} = F^{-1}(\theta)$, ($\theta$ is quorum)

# Large population (herd behaviour)

- behaviour dynamics
  $dx_i = u_i dt + \sigma dB_t$
- penalty
  $$g(x,m) = \beta(x - \underbrace{\int ym(y,t)dy}_{\text{average}})^2$$

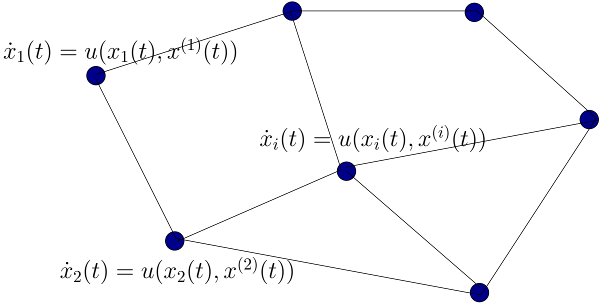
- discounted cost $J = E\int_0^\infty e^{-\rho t}\left[\frac{1}{2}|u(t)|^2 + g(x(t), m(\cdot,t))\right]dt$
- mean field game with discounted cost

$$-\partial_t J + \frac{1}{2}|\nabla_x J|^2 - \frac{\sigma^2}{2}\Delta J + \rho J = g(x,m) \qquad \text{(HJB)}$$

$$u \downarrow \qquad\qquad \uparrow m$$

$$\partial_t m + div(m \cdot u(x)) - \frac{\sigma^2}{2}\Delta m = 0 \qquad \text{(Kolmogorov)}$$

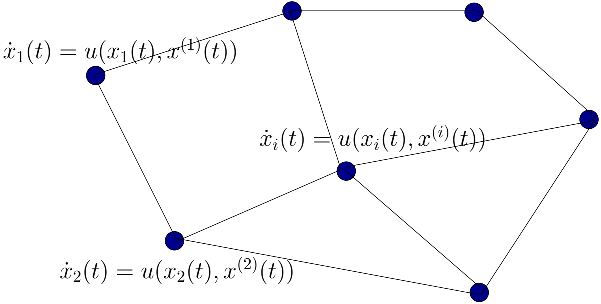
# Consensus




- $N$ "dynamic agents" (vehicles, employes, computers,...)
- ...described by differential (difference) equations
- model interaction through communication graph
- vector $x^{(i)}$ represents neighbors' states
- **main feature**: one agent is influenced only by neighbors:

$$\dot{x}_i(t) = u(x_i(t), x^{(i)}(t))$$

# why "consensus"?



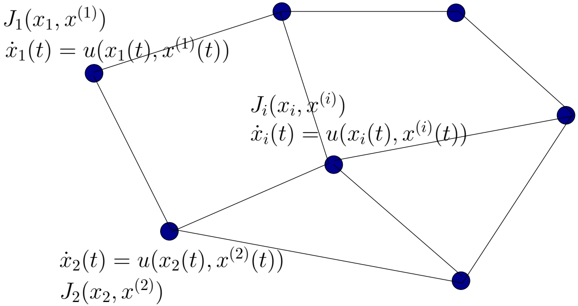

- use local control $u(x_i, x^{(i)})$
- .. to converge to global "consensus value"

$$x(t) \to ave(x(0))$$

- consensus originates from computer science
- shown connections to potential games

[Lynch, *Morgan Kaufmann*, 1996], [Shamma *et al.*, *Trans. on Systems, Man, and Cyb.*, to appear]

# Where is game theory here?




- Assign $N$ local objective functions $J_i(x_i, x^{(i)})$ so that..
- if local control $u(x_i, x^{(i)})$ is optimal w.r.t. $J_i(x_i, x^{(i)})$
- all states converge to global "consensus value"

$$x(t) \to ave(x(0))$$

[Lynch, *Morgan Kaufmann*, 1996], [Shamma *et al.*, *Trans. on Systems, Man, and Cyb.*, to appear]

# Linear quadratic consensus problem

- linear dynamics $\dot{x}_i = a_i x_i + b_i u_i$
- objective functions
  $J_i(x_i, x^{(i)}, u_i) = \frac{1}{2}\int_0^\infty e^{-\rho t}\Big[|u(t)|^2 + (x_i - x^{(i)})^2\Big]dt$
- averaging over neighbors $x^{(i)} = \sum_{j \in N_i} w_{ij} x_j$
- in compact form

$$\dot{x} = Ax + \sum_{i=1}^N B_i u_i$$
$$J_i(x_i, x^{(i)}, u_i) = \frac{1}{2}\int_0^\infty e^{-\rho t}\Big[R_i u_i^2 + x' Q x\Big]dt$$

# Feedback Nash Equilibrium

- Nash equilibrium strategies are linear in state:

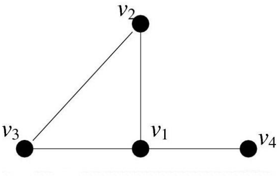
$$u_i = \mu_i(x,t) = -R_i^{-1} B_i' Z_i x$$

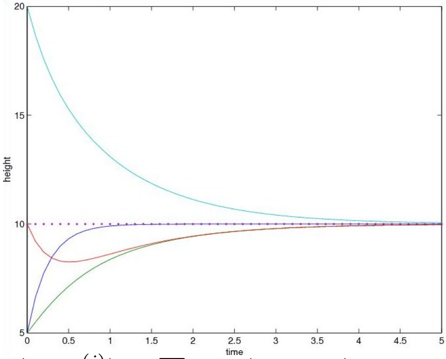
- $Z_i$ are solutions to the coupled Riccati equations

$$Z_i\Big(A - \sum_{i=N} S_i Z_i\Big) + \Big(A - \sum_{i=N} S_i Z_i\Big)' Z_i + Z_i S_i Z_i + Q_i = \rho_i Z_i$$

- $S_i = B_i' R_i^{-1} R_i$
- drawback:
    1. can be convoluted when $N$ is large,
    2. strategies use full state vector

# Linear (averaging): Arithmetic Mean



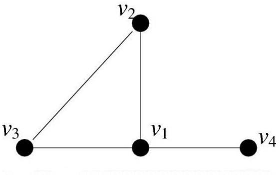



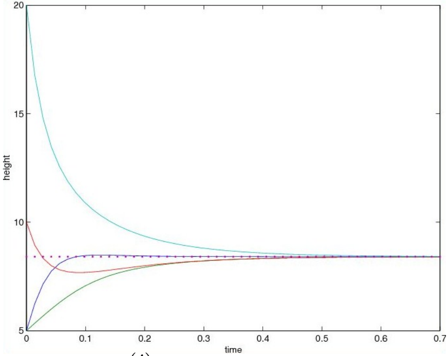

- local linear control: $u(x_i, x^{(i)}) = \sum_{j \in N_i}(x_j - x_i)$.
- local quadratic objective: $F(x_i, x^{(i)}) = \left(\sum_{j \in N_i}(x_j - x_i)\right)^2$;

$$J_i(x_i, x^{(i)}, u_i) = \lim_{T \longrightarrow \infty} \int_0^T \left(F(x_i, x^{(i)}) + \rho u_i^2\right) dt$$

[Olfati-Saber, Fax, Murray, *Proc. of the IEEE*, 2007], [Ren, Beard, Atkins, *ACC 2005*]

# Nonlinear 1/2 (Geometric Mean)



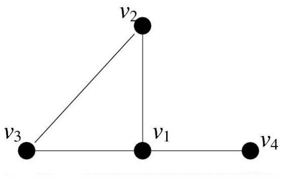



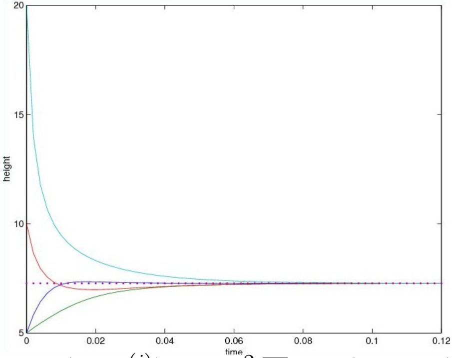

- local nonlinear control: $u(x_i, x^{(i)}) = x_i \sum_{j \in N_i} (x_j - x_i)$.
- local objective: $F(x_i, x^{(i)}) = \left( x_i \sum_{j \in N_i} (x_j - x_i) \right)^2$;

$$J_i(x_i, x^{(i)}, u_i) = \lim_{T \longrightarrow \infty} \int_0^T \left( F(x_i, x^{(i)}) + \rho u_i^2 \right) dt$$

[Bauso, Giarrè, Pesenti, *Systems and Control Letters*, 2006], [Cortes, *Automatica*, 2008]

# Nonlinear 2/2 (Harmonic Mean)

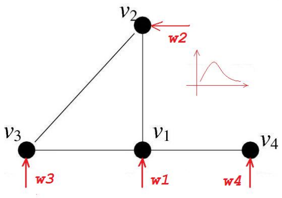


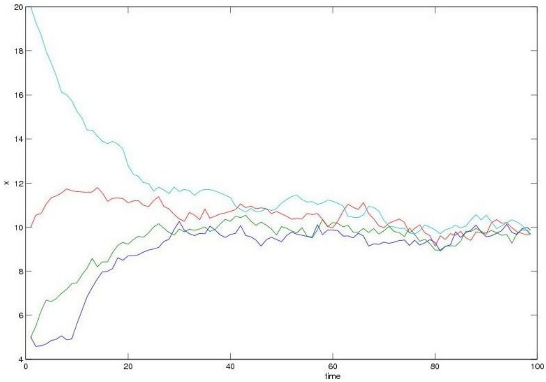




- local nonlinear control: $u(x_i, x^{(i)}) = -x_i^2 \sum_{j \in N_i}(x_j - x_i)$
- local objective: $F(x_i, x^{(i)}) = \left(x_i^2 \sum_{j \in N_i}(x_j - x_i)\right)^2$;
  $J_i(x_i, x^{(i)}, u_i) = \lim_{T \longrightarrow \infty} \int_0^T \left(F(x_i, x^{(i)}) + \rho u_i^2\right) dt$

[Bauso, Giarrè, Pesenti, *Systems and Control Letters*, 2006], [Cortes, *Automatica*, 2008]

# Disturbances 1/2 (Stochastic)

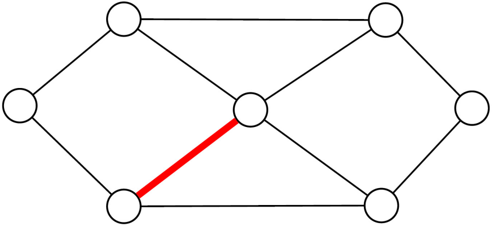



- Least-mean square consensus

[Xiao, Boyd, Kim, *J. Parallel and Distributed Computing*, 2007]

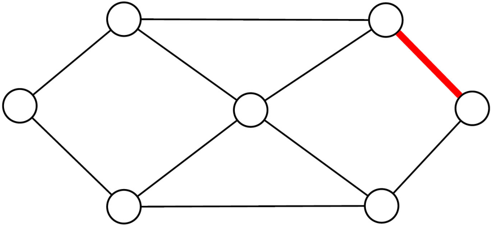
# Time-varying topology 1/3



- discrete-time gossip algorithms
- at $t = 1$
  - Pick (randomly) one edge $(i, j)$ and an increasing odd function $f(.)$
  - $x_i(t+1) = x_i(t) + f(x_i(t) - x_j(t))$
  - $x_j(t+1) = x_j(t) + f(x_j(t) - x_i(t))$

[Boyd *et al.*, *IEEE trans. on Information Theory*, 2006], [Giua *et al.*, *TAC*, in press]

# Time-varying topology 2/3

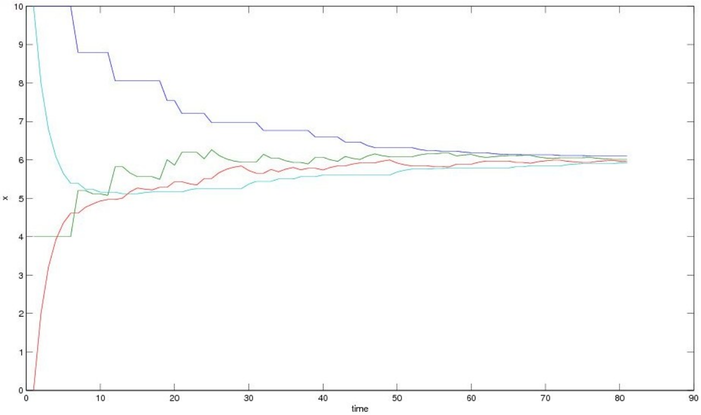


- at $t = 2$
  - Pick a second edge $(k, l)$
  - $x_k(t+1) = x_k(t) + f(x_k(t) - x_l(t))$
  - $x_l(t+1) = x_l(t) + f(x_l(t) - x_k(t))$

and so on for $t = 3, 4, \ldots$

[Boyd *et al.*, *IEEE trans. on Information Theory*, 2006], [Giua *et al.*, *TAC*, in press]

# Time-varying topology 3/3



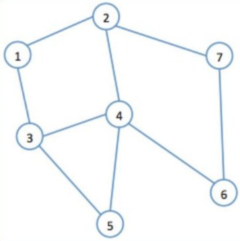

- slow convergence

[Boyd *et al.*, *IEEE trans. on Information Theory*, 2006], [Giua *et al.*, *TAC*, in press]

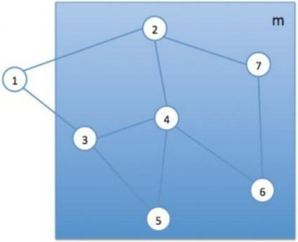
# From state-feedback NE to Mean field-feedback NE

- state feedback NE strategies

$$u_1 = \mu_1(x, t) = \mu_1(x_1, x_2, \ldots, x_7, t)$$




$$\Downarrow$$




- individual state feedback NE strategies $u_1 = \mu_1(x_1, m, t)$
- $m$ is aggregate and exogenous representation of $x_2, \ldots, x_7$

# homogeneous agents and complete graphs

- density $m(x,t) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} I_{x_i}$
- average $\bar{m}(t) = \int x m(x,t) dx$
- track average signal $\bar{m}(t)$ (exogenous)

$$J_i(x_i, \bar{m}, u_i) = \frac{1}{2} \int_0^{\infty} e^{-\rho t} \Big[ |u|^2 + (x_i - \bar{m})^2 \Big] dt$$

- initial states distribution $m(x,0) = m_0$ $(x_i \in \mathbb{R})$
- distribution evolution

$$\partial_t m + \partial_x((ax+bu)m(x,t)) = 0 \quad \text{(advection)}$$

# Feedback mean-field equilibrium

- Solve tracking problem via Riccati method and plug the optimal $u$ in advection (below)

$$u_i = -\frac{r}{b}\left(sx_i + k\right)$$

$$\begin{aligned}(2a-\rho)s + 1 - \frac{b^2}{r}s^2 &= 0\\ \left(a - \rho - \frac{b^2}{r}s\right)k - \bar{m}(t) &= 0.\end{aligned}$$

$$\frac{\partial}{\partial t}m(x,t) + \frac{\partial}{\partial x}\left((ax+bu)\cdot m(x,t)\right) = 0$$

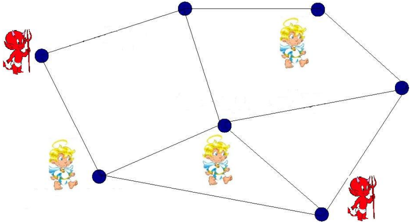
$$m(x,0) = m_0$$

$$\bar{m} = \int_{x\in\mathbb{R}} x m(x,t)dx$$

- solve advection and plug resulting $m$ in offset condition (above)

# Extensions

- Consensus with malicious agents
- applications: communication/social networks



- malicious agent $s$:

$$J_s = \frac{1}{2}\int_0^\infty e^{-\rho t}\Big[|u|^2 + \underbrace{(1-\alpha_s)(x_s - x^{(s)})^2}_{\text{track neighbors}} + \underbrace{\alpha_s(x_s - \bar{x}_s)^2}_{\text{track } \bar{x}_s}\Big]dt$$

- stochastic dynamics

$$dx_i = (a_i x_i + b_i u_i)dt + \sigma_i dB_{it}$$

# Conclusions

- Mean field games require solving coupled partial differential equations (HJB-Kolmogorov)
- consensus translates into a mean field game when infinite homogenous players (large population)
- analyze consensus with malicious agents
- inspect connections with opinion dynamics with stubborn agents in social networks
- mean field stochastic games

# Main references


- J.-M. Lasry and P.-L. Lions. Mean field games. *Japanese Journal of Mathematics*, 2(1), Mar. 2007.
- O. Guéant, J.-M. Lasry and P.-L. Lions. Mean field games and applications. *Lect. notes in math.*, Springer 2011.
- M. Huang, P.E. Caines, and R.P. Malhame. *IEEE TAC*, 52(9), 2007.
- Bauso, Giarrè, Pesenti, *Systems and Control Letters*, 2006.
- H. Tembine, Q. Zhu, and T. Basar, Risk-sensitive mean field stochastic games, IFAC WC 2011, Milano, Italy, 2011.