Optimal Control on Non-Compact Lie Groups: A Projection Operator approach

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Introduction

Why do Trajectory Optimization?

Well known:

- **Optimal control** may be used to provide stabilization, tracking, etc., for nonlinear systems
- **Model predictive/receding horizon** strategies have been used successful for a number of nonlinear systems with constraints

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Also:

- **Trajectory exploration**: What cool stuff can this system do?
  - capabilities
  - limitations
- **Trajectory modeling**: Can the trajectories of this (complex) system be modeled by those of a simpler system? [e.g., reduced order, flat, ...]
- **Objective function design**: needed to exploit system capabilities
- **Systems analysis**: investigate system structure, e.g., controllability

Minimization of Trajectory Functionals

Consider the problem of minimizing a functional

$$h(x(\cdot), u(\cdot)) := \int_0^T l(x(\tau), u(\tau), \tau)\, d\tau + m(x(T))$$

over the set $\mathcal{T}$ of bounded trajectories of the nonlinear system

$$\dot{x}(t) = f(x(t), u(t))$$

with $x(0) = x_0$ (... without additional constraints).

We write this **constrained** problem as

$$\min_{\xi \in \mathcal{T}} h(\xi)$$

where

$\xi = (\alpha(\cdot), \mu(\cdot))$ is a bounded curve with $\alpha(\cdot)$ continuous and $\alpha(0) = x_0$.

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How can we approach this problem?

Unconstrained (?) Optimal Control

- In the usual case, the choice of a **control** trajectory $u(\cdot)$ determines the **state** trajectory $x(\cdot)$ (recall that x_0 has been specified). With such a **trajectory parametrization**, one obtains so-called ***unconstrained* optimal control problem**

$$\min_{u(\cdot)} \; h(x(\cdot; x_0, u(\cdot)), u(\cdot))$$

- Why not just search over control trajectories $u(\cdot)$? If the system described by f is sufficiently stable, then such a **shooting method** may be effective.

- Unfortunately, the modulus of continuity of the map $u(\cdot) \mapsto (x(\cdot), u(\cdot))$ is often so large that such shooting is **computationally useless**:

  small changes in $u(\cdot)$ may give **LARGE** changes in $x(\cdot)$

Projection Operator Approach

Key Idea: a **trajectory tracking controller** may be used to minimize the effects of system instabilities, providing a numerically effective, redundant **trajectory parametrization**.

Let $\xi(t) = (\alpha(t), \mu(t))$, $t \geq 0$, be a bounded curve and let $\eta(t) = (x(t), u(t))$, $t \geq 0$, be the trajectory of f determined by the **nonlinear feedback** system

$$\begin{aligned} \dot{x} &= f(x,u), \qquad\qquad x(0) = x_0, \\ u &= \mu(t) + K(t)(\alpha(t) - x) \; . \end{aligned}$$

The map

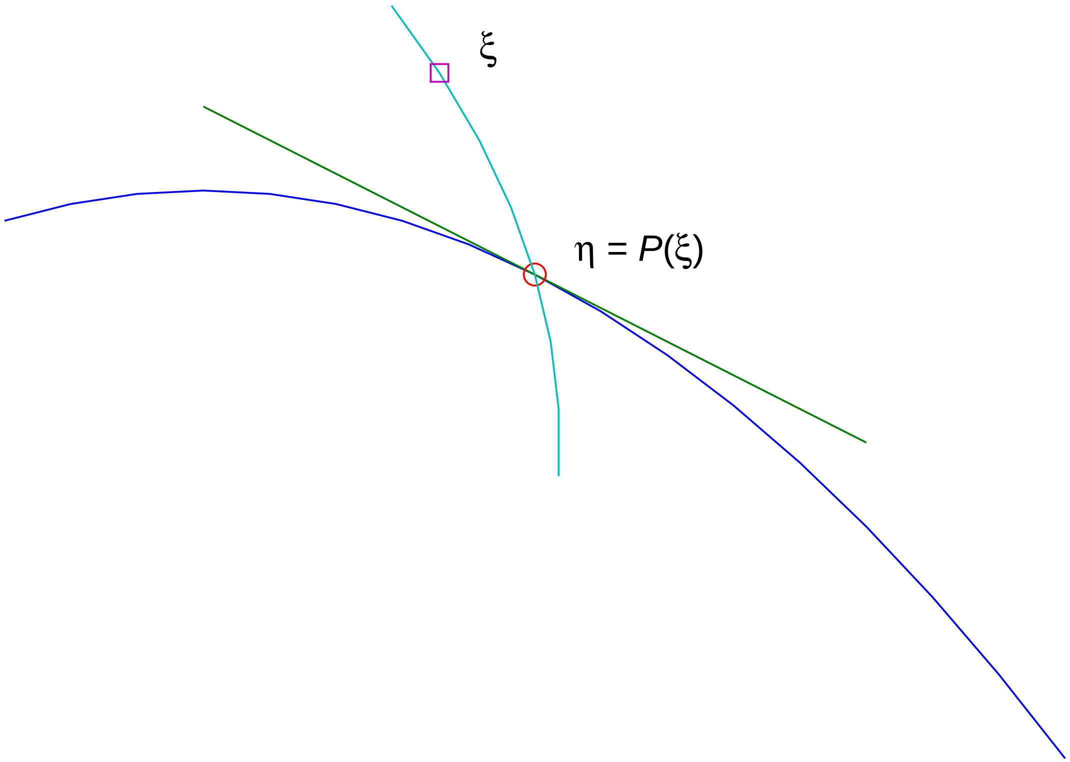
$$\mathcal{P} : \xi = (\alpha(\cdot), \mu(\cdot)) \mapsto \eta = (x(\cdot), u(\cdot))$$

is a continuous, **Nonlinear Projection Operator**.

For each $\xi \in \operatorname{dom} \mathcal{P}$, the curve $\eta = \mathcal{P}(\xi)$ is a trajectory.

Note: the trajectory contains both **state** and **control** curves.

Projection Operator



Projection Operator Properties

Suppose that f is C^r and that K is **bounded** and **exponentially stabilizes** $\xi_0 \in \mathcal{T}$. Then [1]

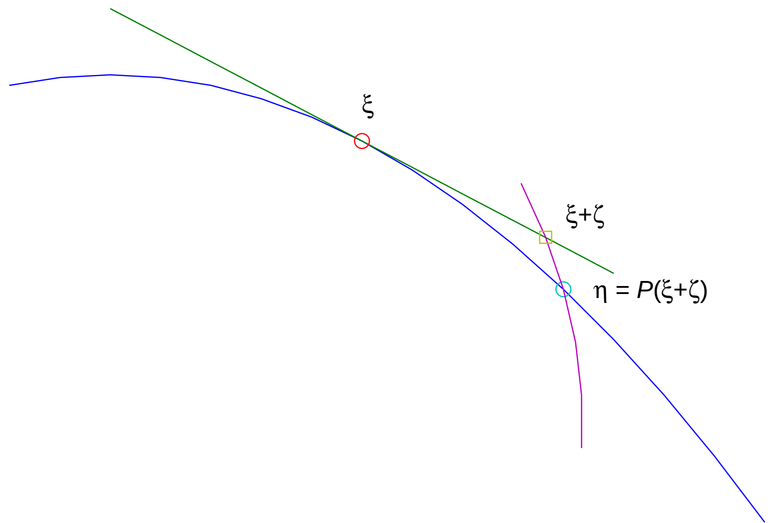
- $\mathcal{P}$ is well defined on an L_∞ neighborhood of ξ_0
- $\mathcal{P}$ is C^r (Fréchet diff wrt L_∞ norm)
- $\xi \in \mathcal{T}$ **if and only if** $\xi = \mathcal{P}(\xi)$
- $\mathcal{P} = \mathcal{P} \circ \mathcal{P}$ (**projection**)

On the finite interval $[0, T]$, choose $K(\cdot)$ to obtain stability-like properties so that the **modulus of continuity** of $\mathcal{P}$ is relatively **small**.

On the infinite horizon, **instabilities must be stabilized** in order to obtain a projection operator; consider $\dot{x} = x + u$.

[1] J. Hauser and D. Meyer,
"The trajectory manifold of a nonlinear control system",
Proceedings of the 37th IEEE Conference of
Decision and Control (CDC), vol. 1, pp.1034-1039, **1998**

Trajectory Manifold



Theorem: $\mathcal{T}$ is a **Banach manifold**: Every $\eta \in \mathcal{T}$ near $\xi \in \mathcal{T}$ can be **uniquely represented** as

$$\eta = \mathcal{P}(\xi + \zeta), \qquad \zeta \in T_\xi \mathcal{T}$$

Key: the **projection** operator $D\mathcal{P}(\xi)$ provides the required **subspace splitting**. Note: $\zeta \in T_\xi \mathcal{T}$ **if and only if** $\zeta = D\mathcal{P}(\xi) \cdot \zeta$

Equivalent Optimization Problems

Using the **projection operator**, we see that

$$\min_{\xi \in \mathcal{T}} h(\xi) = \min_{\xi = \mathcal{P}(\xi)} h(\xi)$$

where

$$h(x(\cdot), u(\cdot)) = \int_0^T l(\tau, x(\tau), u(\tau))\, d\tau + m(x(T))$$

Furthermore, defining

$$\tilde{h}(\xi) := h(\mathcal{P}(\xi))$$

for $\xi \in \mathcal{U}$ with $\mathcal{P}(\mathcal{U}) \subset \mathcal{U} \subset \operatorname{dom} \mathcal{P}$, we see that

$$\underbrace{\min_{\xi \in \mathcal{T}} h(\xi)}_{\text{constrained}} \quad \text{and} \quad \underbrace{\min_{\xi \in \mathcal{U}} \tilde{h}(\xi)}_{\text{unconstrained}}$$

are **equivalent** in the sense that

- if $\xi^* \in \mathcal{T} \cap \mathcal{U}$ is a **constrained** *local minimum* of h, then it is an **unconstrained** *local minimum* of $\tilde{h}$;
- if $\xi^+ \in \mathcal{U}$ is an **unconstrained** *local minimum* of $\tilde{h}$ in $\mathcal{U}$, then $\xi^* = \mathcal{P}(\xi^+)$ is a **constrained** *local minimum* of h.

Projection operator Newton method

given initial trajectory $\xi_0 \in \mathcal{T}$

for $i = 0, 1, 2, \ldots$

redesign feedback $K(\cdot)$ if desired/needed

descent direction $\zeta_i = \arg \min_{\zeta \in T_{\xi_i}\mathcal{T}} Dh(\xi_i)\cdot\zeta + \frac{1}{2} D^2\tilde{h}(\xi_i)\cdot(\zeta, \zeta)$ (LQ)

line search $\gamma_i = \arg \min_{\gamma \in (0,1]} h(\mathcal{P}(\xi_i + \gamma\zeta_i))$

update $\xi_{i+1} = \mathcal{P}(\xi_i + \gamma_i\zeta_i)$

end

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This **direct method** generates a descending trajectory sequence in **Banach space**! Also, **quadratic** convergence rate.

Note that

$$h(\xi) + \varepsilon\, Dh(\xi)\cdot\zeta + \frac{1}{2}\varepsilon^2 D^2\tilde{h}(\xi)\cdot(\zeta,\zeta)$$

is the **second order approximation** of $\tilde{h}(\xi + \varepsilon\zeta) = h(\mathcal{P}(\xi + \varepsilon\zeta))$ when $\xi \in \mathcal{T}$ and $\zeta \in T_\xi\mathcal{T}$.

Derivatives

First and **second** derivative of $\tilde{h}(\xi) = h(\mathcal{P}(\xi))$ are given by

$$D\tilde{h}(\xi) \cdot \zeta = Dh(\mathcal{P}(\xi)) \cdot D\mathcal{P}(\xi) \cdot \zeta$$

$$\begin{aligned} D^2\tilde{h}(\xi) \cdot (\zeta_1, \zeta_2) = \\ & D^2 h(\mathcal{P}(\xi)) \cdot (D\mathcal{P}(\xi) \cdot \zeta_1, D\mathcal{P}(\xi) \cdot \zeta_2) \\ & + Dh(\mathcal{P}(\xi)) \cdot D^2\mathcal{P}(\xi) \cdot (\zeta_1, \zeta_2) \end{aligned}$$

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When $\xi \in \mathcal{T}$ and $\zeta_i \in T_\xi \mathcal{T}$, they specialize into

$$D\tilde{h}(\xi) \cdot \zeta = Dh(\xi) \cdot \zeta$$

$$D^2\tilde{h}(\xi) \cdot (\zeta_1, \zeta_2) = D^2 h(\xi) \cdot (\zeta_1, \zeta_2) + Dh(\xi) \cdot D^2\mathcal{P}(\xi) \cdot (\zeta_1, \zeta_2)$$

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How to compute $D^2\mathcal{P}(\xi) \cdot (\zeta_1, \zeta_2)$?

Computation of $D^2\mathcal{P}$

We may use ODEs to calculate $D^2\mathcal{P}(\xi)\cdot(\zeta_1,\zeta_2)$:

$$\begin{aligned}
\eta &= (x,u) = \mathcal{P}(\xi) = \mathcal{P}(\alpha,\mu)\\
\gamma_i &= (z_i,v_i) = D\mathcal{P}(\xi)\cdot\zeta_i = D\mathcal{P}(\xi)\cdot(\beta_i,\nu_i)\\
\omega &= (y,w) = D^2\mathcal{P}(\xi)\cdot(\zeta_1,\zeta_2)
\end{aligned}$$

$$\begin{aligned}
\eta(t):\quad \dot{x}(t) &= f(x(t),u(t)), \qquad x(0)=x_0\\
u(t) &= \mu(t)+K(t)(\alpha(t)-x(t))
\end{aligned}$$

$$\begin{aligned}
\gamma_i(t):\quad \dot{z}_i(t) &= A(\eta(t))z_i(t)+B(\eta(t))v_i(t), \qquad z_i(0)=0\\
v_i(t) &= \nu_i(t)+K(t)(\beta_i(t)-z_i(t))
\end{aligned}$$

$$\begin{aligned}
\omega(t):\quad \dot{y}(t) &= A(\eta(t))y(t)+B(\eta(t))w(t)+D^2 f(\eta(t))\cdot(\gamma_1(t),\gamma_2(t))\\
w(t) &= -K(t)y(t), \qquad y(0)=0
\end{aligned}$$

- The derivatives are about the **trajectory** $\eta=\mathcal{P}(\xi)$
- The feedback $K(\cdot)$ stabilizes the state at each level

This was the introduction...

What if the system evolves on a Lie group?

Mathematical Preliminaries

Smooth manifolds

A **smooth manifold** M is a set which "**locally looks like** $\mathbb{R}^n$".
Think about, e.g., the 2-sphere $\mathbb{S}^2$ in $\mathbb{R}^3$.

- Manifolds with be indicated with capital letters, usually M or N.
- A **point** on the manifold will be denoted simply by x.
- $T_x M$ and $T_x^* M$ denote, respectively, the **tangent** and **cotangent spaces** of M at x.
- A generic **tangent vector** is usually written as v_x or w_x.
- The **tangent** and **cotangent bundles** of M are denoted by TM and T^*M, respectively.

Vector fields on a manifold

- The **natural bundle projection** from TM to M is the mapping

$$\begin{aligned} \pi: \quad TM &\rightarrow M \\ \mathrm{v}_x &\mapsto x \end{aligned}$$

- A **vector field** on a manifold M is a mapping

$$\begin{aligned} X: \quad M &\rightarrow TM \\ x &\mapsto X(x) \end{aligned}$$

which is a **section** of the tangent bundle TM, that is, it satisfies

$$\pi X(x) = x$$

Lie groups

- A **Lie group** is a smooth manifold endowed with a group structure. The group operation must be **smooth**.
- A generic **Lie group** is denoted by G.
- Typical examples are the groups SO(3), SE(2), SE(3), and U(n)...
- ...but also TSO(3), TSE(2), TSE(3) are Lie groups!

  These are called the **tangent groups**.
  Our theory apply to mechanical systems.

Lie groups (cont'd)

- **Left** and **right translations** of $x \in G$ (a group element) by the group element $g \in G$ are denoted by

$$L_g x \quad \text{and} \quad R_g x,$$

respectively.

- When convenient, we will adopt also the **shorthand notation**

$$gx, \quad xg, \quad g\mathrm{v}_x, \quad \mathrm{v}_x g$$

for, in the same order,

$$L_g x, \quad R_g x, \quad T_x L_g(\mathrm{v}_x), \quad T_x R_g(\mathrm{v}_x)$$

.

Lie Algebras

- A **left-invariant vector field** on G is a vector field X that satisfies

$$X(L_g x) = (T_x L_g) X(x).$$

- Given $\varrho \in T_e G$, the symbol X_ϱ is the associated left-invariant vector field

$$X_\varrho(g) := T_e L_g(\varrho).$$

- The **Lie algebra** $\mathfrak{g}$ is identified with the **tangent space** $T_e G$ endowed with the **Lie bracket** operation

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

  defined by

$$[\varrho, \varsigma] := [X_\varrho, X_\varsigma](e),$$

  where the later bracket is the **Jacobi-Lie bracket** evaluated at the group identity.

Triviality and exponential map

- The tangent bundle TG of Lie groups G is trivial. That is,

$$TG \approx G \times \mathfrak{g}.$$

- The **exponential map** $\exp : \mathfrak{g} \to G$ is a diffeomorphism between a neighborhood of the origin of the Lie Algebra $\mathfrak{g}$ and a neighborhood of the identity of the Lie group G.

- The exponential map $\exp : \mathfrak{g} \to G$ can be used to parameterize the neighborhood of any point $g \in G$.

  Using left translation, we parameterize a neighborhood of $g \in G$ as

$$g \exp(\xi)\,, \quad \xi \in \mathfrak{g}$$

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- **Key idea:** On a Lie group, the expansion of a function $f : G_1 \to G_2$ is written as

$$f(g \exp_{G_1}(tv)) = f(g) \exp_{G_2}(n_v(t)).$$

  This generalized on a vector space

$$f(x + tv) = f(g) + n_v(t)$$

Control systems on Lie groups

Control systems on a Lie group

- A **control system on a Lie group** G is a mapping

$$\begin{aligned} f: \quad G \times \mathbb{R}^m \times \mathbb{R} &\to TG \\ (g, u, t) &\mapsto f(g, u, t)\,, \end{aligned}$$

such that $\pi f(g, u, t) = g$ for each $(g, u, t) \in G \times \mathbb{R}^m \times \mathbb{R}$

- A **state trajectory** $g(t)$, $t \geq 0$, of f is an absolutely continuous curve in G that satisfies (a.e.), for an assigned input $u(t)$,

$$\dot{g}(t) = f(g(t), u(t), t)\,.$$

We will assume f is sufficiently smooth, Lipschitz, ... to guarantee **existence** and **uniqueness** of solutions.

- We can rewrite $\dot{g}(t) = f(g(t), u(t), t)$ as

$$\dot{g}(t) = g(t)\lambda(g(t), u(t), t)\,,$$

where $\lambda : G \times \mathbb{R}^m \times \mathbb{R} \to \mathfrak{g}$, $\lambda(g, u, t) := g^{-1} f(g, u, t)$ is the **left trivialization** of the control vector field f.

The Projection Operator approach on Lie groups

Minimization of Trajectory Functionals

Consider the problem of minimizing a functional

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over the set $\mathcal{T}$ of (bounded) trajectories of the nonlinear system

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with $g(0) = g_0$.

As in the vector case, we write this **constrained** problem as

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How can we generalize the Projection Operator approach to Lie groups?

Projection operator Newton method

The Newton algorithm is structurally the same:

given initial trajectory $\xi_0 \in \mathcal{T}$

for $i = 0, 1, 2, \ldots$

redesign feedback $K(\cdot)$ if desired/needed

descent direction

$$\zeta_i = \arg \min_{\xi_i \zeta \in T_{\xi_i} \mathcal{T}} Dh(\xi_i) \cdot \xi_i \zeta + \frac{1}{2} \mathbb{D}^2 \tilde{h}(\xi_i) \cdot (\xi_i \zeta, \xi_i \zeta) \quad \text{(LQ)}$$

line search $\gamma_i = \arg \min_{\gamma \in (0,1]} h(\mathcal{P}(\xi_i \exp(\gamma \zeta_i)))$

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end

- What is the linearization of a system evolving of a Lie group ? $\xi_i \zeta \in T_{\xi_i} \mathcal{T}$.
- What does it mean to compute a second derivative on a Lie groups ? $Dh(\xi_i) \cdot \xi_i \zeta + \frac{1}{2} \mathbb{D}^2 \tilde{h}(\xi_i) \cdot (\xi_i \zeta, \xi_i \zeta)$

Left-trivialized linearization around a trajectory

Left-trivialized linearization around a trajectory

- Let

$$\eta(t) = (g(t), u(t)), \quad t \in [0, \infty)$$

  be a the state-input trajectory of f.

- Consider the **linear perturbation** of the input defined as

$$u_\varepsilon(t) := u(t) + \varepsilon v(t)$$

- Indicate with g_ε the **perturbed state trajectory** associated with u_ε.
- The state trajectory g_ε satisfies

$$\dot{g}_\varepsilon(t) = g_\varepsilon(t)\lambda(g_\varepsilon(t), u_\varepsilon(t), t)\,,$$
$$g_\varepsilon(0) = g_0\,.$$

Left-trivialized perturbed trajectory

- Define the **left-trivialized perturbed trajectory**

$$z_\varepsilon(t), \quad t \in [0, T(\varepsilon)),$$

so that

$$g_\varepsilon(t) = g(t)\exp(z_\varepsilon(t)), \quad t \in [0, T(\varepsilon))$$

- Define $x_\varepsilon(t) := \exp z_\varepsilon(t)$.
- The left trivialized perturbed trajectory satisfies

$$\dot{z}_\varepsilon = \mathbf{d}\log_{z_\varepsilon}\Big(\mathrm{Ad}_{x_\varepsilon}\lambda\big(g x_\varepsilon, u_\varepsilon, t\big) - \lambda\big(g, u, t\big)\Big)$$
$$z_\varepsilon(0) = 0\,.$$

where

$$\mathbf{d}\log_\varrho \varsigma = \mathbf{D}\log(\exp(\varrho)) \cdot \exp(\varrho)\varsigma \qquad \textbf{(trivialized tangent)}$$

and

Ad is the **adjoint action** of G on $\mathfrak{g}$.

Left-trivialized linearization around a trajectory

The left-trivialized perturbed trajectory $z_\varepsilon(t)$, $t \geq 0$, can be expanded to first order as $z_\varepsilon(t) = \varepsilon z(t) + o(\varepsilon)$, where $z(t)$ is given by the left-trivialized linearization

$$\dot{z}(t) = A(\eta(t), t)\, z(t) + B(\eta(t), t)\, v(t)\,,$$
$$z(0) = z_0\,,$$

with

$$A(\eta, t) := \mathbf{D}_1\lambda(g, u, t) \circ TL_g - \mathrm{ad}_{\lambda(g,u,t)}\,,$$
$$B(\eta, t) := \mathbf{D}_2\lambda(g, u, t)\,,$$

where

ad is the **adjoint action** of $\mathfrak{g}$ on itself.

Projection Operator

Projection Operator on a Lie Group

- Vector space $\mathbb{R}^n$
  The Projection Operator $\eta = (x, u) = \mathcal{P}(\alpha, \mu) = \mathcal{P}(\xi)$ is given by

$$\dot{x} = f(x, k(x, \xi, t))$$
$$u = k(x, \xi, t) = \alpha + K(t)(\mu - x)$$

- Lie group G
  The Projection Operator $\eta = (x, u) = \mathcal{P}(\alpha, \mu) = \mathcal{P}(\xi)$ is given by

$$\dot{g} = f(g, k(g, \xi, t)) = g\lambda(g, k(g, \xi, t))$$
$$u = k(g, \xi, t) = \alpha + K(t)[\log(g^{-1}\mu)]$$

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- Note that $(\mathbb{R}^n, +)$ **is** an abelian Lie group!
  Given $x_1, x_2 \in \mathbb{R}^n$, $x_2^{-1}x_1 = x_1 - x_2 = -x_2 + x_1$.
  Also, $\exp(v) = v$, Ad $= \mathrm{id}$, and ad $= \mathrm{id}$.

  The theory on $\mathbb{R}^n$ is a **special case** of the general theory!

Linearization of the Projection Operator

				Vector Space	Lie Group
Curve	ξ	$=$	(α, μ)	$\mathbb{R}^n \times \mathbb{R}^m$	$G \times \mathbb{R}^m$
Perturbation	ζ	$=$	(β, ν)	$\mathbb{R}^n \times \mathbb{R}^m$	$\mathfrak{g} \times \mathbb{R}^m$
Trajectory	η	$=$	(g, u)	$\mathbb{R}^n \times \mathbb{R}^m$	$G \times \mathbb{R}^m$
Traj. perturbation	γ	$=$	(z, v)	$\mathbb{R}^n \times \mathbb{R}^m$	$\mathfrak{g} \times \mathbb{R}^m$

- Vector space $\mathbb{R}^n$
  $\mathcal{P}(\xi + \varepsilon\zeta) = \eta + \varepsilon\gamma + o(\varepsilon)$. We obtain

$$\dot{z} = A(\eta(t))z + B(\eta(t))v\,, \qquad z(0) = 0$$
$$v = \nu + K(t)(\beta - z)$$

- Lie group G
  $\mathcal{P}(\xi \exp(\varepsilon\zeta)) = \mathcal{P}(\xi)\exp(\varepsilon\gamma + o(\varepsilon))$. We obtain, recall $\mathcal{P}(\xi) = \eta$,

$$\dot{z} = A(\eta(t))\, z + B(\eta(t))\, v\,, \qquad z(0) = 0$$
$$v = \nu + K(t)\,\mathbf{d}\log_{\log(g^{-1}\alpha)}(\mathrm{Ad}_{g^{-1}\alpha}\beta - z)$$

Linearization of the Projection Operator

				Vector Space	Lie Group
Curve	ξ	$=$	(α, μ)	$\mathbb{R}^n \times \mathbb{R}^m$	$G \times \mathbb{R}^m$
Perturbation	ζ	$=$	(β, ν)	$\mathbb{R}^n \times \mathbb{R}^m$	$\mathfrak{g} \times \mathbb{R}^m$
Trajectory	η	$=$	(g, u)	$\mathbb{R}^n \times \mathbb{R}^m$	$G \times \mathbb{R}^m$
Traj. perturbation	γ	$=$	(z, v)	$\mathbb{R}^n \times \mathbb{R}^m$	$\mathfrak{g} \times \mathbb{R}^m$

- Vector space $\mathbb{R}^n$
  $\mathcal{P}(\xi + \varepsilon\zeta) = \eta + \varepsilon\gamma + o(\varepsilon)$. We obtain

$$\dot{z} = A(\eta(t))z + B(\eta(t))v\,, \qquad z(0) = 0$$
$$v = \nu + K(t)(\beta - z)$$

- Lie group G
  $\mathcal{P}(\xi \exp(\varepsilon\zeta)) = \mathcal{P}(\xi)\exp(\varepsilon\gamma + o(\varepsilon))$. We obtain, recall $\mathcal{P}(\xi) = \eta$,

$$\dot{z} = A(\eta(t))\,z + B(\eta(t))\,v\,, \qquad z(0) = 0$$
$$v = \nu + K(t)\mathbf{d}\log_{\log(g^{-1}\alpha)}(\mathrm{Ad}_{g^{-1}\alpha}\beta - z)$$

When $\xi = \mathcal{P}(\xi) = \eta$, $\mathbf{d}\log_{\log(g^{-1}\alpha)} = \mathrm{id}$ and $\mathrm{Ad}_{g^{-1}\alpha} = \mathrm{id}$!

Quadratic approximation of the cost function

Derivatives

We can expand $\tilde{h}(\xi \exp(\varepsilon\zeta)) := h(\mathcal{P}(\xi \exp(\varepsilon\zeta))$ as

$$\begin{aligned}\tilde{h}(\xi \exp(\varepsilon\zeta)) &= h(\mathcal{P}(\xi)) + \varepsilon\, D\tilde{h}(\xi) \cdot \xi\zeta \\ &\quad + 1/2\, \varepsilon^2\, \mathbb{D}^2 \tilde{h}(\xi) \cdot (\xi\zeta, \xi\zeta) + o(\varepsilon^2)\end{aligned}$$

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First and second derivative of $\tilde{h}(\xi) = h(\mathcal{P}(\xi))$ are given by

$$D\tilde{h}(\xi) \cdot \xi\zeta = Dh(\mathcal{P}(\xi)) \cdot D\mathcal{P}(\xi) \cdot \xi\zeta$$

$$\begin{aligned}&\mathbb{D}^2\tilde{h}(\xi) \cdot (\xi\zeta_1, \xi\zeta_2) = \\ &\quad \mathbb{D}^2 h(\mathcal{P}(\xi)) \cdot (D\mathcal{P}(\xi) \cdot \xi\zeta_1, D\mathcal{P}(\xi) \cdot \xi\zeta_2) \\ &\quad + Dh(\mathcal{P}(\xi)) \cdot \mathbb{D}^2\mathcal{P}(\xi) \cdot (\xi\zeta_1, \xi\zeta_2)\end{aligned}$$

Derivatives

We can expand $\tilde{h}(\xi \exp(\varepsilon\zeta)) := h(\mathcal{P}(\xi \exp(\varepsilon\zeta))$ as

$$\begin{aligned}\tilde{h}(\xi \exp(\varepsilon\zeta)) &= h(\mathcal{P}(\xi)) + \varepsilon\, D\tilde{h}(\xi)\cdot\xi\zeta \\ &\quad + 1/2\,\varepsilon^2\,\mathbb{D}^2\tilde{h}(\xi)\cdot(\xi\zeta,\xi\zeta) + o(\varepsilon^2)\end{aligned}$$

First and second derivative of $\tilde{h}(\xi) = h(\mathcal{P}(\xi))$ are given by

$$D\tilde{h}(\xi)\cdot\xi\zeta = Dh(\mathcal{P}(\xi))\cdot D\mathcal{P}(\xi)\cdot\xi\zeta$$

$$\begin{aligned}&\mathbb{D}^2\tilde{h}(\xi)\cdot(\xi\zeta_1,\xi\zeta_2) = \\ &\quad \mathbb{D}^2 h(\mathcal{P}(\xi))\cdot(D\mathcal{P}(\xi)\cdot\xi\zeta_1, D\mathcal{P}(\xi)\cdot\xi\zeta_2) \\ &\qquad + Dh(\mathcal{P}(\xi))\cdot\mathbb{D}^2\mathcal{P}(\xi)\cdot(\xi\zeta_1,\xi\zeta_2)\end{aligned}$$

When $\xi \in \mathcal{T}$ and $\xi\zeta_i \in T_\xi\mathcal{T}$, they specialize into

$$D\tilde{h}(\xi)\cdot\xi\zeta = Dh(\xi)\cdot\xi\zeta$$

$$\mathbb{D}^2\tilde{h}(\xi)\cdot(\xi\zeta_1,\xi\zeta_2) = \mathbb{D}^2 h(\xi)\cdot(\xi\zeta_1,\xi\zeta_2) + Dh(\xi)\cdot\mathbb{D}^2\mathcal{P}(\xi)\cdot(\xi\zeta_1,\xi\zeta_2)$$

Derivatives

We can expand $\tilde{h}(\xi \exp(\varepsilon\zeta)) := h(\mathcal{P}(\xi \exp(\varepsilon\zeta))$ as

$$\begin{aligned}\tilde{h}(\xi \exp(\varepsilon\zeta)) &= h(\mathcal{P}(\xi)) + \varepsilon\, D\tilde{h}(\xi)\cdot\xi\zeta \\ &\quad + 1/2\,\varepsilon^2\, \mathbb{D}^2\tilde{h}(\xi)\cdot(\xi\zeta,\xi\zeta) + o(\varepsilon^2)\end{aligned}$$

First and second derivative of $\tilde{h}(\xi) = h(\mathcal{P}(\xi))$ are given by

$$D\tilde{h}(\xi)\cdot\xi\zeta = Dh(\mathcal{P}(\xi))\cdot D\mathcal{P}(\xi)\cdot\xi\zeta$$

$$\begin{aligned}\mathbb{D}^2\tilde{h}(\xi)\cdot(\xi\zeta_1,\xi\zeta_2) &= \\ &\mathbb{D}^2 h(\mathcal{P}(\xi))\cdot(D\mathcal{P}(\xi)\cdot\xi\zeta_1, D\mathcal{P}(\xi)\cdot\xi\zeta_2) \\ &+ Dh(\mathcal{P}(\xi))\cdot\mathbb{D}^2\mathcal{P}(\xi)\cdot(\xi\zeta_1,\xi\zeta_2)\end{aligned}$$

When $\xi \in \mathcal{T}$ and $\xi\zeta_i \in T_\xi\mathcal{T}$, they specialize into

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How to compute $\mathbb{D}^2\mathcal{P}(\xi)\cdot(\zeta_1,\zeta_2)$?

Second order approximation of the Projection Operator

- Vector space $\mathbb{R}^n$.

$$\omega = \mathbf{D}\mathcal{P}^2(\xi)\cdot(\zeta_1,\zeta_2)$$

with $\xi \in \mathcal{T}$ and $\gamma_i = \mathbf{D}\mathcal{P}(\xi)\cdot\zeta_i$,

$$\begin{aligned} \dot{y} &= A(\eta)y + B(\eta)w + \mathbf{D}^2\lambda(\eta)\cdot(\gamma_1,\gamma_2)\,, \qquad y(0) = 0\,, \\ w &= -K(t)y\,, \end{aligned}$$

- Lie group G.

$$\mathcal{P}(\xi)\omega = \mathbb{D}\mathcal{P}^2(\xi)\cdot(\xi\zeta_1,\xi\zeta_2)$$

with $\xi \in \mathcal{T}$ and $\mathcal{P}(\xi)\gamma_i = \mathbf{D}\mathcal{P}(\xi)\cdot\xi\zeta_i$,

$$\begin{aligned} \dot{y} &= A(\eta)y + B(\eta)w \qquad\qquad y(0) = 0\,, \\ &\quad - 1/2\big[(\mathrm{ad}_{z_1}\mathrm{ad}_{z_2} + \mathrm{ad}_{z_2}\mathrm{ad}_{z_1})\lambda(\eta) \\ &\qquad - \mathrm{ad}_{z_1}(A(\eta)z_2 + B(\eta)v_2) \\ &\qquad - \mathrm{ad}_{z_2}(A(\eta)z_1 + B(\eta)v_1)\big] \\ &\quad + \mathbb{D}^2\lambda(\eta)\cdot(\eta\gamma_1,\eta\gamma_2)\,, \\ w &= -K(t)\big[y + 1/2\left([z_1,\beta_2] + [z_2,\beta_1]\right)\big] \end{aligned}$$

Recall $\gamma_i = (z_i, v_i)$, $\zeta_i = (\beta_i, \nu_i)$.

Second geometric derivative

Let M_1 and M_2 be two smooth manifolds endowed with **affine connections** ${}^1\nabla$ and ${}^2\nabla$, respectively. Let $f : M_1 \to M_2$ be a smooth mapping.

The second geometric derivative is a tool to extend the classical (Leibniz's) product rule to the covariant derivative of the "product" $Df(\gamma_1(t)) \cdot V_1(t)$, for a curve γ_1 and a vector field V_1 along γ_1 in M_1.

Chosen $x \in M_1$ and two tangent vectors v_x and $\mathrm{w_x} \in T_x M_1$. Let $\gamma_1 : I \to M_1$ be a smooth curve in M_1 such that

$$\gamma_1(t_0) = x \quad \text{and} \quad \dot{\gamma}_1(t_0) = \mathrm{w}_x \,.$$

Let V_1 a smooth vector field along γ_1 such that

$$V_1(t_0) = \mathrm{v}_x \,,$$

and

$$V_2(t) := \mathbf{D}f(\gamma_1(t)) \cdot V_1(t) \in T_{f(\gamma_1(t))} M_2$$

a smooth vector field along the curve $\gamma_2(t) := f(\gamma_1(t))$ in M_2.

Second geometric derivative (cont'd)

The **second geometric derivative** of the map $f : M_1 \to M_2$ at $x \in M_1$ in the directions v_x and $\mathrm{w}_x \in T_x M_1$ is the bilinear mapping $\mathbb{D}^2 f(x) : T_x M_1 \times T_x M_1 \to T_{f(x)} M_2$ defined as

$$\mathbb{D}^2 f(x) \cdot (\mathrm{v}_x, \mathrm{w}_x) := D_t V_2(t_0) - \mathbf{D} f(\gamma_1(t_0)) \cdot D_t V_1(t_0)\,, \qquad (1)$$

where $D_t V_1$ and $D_t V_2$ denote the covariant differentiation with respect to ${}^1\nabla$ and ${}^2\nabla$, respectively.

Second geometric derivative (cont'd)

The **second geometric derivative** of the map $f : M_1 \to M_2$ at $x \in M_1$ in the directions v_x and $\mathrm{w}_x \in T_x M_1$ is the bilinear mapping $\mathbb{D}^2 f(x) : T_x M_1 \times T_x M_1 \to T_{f(x)} M_2$ defined as

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where $D_t V_1$ and $D_t V_2$ denote the covariant differentiation with respect to ${}^1\nabla$ and ${}^2\nabla$, respectively.

Denote by ${}^1P$ and ${}^2P$ the *p*arallel displacements associated to ${}^1\nabla$ and ${}^2\nabla$, respectively. Then, equation (1) is equal (for $t = t_0$) to

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big({}^2P_{\gamma_2}^{\,t \leftarrow t+\varepsilon}\, \mathbf{D} f(\gamma_1(t+\varepsilon)) \cdot {}^1P_{\gamma_1}^{\,t+\varepsilon \leftarrow t} X_1(\gamma_1(t)) - \mathbf{D} f(\gamma_1(t)) \cdot X_1(\gamma_1(t)) \Big) \,, \tag{2}$$

Second geometric derivative (cont'd)

The **second geometric derivative** of the map $f : M_1 \to M_2$ at $x \in M_1$ in the directions v_x and $\mathrm{w}_x \in T_x M_1$ is the bilinear mapping $\mathbb{D}^2 f(x) : T_x M_1 \times T_x M_1 \to T_{f(x)} M_2$ defined as

$$\mathbb{D}^2 f(x) \cdot (\mathrm{v}_x, \mathrm{w}_x) := D_t V_2(t_0) - \mathbf{D} f(\gamma_1(t_0)) \cdot D_t V_1(t_0)\,, \qquad (1)$$

where $D_t V_1$ and $D_t V_2$ denote the covariant differentiation with respect to ${}^1\nabla$ and ${}^2\nabla$, respectively.

Denote by ${}^1P$ and ${}^2P$ the *p*arallel displacements associated to ${}^1\nabla$ and ${}^2\nabla$, respectively. Then, equation (1) is equal (for $t = t_0$) to

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Those concepts need to be specialized for Lie groups.
We used the **symmetric (0)-Cartan-Shouten connection...**
no time for the details, unfortunately!

Conclusions

- we have extended the **projection operator based trajectory optimization approach** to the class of nonlinear systems that evolve on **non-compact Lie groups** [2].
- This required the introduction of a **geometric derivative** notion for the repeated differentiation of a mapping between two Lie groups, endowed with affine connections.
  (*Not explained for time constraints...*)
- With this tool, chain rule like formulas where used to develop expressions for the basic objects needed for trajectory optimization.
- **Coding of the algorithm and numerical tests are under development!**

[2] A. Saccon, J. Hauser and A. P. Aguiar,
"Optimal Control on Non-Compact Lie Groups: A Projection Operator Approach",
Submitted to the IEEE Conference of Decision and Control (CDC), 2010

Obrigado pela vossa atenção!