Progetto 10 - Real Time pricing of Electricity Markets

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1 Introduction and Motivation

Due to the enormous increasing in the demand for power supply, in the last decades the scientific and industrial communities have put a huge effort in research and development of efficient techniques to produce, store and distribute energy. Due to the shift from a fully centralized scenario, in which there are a few large power plants which supply in a fixed way residential and industrial areas, to the so called *smart grids*, a growing interest is put in energy market control algorithms (see also *1st IEEE International Conference on Smart Grid Communications (October 2010, Gaithersburg, MD)*).

Such a type of algorithms, in which each producer and consumer is thought as an autonomous agent able to take decisions, aims to solve several problems which are typical of the centralized policy. A first goal is to reduce the annual summer demand peak, which is mainly due to the large use for air conditioning. The huge demand sets a maximum required capacity for the system, and entails the maintenance of expensive and polluting mega power plants, which are used just a few days or hours per year.
A second rationale for a dynamic control of the energy market is related to the increasing amount of energy produced on the basis of renewable resources, such as wind–power and solar energy. These are affected by a high degree of uncontrolled and sometimes hardly predictable stochastic uncertainty, since weather forecast could be erroneous in timing as well as in "quantity" of wind or sun, and thus on the produced energy. Moreover, the market has been recently flowed by relatively cheap small solar plants for buildings' roofs, so that we have a plethora of small producers whose role cannot be controlled in a centralized way. Rather, again, a dynamic control algorithm allows all the agents to autonomously and actively participate in the energy market, providing the system operators a mechanism for matching supply and demand by choosing the price of the energy. Such a mechanism must, however, by cleverly designed in order to guarantee the stability and reliability of the network despite stochastic uncertainties on the amount of supply or demand, and, more generally, to steer the amount of consumed energy to a value which is "optimal". Optimality, in this research field, is usually defined as the maximization of the sum of the benefit for all the users. Clearly, in a realistic scenario, the agents could be weighted differently for some reasons, but, as it will be clear in what follows, the model we use does not allow to go in such a detail in the distribution of the energy among the users, so that we won't address this problem.

Theoretical consequences of dynamic pricing have been investigated by several authors. In Borenstein at al. [6] a comparison is done among different dynamic pricing strategies. The authors conclude in favour of real–time pricing, which is characterized by electric rates that reflect the actual wholesale market prices to the customers. In other words, the idea is that of minimizing the gap between energy production cost for the suppliers and retail prices of the market, in order to get as close as possible to the energy consumption that maximize the total benefit for the users.

The dynamic pricing has already been tested on the market by several companies, such as ComEd (which is an energy delivery subsidiary of Exelon Corporation); in most commercial applications, the energy price changes every hour, and the electric rate is calculated by applying different prices depending on the actual cost of the energy at every time step. It has been proved [1] that if the real–time retail prices are communicated to the customers, who will then myopically adjust their consumption of energy in function of their own advantage, the direct linking between the consumer prices and the wholesale market prices may cause an unstable close–loop feedback situation. In a later paper [2], a stabilizing pricing algorithm has been presented for a simple market model, under opportune assumptions on the behavior of the market participants. Such a procedure steers the price to a value which is optimal in the sense presented above.

This last paper is the basis of our analysis. We reduced the algorithm the authors present, which holds for a network of agents, performing a well–known consumer and producer aggregation in order to deal with a simple network of only three agents, an aggregate consumer, an aggregate producer, and a system operator, the ISO, charged of modifying the prices according to some pre–designed policy. Once this is done, we provided a bound on the algorithm parameter which guarantees local stability, offering some simulation to show that the algorithm is also globally stable. Given such bound, we improved the algorithm by introducing a clever choice of the parameter which ensures the maximization of the convergence rate to the optimal value. The choice is done according to different interpolation curves which estimate the real characteristics of the consumers, and which are computed using the past available data on the consumers' behavior.
A further problem which has been addressed is to balance the budget of the system operator which provides the control algorithm to the market participants. This is, in fact, a non–for–profit agent which should not overall gain or loose money for its service. Unfortunately, this is something intrinsically unavoidable during the transient of the algorithm, so that it develops a total income, or loss. We proposed a sort of Integral control algorithm to gradually control to zero this income or loss, and we provided a sufficient condition, based on Lyapunov Linearization criterion, on the parameters of

the algorithm to ensure stability.
The last problem faced has been introducing a dynamic evolution of the customer's behavior, which should reflect, for the particular case considered, the different activity during a day. This was done in order to test the performance of our algorithm while chasing a time–variant optimal equilibrium point. Numerical simulation are also given for the case in which the consumers are supposed not to know exactly their value function. This situation has been modeled by adding a stochastic disturb to the available data from the market model.

1.1 Summary

The paper is organized as follows:

- In **Section 2** we present the model, the market participants and the assumptions on their behavior. We introduce the market dynamics and we mention some theoretical result from [2], which will be used in the following section.

- In **Section 3** we analyze the stability bounds for the algorithm proposed in [2], and provide the proof of local stability (in both linear and non–linear expression of the customers' value function). Moreover, we use several functions to interpolate the available data, in order to maximize the convergence rate. Numerical results are also shown.

- In **Section 4** we address the problem of balancing the total revenues of the system operator, and propose a control algorithm, giving also theoretical bounds to the parameters.

- In **Section 5** we add a stochastic disturb to the customer's value function in order to simulate the uncertainties that may affect the consumers' behavior.

- In **Section 6** we propose a simplified model for the dynamic evolution of the system. We suppose the customers' value functions to change four times a day, and show the simulative results of the algorithm, trying to chase the equilibrium.

- In **Section 7** we sum up the main results of the article and show possible directions for future work.

2 Preliminaries

2.1 Market participants

The aim of this section is to give some details on the model we consider, describing how the various agents act in the energy market according to their own, different goals. As we will see, the model of the electricity market we consider in this work is a simple one made up of three groups of participants: the suppliers, or producers, the consumers and an independent system operator (ISO). The goal of the participating consumers and suppliers is to maximize their own benefit - i.e. the value they can draw from their engagement in the market. On the other hand, the ISO is a non–for–profit player which aims to maximize the total social welfare (the aggregate surplus of consumers and producers, which descend from the notion of optimality we consider) under the constraint of matching supply and demand. The ISO obviously has also to take into account the physical constraints of the network, since it is charged also of the distribution of the electricity. Now we describe in detail the characteristics of these market participants.

1. *The Consumers and the Producers*: Let's assume we have two index sets $S := \{1, \ldots, n_s\}$ and $D := \{1, \ldots, n_d\}$ which represent the suppliers and consumers respectively. A value function $v_j(x)$ is associated to each customer $j \in D$, and it represents the economical value deriving from the consumption of x units of electricity. Similarly, a cost function $c_i(x)$ represents the cost for each producer $i \in S$ to supply x energy units.
   We assume the cost function $c_i(\cdot) \in \mathcal{C}^2[0, \infty)$, is strictly increasing and strictly convex, $\forall i \in S$. Analogously, $\forall j \in D$, the value function $v_j(\cdot) \in \mathcal{C}^2[0, \infty)$ is supposed to be strictly increasing and strictly concave.
   It is assumed that the utility function of supplier $i \in S$ is given by $u_i(\lambda, x) := \lambda x - c_i(x)$, where λ is the clearing prize. Equivalently, the utility function of the consumer $j \in D$ is given by $u_j(\lambda, x) := v_j(x) - \lambda x$, therefore the utility functions are quasi-linear. In this context, the utility functions represent the net benefit that an agent can draw from the engagement in the market, i.e. from producing or consuming x units of electricity when λ is the market price per unit. Let $d_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $j \in D$, and $s_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $i \in S$, denote $\mathcal{C}^1$ functions mapping price to consumption and production respectively. According to the framework of utility maximizing agents, we assume that each

agent maximizes the net benefit that can be achieved from the market. Consequently:

$$\begin{aligned} d_j(\lambda) &= \arg\max_{x\in\mathbb{R}_+} v_j(x) - \lambda x, \quad j \in D & (1) \\ s_i(\lambda) &= \arg\max_{x\in\mathbb{R}_+} \lambda x - c_i(x), \quad i \in S & (2) \end{aligned}$$

When $\lambda \in [0, \infty)$, the maximization problems defined in (1) e (2) have a unique solution in $\mathbb{R}_+$ and the functions $d_j(\cdot)$ and $s_i(\cdot)$ are well-defined, thanks to the hypothesis we made on $v_j(\cdot)$ and $c_i(\cdot)$. Hence:

$$\begin{aligned} d_j(\lambda) &= \max\{0, \arg\{\dot{v}_j(x) = \lambda\}\} = \max\{0, \dot{v}_j^{-1}(\lambda)\} \\ s_i(\lambda) &= \max\{0, \arg\{\dot{c}_i(x) = \lambda\}\} = \max\{0, \dot{c}_i^{-1}(\lambda)\} \end{aligned}$$

In the interest of simplicity we assume that $d_j(\lambda) = \dot{v}_j^{-1}(\lambda)$ and $s_i(\lambda) = \dot{c}_i^{-1}(\lambda)$. This is mathematically justified by adding the assumptions $\dot{c}_i(0) = 0$ and $\dot{v}_j(0) = \infty$, or by assuming that $\lambda \in [\min \dot{c}_i(0), \max \dot{v}_j(0)]$, or even extending the inverse functions to define $\dot{v}_j^{-1}(\lambda) = 0, \forall\lambda > \dot{v}_j(0)$, and $\dot{c}_i^{-1}(\lambda) = 0, \forall\lambda < \dot{c}_i(0)$.

Definition The *social welfare* is defined as the aggregate benefit of the producers and the customers:

$$\mathcal{S} = \sum_{j\in D} u_j(\lambda, d_j) + \sum_{i\in S} u_i(\lambda, s_i). \tag{3}$$

When the system is at the equilibrium, i.e. the total supply equals the total demand and there is a unique clearing prize λ for the entire system, then:

$$\mathcal{S} = \sum_{j\in D} v_j(d_j) - \sum_{i\in S} c_i(s_i). \tag{4}$$

2. *The Independent System Operator*: The system operator is an independent non-for-profit organization, whose primary function is to optimally match supply and demand, subject to network constraints. In particular, real-time market balancing involves solving a constrained optimization problem, with the goal of maximizing the social welfare as shown in (4). The network constraints are referred to power flow, transmission lines, generators' capacity and local and system-wide reserve capacity requirements. A set of the Locational Marginal

Prices (LMPs), corresponding to the nodal power balance constraints, emerge. When the transmission lines are congested, the prices may vary from location to location as they represent the marginal cost of supplying electricity at each particular node. When there is sufficient capacity in the network, then no transmission line will be congested, and therefore the entire network will have a unique price per unit of energy.
In order to develop simple mathematical models, we make the following simplifying assumptions:

- Resistive losses in transmission/distribution lines are negligible.
- The line capacities are high enough, so that no congestion will occur.
- There are no capacity constraints on the generators.
- There are no reserve capacity requirements.

The ISO's optimization problem can be written as:

$$\begin{aligned} \max \quad & \sum_{j \in D} v_j(d_j) - \sum_{i \in S} c_i(s_i) \\ s.t. \quad & \sum_{j \in D} d_j = \sum_{i \in S} s_i \end{aligned} \tag{5}$$

The following Lemma is taken from [1], adopted from [8].

Lemma 2.1 *Let $d^* = [d_1^*, \dots, d_{n_d}^*]$, and $s^* = [s_1^*, \dots, s_{n_s}^*]$ where d_j^*, $j \in D$ and s_i^*, $i \in S$, solve (5). There exists a price $\lambda^* \in [0, \infty)$, such that (d^*, s^*) solves (1) e (2). Furthermore, λ^* is the Lagrangian multiplier corresponding to the balance constraint.*

Proof The proof, based on Lagrangian duality, in [8] would be applicable here with some minor adjustments.

The implication of Lemma 2.1 is that by setting the market price to λ^*, the Lagrangian multiplier corresponding to (5), the ISO creates an environment in which the collective selfish behavior of the participants results in a system-wide optimal condition. In other words, the aggregate surplus is maximized while each agent maximizes his own profit.

3. *Representative Agent Model*: We now develop an abstraction of the model in (5), with only one producer agent and one consumer agent, representing the whole group of producers and customers respectively. In fact, when the price is uniform all over the system, the ISO is not interested into the individual consumer/producer reaction to real-time prices, as the quantity of interest is the aggregate response. In multi-agent systems, especially in the economics context, a representative agent is an abstract agent whose decisions and responses to signals and events are mathematically equivalent to the aggregate decision of a group of agents. This result is a well studied subject in economics [9].

Lemma 2.2 *Let functions v_j, $j \in D$, and c_i, $i \in S$, satisfying our hypothesis, and $\dot{v}_j(0) = \infty$, $\forall j$, and $\dot{c}_i(0) = 0$, $\forall i$. Suppose that there exists functions $\bar{v}$ and $\bar{c}$ satisfying the assumptions of concavity and convexity respectively, such that:*

$$\lambda = \dot{\bar{v}}\left(\sum_{i=1}^{n_d} \dot{v}_i^{-1}(\lambda)\right), \quad \forall \lambda \in \mathbb{R}_+ \tag{6}$$

and

$$\lambda = \dot{\bar{c}}\left(\sum_{i=1}^{n_s} \dot{c}_i^{-1}(\lambda)\right), \quad \forall \lambda \in \mathbb{R}_+ \tag{7}$$

Then:

- *If (d^*, s^*) solves (5), then $\bar{d}^* := \sum \bar{d}_j^*$ and $\bar{s}^* := \sum \bar{s}_i^*$ satisfy:*

  $$\bar{d}^* = \bar{s}^* = x^*$$

  where x^* solves:

  $$\max_x \quad \bar{v}(x) - \bar{c}(x) \tag{8}$$

- *If λ^* and $\bar{\lambda}^*$ are the optimal clearing prices corresponding to (5) and (8) respectively, then $\lambda^* = \bar{\lambda}^* = \dot{\bar{v}}(x^*) = \dot{\bar{c}}(x^*)$.*

Lemma 2.2 presents a construction for the representative agent model applicable to the development in this article.

Remark Consider the case where $v_i(x) := \alpha_i \log(1 + x)$, and define $\bar{v}(x) := \bar{\alpha} \log(n_d + x)$, where $\bar{\alpha} = \sum \alpha_i$. Then $\bar{v}$ satisfies (6). However, since $\dot{v}_i(0) = \alpha_i < \infty$, the response to a price λ of the representative agent with value function $\bar{\alpha}\log(n_d + x)$ is equal to the sum of the responses of the individual agents only when $\lambda \leq \min_i \alpha_i$. So if α_i are sufficient big, then there will be no problems in pratical applications.

2.2 Dynamic Supply-Demand Model

In this section we review the a dynamical system model proposed to simulate the interaction of wholesale supply and retail demand in electricity markets, under the assumption of modeling all customers as a single representative agent, as well as we will do for all producers. The cost and value functions $c(\cdot)$ and $v(\cdot)$ represent supply and demand respectively. In a power grid, the aggregate supply has to match exactly the aggregate demand at each time step. Therefore, in real-time pricing, supply must follow the demand. In other words, the exact amount of power requested by the consumers needs to be supplied at each instant of time.
At discrete time intervals, the ISO computes and announces the price for the next time interval, according to a mechanism which is called *exanté pricing*.

Remark A different price assignment is also possible, in the sense that the price announced at time $t = k$ could correspond to the time interval $[k-1, k]$. In this case, we have the so–called *expost pricing*, and of course the demand is affected by some uncertainty, as the actual price per unit of energy will be revealed only at the end of the period of consumption. On the contrary, in *exanté pricing*, it is up to the ISO to face the economic risk deriving from the gap between the predicted and real price at each time interval, as the suppliers are supposed to get payed according to the actual marginal cost of production. Anyway, it can be proved (see [1]) that this second possible choice leads to the same dynamics as the previous one, i.e. the exanté pricing.

Let λ_t^r be the exanté retail price per unit of electricity announced by the ISO at time t for the time interval $[t, t+1]$, and let d_t be the actual consumption that will occur during this time interval. As a consequence of the myopic autonomy given to the consumers, the aggregate demand is set to

$$d_t = \arg\max_{x \in \mathbb{R}_+} v(x) - \lambda_t^r x.$$

It is reasonable to assume that the ISO does not know the value function, hence the utility function, of the consumers. Moreover, even the consumers could not have complete information about the exact form of their value function (see Section 5). Under these conditions, an exact model of the consumers' response to the announced price is not available, and the ISO can only estimate the demand, and consequently calculate the retail price. During the time interval $[t, t+1]$, the producers supply an amount of energy which is exactly the total demand d_t for this time period, and get paid by the ISO according to their actual marginal cost of production λ_t^w. At time $t+1$ the ISO needs to announce λ_{t+1}^r. The following equations describe the dynamics of the market:

$$\begin{aligned} d_t &= \arg\max v(x) - \lambda_t^r x = \dot{v}^{-1}(\lambda_t^r) \\ s_t &= d_t = x_t \\ \lambda_t^w &= \dot{c}(x_t) \\ \lambda_{t+1}^r &= \Pi(\lambda_t^r, \lambda_t^w) \end{aligned} \tag{9}$$

where $\Pi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is the pricing function.
This algorithm is depicted in the block diagram in Figure 1.

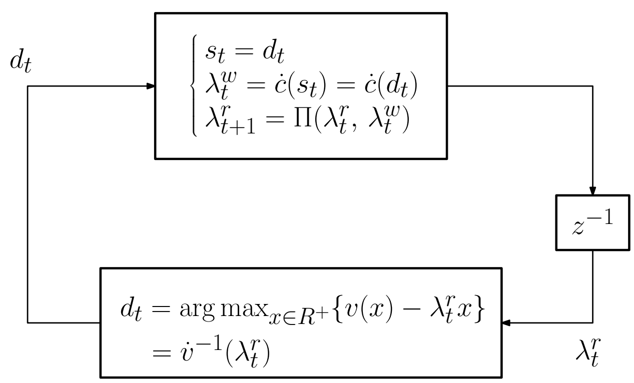

Figure 1: Block diagram describing the prices' dynamic control algorithm.

2.3 ISO's risk

The ISO commits to a price $\lambda_t^r = \Pi(\lambda_{t-1}^r, \lambda_{t-1}^w)$ for the consumers, while it has to pay the generators $\lambda_t^w = \dot{c}(x_t)$. The ISO's risk or revenue differential is therefore given by:

$$i(t) = (\lambda_t^r - \lambda_t^w)\, x_t. \tag{10}$$

3 Stability Analysis

The goal of this section is to present a simple stabilizing algorithm, based on the main outcome proposed in [2], which leads to the same value both the retail and wholesale prices, value which is exactly the optimal value in the sense of the global benefit. The steady assumption is that the consumers' behavior is static, namely the value function is time–independent. The proof is very simple and is based on the linearization method for discrete time systems of the Lyapunov Stability Analysis.

Theorem 3.1 *Suppose that there exist a maximum value of the demand* $d_{max} > 0$*, and for each trajectory satisfying the model equations 9 consider the demand* $d_t \in [0, d_{max}]$*. Let the pricing function* Π *be defined as:*

$$\Pi(\lambda^r, \lambda^w) = \lambda^r + \gamma(\lambda^w - \lambda^r) = \gamma\lambda^w + (1 - \gamma)\lambda^r \tag{11}$$

where $\gamma > 0$*. Then for sufficiently small* γ*, the pricing function* Π *stabilizes the price's dynamic:*

$$\lambda^r_{t+1} = \Pi(\lambda^r_t, \lambda^w_t) = \lambda^r_t + \gamma(\lambda^w_t - \lambda^r_t) \tag{12}$$

in the sense that λ^r_t *and* λ^w_t *converge to the same value* $\lambda^{r*} = \lambda^{w*} = \lambda^*$*, and the corresponding supply and demand converge also to equilibrium:* $s^* = d^* = x^*$*.*

Let's assume that the equations for the dynamic of the retail and wholesale prices are given by:

$$\lambda^w = \dot{c}(x) = c \cdot x, \tag{13}$$

$$\lambda^r = \dot{v}(x) = a(x), \tag{14}$$

where $c > 0$*, because the cost function is strictly convex, while* $a(x) > 0$ *and its derivative* $\dot{a}(x) < 0$*, because the value function is strictly concave and so its derivative is monotonically decreasing. In such a context the interval of admissible values for the equilibrium's stability is* $\gamma \in [0, \gamma^*)$*, where:*

$$\gamma^* = \frac{2\,\dot{a}(x^*)}{\dot{a}(x^*) - c}. \tag{15}$$

Proof Let's start giving an expression of the dynamics of the retail price as function of the same variable λ_t^r only. In particular we have:

$$\begin{aligned}
\lambda_{t+1}^r &= \lambda_t^r + \gamma(\lambda_t^w - \lambda_t^r) \\
&= \lambda_t^r(1-\gamma) + \gamma\lambda_t^w \\
&= \lambda_t^r(1-\gamma) + \gamma\, c\, x(t) \\
&= \lambda_t^r(1-\gamma) + \gamma\, c\, a^{-1}(\lambda_t^r) \\
&= f(\lambda_t^r). \qquad (16)
\end{aligned}$$

We note that $f(\cdot)$ is not a linear function of the variable λ_t^r, which is continuous with its derivatives, and λ^* is the point of equilibrium. We calculate now the derivative of $f(\cdot)$ w.r.t. λ^r, (the Jacobian matrix in a multivariable case), writing for simplicity λ in place of λ^r:

$$\begin{aligned}
\left.\frac{\partial f}{\partial \lambda}\right|_{\lambda^*} &= \left. 1-\gamma+\gamma\, c\,\frac{1}{\dot{a}(a^{-1}(\lambda))}\right|_{\lambda^*} \\
&= 1-\gamma+\gamma\, c\,\frac{1}{\dot{a}(x^*)}. \qquad (17)
\end{aligned}$$

To apply the linearization method for discrete time systems of Lyapunov Analysis ([3] Chap.4 pg. 179), we now have to check if the modulus of the eigenvalues of the Jacobian Matrix, which is in this case given only by the value in (17), is smaller than 1. So we impose:

$$|1-\gamma+\gamma\, c\,\frac{1}{\dot{a}(x^*)}| < 1$$

which is equivalent to:

$$\begin{aligned}
-1 &< 1-\gamma+\gamma\, c\,\frac{1}{\dot{a}(x^*)} < 1 \\
-2 &< -\gamma+\gamma\, c\,\frac{1}{\dot{a}(x^*)} < 0 \\
-2 &< \gamma\,(-1+c\,\frac{1}{\dot{a}(x^*)}) < 0. \qquad (18)
\end{aligned}$$

The second bound in (18) is always verified, because the term within brackets is negative by assumption.
On the other side, considering the first bound in (18), we find:

$$\gamma < \frac{2\,\dot{a}(x^*)}{\dot{a}(x^*)-c} \qquad (19)$$

which is exactly the equation (15) of the Theorem. □

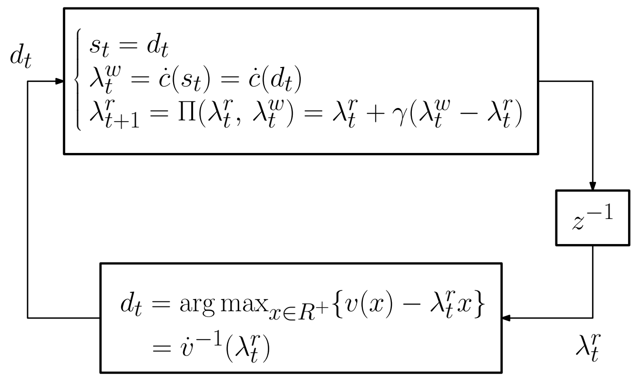

Figure 2: Block diagram describing the stabilizing pricing algorithm.

The dynamic of prices, subject to the presented stabilizing algorithm, can be represented with the scheme in Figure 2.

Remark We have found an interesting upper bound for the control parameter γ in the dynamics of prices that ensures the stability of the system. In the limiting case of choosing $\gamma = \gamma^*$, which corresponds to an eigenvalue on the unit circle, one should check the value of the higher order derivatives of the function $f(\cdot)$ of equation (16), to have some other information about the stability of the equilibrium. There are in fact some criteria for first order discrete time systems, that allow to verify the asymptotic stability of the equilibrium in the case of an eigenvalue with modulus equal to 1 ([3] chap.4 pg. 181). A special remark has to be made regarding the application of this analytic result. In fact, as already mentioned, in real applications we do not suppose to know the behaviour of the costumers, and so the value function is not available. So the exact value of the admissible bound for γ is not computable, but anyway the theory demonstrates that an exact bound exists. The result could anyway taken into account if at least a rough approximation of the value function is known. Clearly, the rougher the information of the value function, the more conservative the bound on γ will be.

The dynamics of the system has been simulated in MATLAB, assuming that the cost and value functions of producers and consumers are:

$$c(x) = 0.5\,x^2, \tag{20}$$

$$v(x) = 20\,\log(5x+1), \tag{21}$$

whose derivatives are:

$$\begin{aligned} \dot{c}(x) &= x, \\ \dot{v}(x) &= 100\,(5x+1)^{-1}. \end{aligned}$$

Therefore the functions respect the hypothesis of the theorem and we can calculate analytically the superior bound for the control parameter γ, which is equal to $\gamma^* = 0.9776$.

First of all we simulated the dynamics of the system with two different values of the parameter, first with $\gamma = 0.1$ and then with $\gamma = 0.8$. The stable dynamics of the retail prices are represented in Figures 3 and 4. The increasing line is the function $\dot{c}(x)$, which gives the wholesale prices w.r.t. the energy units consumption. Analogously, the decreasing curve is the $\dot{v}(x)$, which gives the retail prices. The red line represents the trajectory of the retail prices, which are, at every step, a convex combination of the wholesale and retail prices at the previous time step.

In Figures 5 and 6 the trend of the difference between the retail and wholesale prices, in both cases, is represented. We notice that, for a small value of the parameter, γ the price difference smoothly converges to zero, while, for a value which is next to the superior bound of the parameter γ^*, the trend begins to oscillate, before settling to the equilibrium.

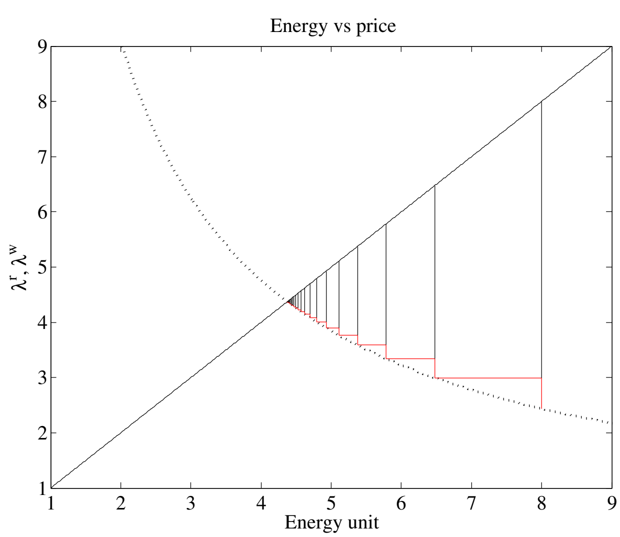

Figure 3: Dynamics of prices with $\gamma = 0.1$

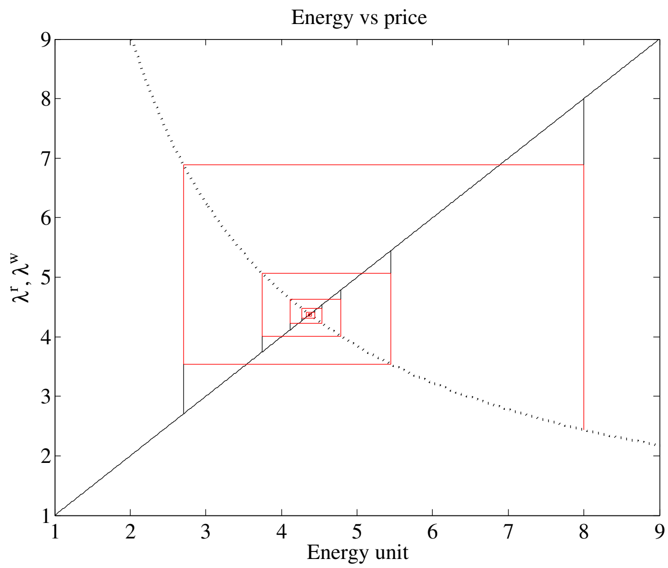

Figure 4: Dynamics of prices with $\gamma = 0.8$

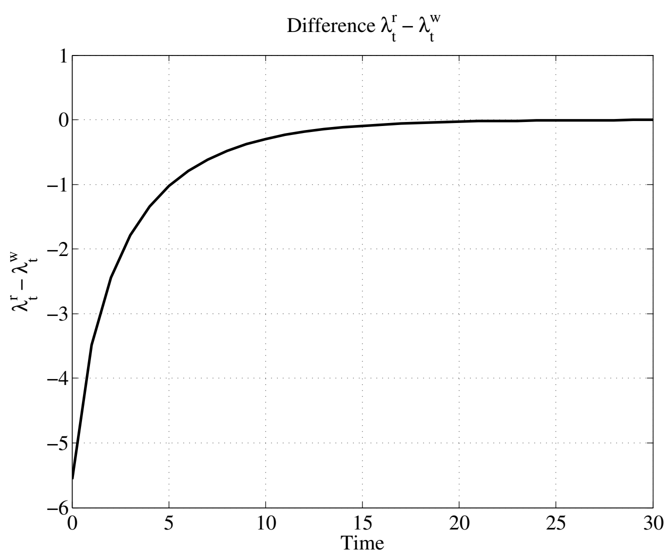

Figure 5: Discrepancies between retail and wholesale prices, with $\gamma = 0.1$

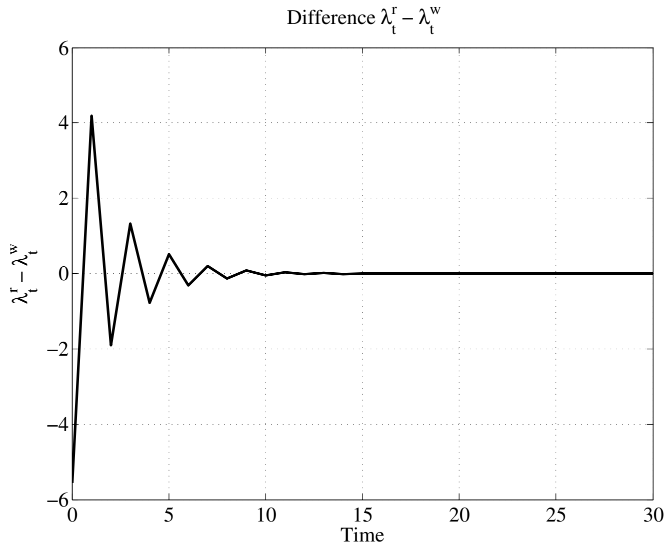

Figure 6: Discrepancies between retail and wholesale prices, with $\gamma = 0.8$

In Figures 7 and 8 it is depicted respectivly the dynamics of the prices when the parameter γ is chosen slightly smaller or bigger than the critical value. As it is expected, in the first case the system is stable, but exhibits big oscillations, while in the second it exits from the linear region, namely the equilibrium point is unstable.

3.1 Estimation of the value function and convergence rate maximization

In the previous section we provided a control algorithm which forces the retail price to converge to the equilibrium, i.e. the value corresponding to the consumption of x^* units of energy, such that $\dot{v}(x^*) = \dot{c}(x^*)$. The goal of this section is to provide a design algorithm to maximize the rate of convergence by a clever choice of γ. Recalling that the derivative of the cost function for the suppliers $\dot{c}(x)$ is supposed to be known, it is trivial that, under the usual hypothesis on the concavity of the value function, there exists an optimal value of γ. Moreover, if we knew the exact expression of the $\dot{v}(x)$ function, we would also be able to compute the intersection of the functions and therefore to reach the equilibrium point in one step. In other words, there always exists a $\bar{\gamma}$ such that the ISO controls the system to the

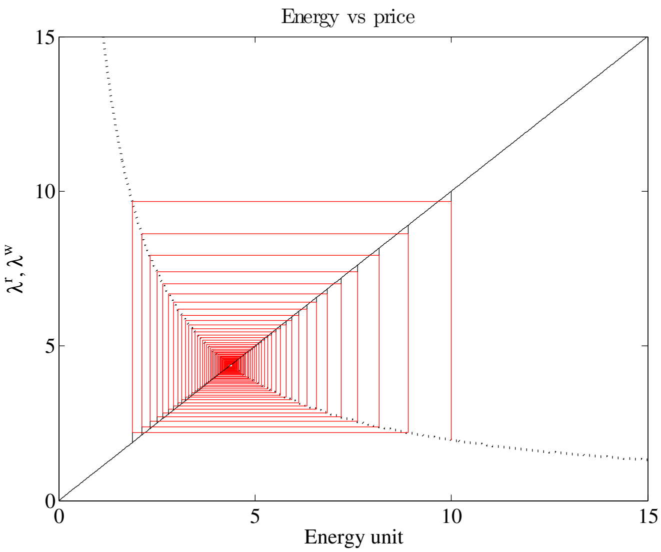

Figure 7: Dynamics of prices with $\gamma = 0.98\,\gamma^*$

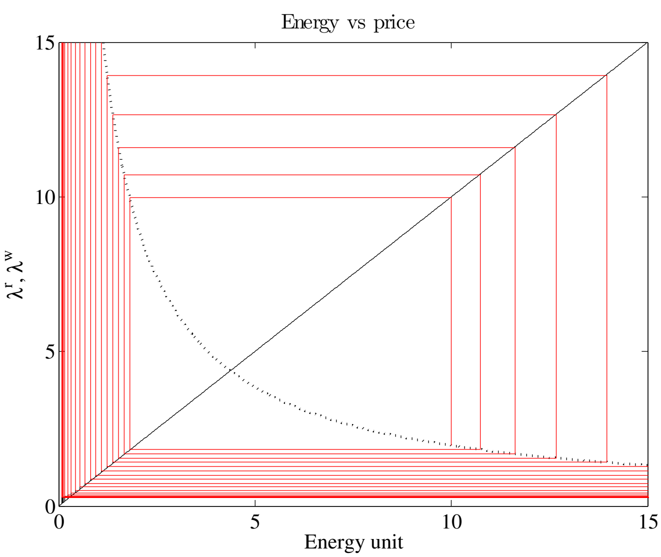

Figure 8: Dynamics of prices with $\gamma = 1.02\,\gamma^*$

optimal price in a single time step using the proposed algorithm.
In fact:

$$\lambda_r(t+1) \quad = \quad \lambda_r(t) + \gamma\left(\lambda_w(t) - \lambda_r(t)\right)$$

$$\exists\bar{\gamma} \quad : \quad \bar{\gamma} = \frac{\lambda_{r,ott} - \lambda_r(t)}{\lambda_w(t) - \lambda_r(t)};$$

and $0 < \bar{\gamma} < 1$.
Unfortunately, in most practical applications the function $\dot{v}(x)$ is unknown, so it is not possible to find the optimal $\bar{\gamma}$ without an a priori knowledge. Still, given partial information about the customers' response to the previously proposed prices on earlier steps, it is possible to get an estimate of the real curve by interpolation with a parameterized curve.
At the first step, only one point of the $\dot{v}$ function is known. In this case there is no other way to proceed than running the algorithm with a sufficient small value of γ. At the second step, it is possible to interpolate the two points with a line - which is unique - and take $\dot{c}(\bar{x})$ as the next retail price, where $\bar{x}$ is such that $\dot{v}(\bar{x}) = \dot{c}(\bar{x})$. At this stage, three points of the value function derivative can be used, and therefore sthere are several possible fitting curves, such as: 1) a line, 2) a parabola and 3) an hyperbola, all achieved with Ordinary Least Squares (OLS).

1. *Line*: as a first attempt, we approximated the $\dot{v}$ function with a decreasing line ($f(x) = -ax + b$ with $a > 0$, $b > 0$). The resulting fit is quite different from the original curve, except for a limited region around the equilibrium point. Moreover, if all of the previous data are used at every step, it takes a very long time to converge to the equilibrium. The effect of taking into account all the previous points without considering a suitable forgetting factor is to slow down the algorithm, as it is equivalent to weighting too much the initial data. Therefore it is expedient to fit the latest points only. In this case the convergence rate to the equilibrium point is much higher.

2. *Parabola*: there is a basic problem when the $\dot{v}$ function is approximated with a convex parabola, namely $f(x) = ax^2 + x + c$ with $a > 0$. The curve, for $x \to \infty$, diverges, while $\dot{v}(\infty) = 0$. It is easy to notice that this fact can lead to two different scenarios. If there are two different intersections between $\dot{c}$ and $\dot{v}$, it is sufficient to take that corresponding to the smaller amount of energy, which is the left one. Otherwise, there can be no intersection at all, in which case the algorithm stops.

3. *Hyperbola*: the approximation throught an equilateral hyperbola ($f(x) = a/x$ with $a > 0$) is the most performant among the presented functions. The hyperbola shares many characteristics with the original $\dot{v}$ function, as it is strictly positive, monotonically decresing, convex, with $f(0) \gg 0$ and $f(\infty) = 0$.

Remark As in this case all the available points belong exactly to the original $\dot{v}$ function, a spline interpolation could have been used. Nevertheless, in most pratical applications the data are affected by some kind of noise, and because of this the spline interpolation could lead to a completely different function.

3.2 Numerical results

We now present the simulation results with the proposed interpolation functions. We set $c(x) = 0.5\,x^2$ as the cost function, and $v(x) = 40\,\log(5x+1)$ as the value function. This yields to $\dot{c}(x) = x$ and $\dot{v}(x) = 200\,(5x+1)^{-1}$ respectively. At the first step we do not have any a priori information or bound for γ, therefore the best we can do is to choose a sufficiently small value for the parameter (in this case, $\gamma = 0.1$).

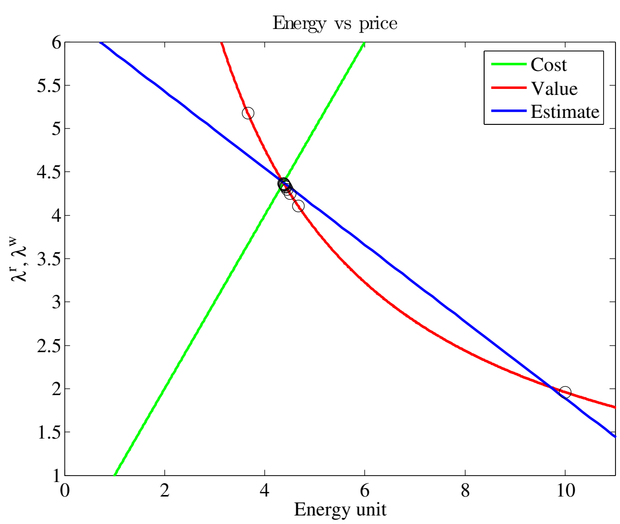

Figure 9: Line interpolation

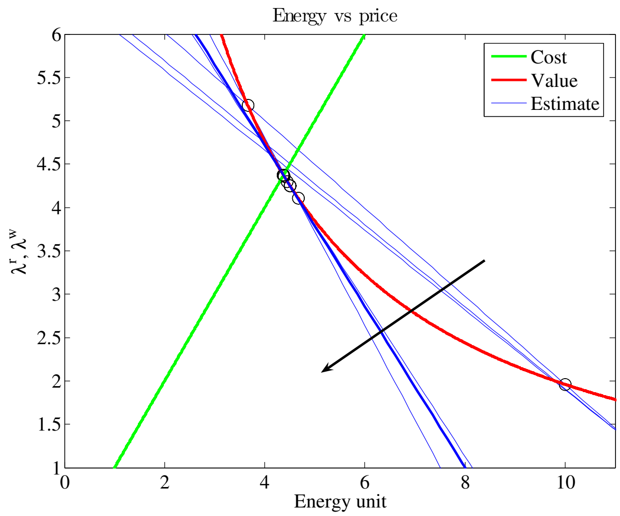

Figure 10: Line interpolation (last points only)

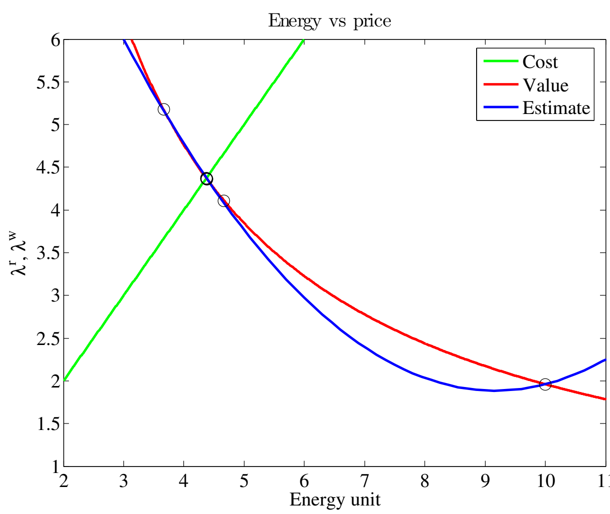

Figure 11: Parabola interpolation

The starting price for the simulations is the one that let the customers consume exactly $x(0) = 10$ units of energy. As we can see in Figure 9, the presence of the first point prevents the line from converging to the equilib-

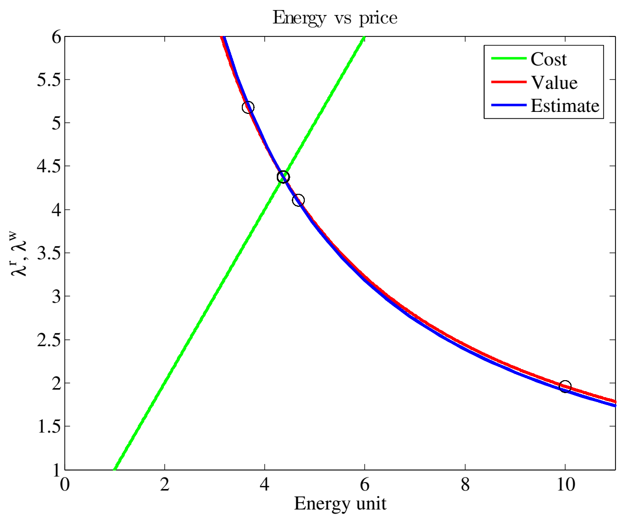

Figure 12: Hyperbola interpolation

rium. The convergence rate is much higher when only the last four points are used for interpolation (as shown in Figure 10, as the algorithm is allowed not to take into permanent account the initial values. The parabola interpolation (Figure 11) is quite fast, but it is not possible to guarantee the convergence for every initial value. As we can see from Figure 12, the hyperbolic interpolation is the most accurate, as it provides both high performance and convergence guarantee.
In Table 1 we show how many steps are needed to reach a retail price within a range of 5% (or 2%) to the optimal retail price. The starting point is the price that let the customers consume exactly $x(0) = 10$ units of energy.

	5%	2%
Line	3	4
Line (only with the last points)	3	4
Parabola	3	3
Hyperbola	3	3
No interpolation (fixed γ)	9	13

Table 1: Number of steps to reach 5% and 2% to the equilibrium ($x(0) = 10$).

In Table 2 are shown the results obtained from a starting price that let the customers consume exactly $x(0) = 2$ units of energy.

	5%	2%
Line	4	7
Line (only with the last points)	4	6
Parabola	∞	∞
Hyperbola	3	4
No interpolation (fixed γ)	16	20

Table 2: Number of steps to reach 5% and 2% to the equilibrium ($x(0) = 2$).

Similar results can be obtained using a different value function, for example $v(x) = 40 \log(5x + 1)$, as shown in Table 3 (where the initial energy consumption is guaranteed to be $x(0) = 10$ units) and Table 4 (initial energy consumption set to $x(0) = 2$ units).

	5%	2%
Line	3	34
Line (only with the last points)	3	3
Parabola	3	3
Hyperbola	3	3
No interpolation (fixed γ)	8	12

Table 3: Number of steps to reach 5% and 2% to the equilibrium ($x(0) = 10$).

	5%	2%
Line	10	34
Line (only with the last points)	6	7
Parabola	∞	∞
Hyperbola	3	10
No interpolation (fixed γ)	20	24

Table 4: Number of steps to reach 5% and 2% to the equilibrium ($x(0) = 2$).

4 ISO budget balancing

In the previous sections we addressed the problem of stabilizing the wholesale and retail prices to a common value which guarantees the maximization of social welfare. In the proposed solution, the ISO only decides the next retail price for the consumers according to the energy request during the previous period. Due to the difference between this prediction and the actual consume, at each time step there exists a gap between the amount of money to be paid to the producers and the amount of money received from the consumers. This means that the ISO is forced to fill this gap either by adding money or keeping it, which yields to a non–zero total income or loss, in contrast with the non–for–profit nature of the ISO. The goal of this section is to provide an effective algorithm to solve this problem, namely, denoting by $I_t, t \geq 0$ the total revenue of the ISO, to steer it asymptotically to zero.

Recalling that λ_t^r, λ_t^w and x_t denote respectively the retail price, the wholesale price and the amount of energy flowing in the network in the period $[t, t+1)$, we define the interval revenue:

$$i_t \stackrel{\triangle}{=} x_t(\lambda_t^r - \lambda_t^w)$$

which is positive if the amount of money received from the consumers exceeds that to be paid to the producer, and negative otherwise. The total revenue of the ISO is thus simply:

$$I_t \stackrel{\triangle}{=} \sum_{k=1}^{t} i_k = \sum_{k=1}^{t} x_k(\lambda_k^r - \lambda_k^w).$$

The update law for the retail price proposed in the previous sections was:

$$\lambda_{t+1}^r = \Pi(\lambda_t^r,\, \lambda_t^w) = \lambda_t^r + \gamma(\lambda_t^w - \lambda_t^r)$$

which is now modified to:

$$\lambda_{t+1}^r = \Pi(\lambda_t^r,\, \lambda_t^w,\, I_t) = \lambda_t^r + \gamma(\lambda_t^w - \lambda_t^r) - \rho I_t$$

where $\rho > 0$ is a real constant whose value is to be decided during the design of the algorithm. Formally, this is equivalent to an integral control over the difference of the prices weighted according to the energy consumption. Intuitively, what happens is that the larger is the total income of the ISO, the less will be the retail price in the following periods, since the ISO has a positive amount of money which is to be given back to the consumers. On the contrary, if the ISO paid in the past to the producers more than what it received by the consumers, then the retail price will be increased in order

to gain more money and fill the cash deficit.
The equations of the algorithm in the time period $[t,\, t+1)$ are thus:

$$\begin{cases} s_t = d_t \\ \lambda^w_{t+1} = \dot{c}(s_t) \\ \lambda^r_{t+1} = \Pi(\lambda^r_t,\, \lambda^w_t,\, I_t) = \lambda^r_t + \gamma(\lambda^w_{t+1} - \lambda^r_t) - \rho I_t \\ d_{t+1} = \dot{v}^{-1}(\lambda^r_{t+1}) \\ I_{t+1} = I_t + i_{t+1} = I_t + d_{t+1}(\lambda^r_{t+1} - \lambda^w_{t+1}) \end{cases} \tag{22}$$

with initialization $I(0) = 0$.
The following result gives a local criterion for the stability of the system.

Theorem 4.1 *Assume that, around the equilibrium point for the algorithm λ^*, it holds:*

$$\dot{v}(x) = -ax + b$$
$$\dot{c}(x) = cx + d.$$

Then the equilibrium point is asymptotically stable if:

$$\begin{cases} \gamma < \frac{2a}{c+a} \\ \frac{\rho}{2} > \gamma\frac{c+a}{b-d} - \frac{2a}{b-d} \end{cases} \tag{23}$$

while a couple of conditions which are only sufficient is:

$$\begin{cases} \gamma < \frac{2a}{c+a} \\ \rho < \gamma\frac{c+a}{b-d} \end{cases}. \tag{24}$$

Proof For simplicity, set $u_t = \lambda^r_t$. By exploiting the assumptions on $\dot{v}$ and $\dot{c}$, we can rewrite the equations of the system as:

$$\begin{cases} u_{t+1} = u_t + \gamma(cx_t + d - u_t) - \rho I_t \\ I_t = I_{t-1} + (u_t - cx_t - d)x_t \end{cases}$$

whence, since $x_t = -\frac{u_t - b}{a}$,

$$\begin{cases} u_{t+1} = u_t + \gamma(-\frac{c}{a}(u_t - b) + d - u_t)) - \rho I_t \\ I_t = I_{t-1} - (u_t + \frac{c}{a}(u_t - b) - d)\frac{u_t - b}{a} \end{cases}.$$

The first equation can be rewritten as:

$$\begin{aligned} u_{t+1} &= u_t - \gamma \frac{c}{a}(u_t - b) + \gamma d - \gamma u_t - \rho I_t \\ &= u_t - \gamma \frac{c}{a} u_t + \gamma \frac{cb}{a} + \gamma d - \gamma u_t - \rho I_t \\ &= u_t(1 - \gamma \frac{c}{a} - \gamma) + \gamma(\frac{bc}{a} + d) - \rho I_t \\ &= f_1(u_t,\, I_t), \end{aligned}$$

while the second equation yields:

$$\begin{aligned} I_{t+1} &= \left(u_t + \frac{c}{a}u_t - \frac{bc}{a} - d\right)\frac{b - u_t}{a} + I_t \\ &= \left[u_t\left(1 + \frac{c}{a}\right) - \left(\frac{bc}{a} + d\right)\right]\frac{b - u_t}{a} + I_t \\ &= -\frac{1}{a}\left(1 + \frac{c}{a}\right)u_t^2 + \frac{b}{a}\left(1 + \frac{c}{a}\right)u_t + \frac{1}{a}\left(\frac{bc}{a} + d\right)u_t - \frac{b}{a}\left(\frac{bc}{a} + d\right) + I_t \\ &= -\frac{1}{a}\left(1 + \frac{c}{a}\right)u_t^2 + \left(\frac{b}{a}\left(1 + \frac{c}{a}\right) + \frac{1}{a}\left(\frac{bc}{a} + d\right)\right)u_t - \frac{b}{a}\left(\frac{bc}{a} + d\right) + I_t \\ &= f_2(u_t,\, I_t). \end{aligned}$$

The equilibrium point of this system is given, in terms of amount of energy and total revenue of the ISO, by:

$$(x^*,\, I^*) = \left(\frac{b-d}{c+a},\, 0\right)$$

so that we have:

$$u^* = \lambda^{r*} = -a\frac{b-d}{c+a} + b = \frac{ad + bc}{c+a}.$$

The next step in the proof is to use the Lyapunov's Linearization Theorem [3] to prove the claim. In order to do this, we consider the Jacobian of the system near $(u^*,\, I^*)$, which is given by:

$$J(u^*,\, I^*) = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial I} \\ \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial I} \end{bmatrix} = \begin{bmatrix} 1 - \gamma\frac{c}{a} - \gamma & -\rho \\ -\frac{2}{a}(1 + \frac{c}{a})u + \left(\frac{b}{a}(1 + \frac{c}{a}) + \frac{1}{a}(\frac{bc}{a} + d)\right) & 1 \end{bmatrix}|_{(u^*,\, I^*)}.$$

Since:

$$
\begin{aligned}
-\frac{2}{a}\left(1+\frac{c}{a}\right)u+\left(\frac{b}{a}\left(1+\frac{c}{a}\right)+\frac{1}{a}\left(\frac{bc}{a}+d\right)\right)|_{u^*} = \\
&= -\frac{2}{a}\frac{c+a}{a}\frac{ad+bc}{c+a}+\left(\frac{b}{a}\left(1+\frac{c}{a}\right)+\frac{1}{a}\left(\frac{bc}{a}+d\right)\right)\\
&= -2\frac{ad+bc}{a^2}+\frac{b(c+a)}{a^2}+\frac{bc+dc}{a^2}\\
&= \frac{b(c+a)}{a^2}-\frac{bc+da}{a^2}=\frac{1}{a^2}(bc+ba-bc-da)\\
&= \frac{b-d}{a}
\end{aligned}
$$

we obtain:

$$
J(u^*,\, I^*) = \begin{bmatrix} 1-\frac{\gamma}{a}(c+a) & -\rho \\ \frac{b-d}{a} & 1 \end{bmatrix}.
$$

To conclude for the local stability of the algorithm using Lyapunov's Linearization Theorem we must provide conditions so that the two eigenvalues of $J(u^*,\, I^*)$ are strictly stable, namely inside the unit circle. The characteristic polynomial of the matrix is:

$$
P_J(z) = z^2 + z\underbrace{\left(-1-1+\frac{\gamma}{a}(c+a)\right)}_{=a_1}+\underbrace{1-\frac{\gamma}{a}(c+a)-\rho\frac{d-b}{a}}_{=a_0},
$$

and it is known that in order $P_J(z)$ to have strictly stable roots it is necessary that:

$$
\begin{aligned}
&|a_0| < 1\\
&(1+a_0)^2 - a_1^2 > 0.
\end{aligned}
$$

Concerning the first condition, it must be true that $a_0 < 1$, so:

$$
\begin{aligned}
&1-\frac{\gamma}{a}(c+a)+\rho\frac{b-d}{a} < 1 \Rightarrow -\frac{\gamma}{a}(c+a)+\rho\frac{b-d}{a} < 0\\
&\Rightarrow \rho(b-d) < \gamma(c+a) \Rightarrow \rho < \gamma\frac{c+a}{b-d}
\end{aligned}
$$

and $a_0 > -1$, which is:

$$
\begin{aligned}
&1-\frac{\gamma}{a}(c+a)+\rho\frac{b-d}{a} > -1 \Rightarrow \rho\frac{b-d}{a} > -2+\frac{\gamma}{a}(c+a)\\
&\Rightarrow \rho > \gamma\frac{c+a}{b-d}-\frac{2a}{b-d}
\end{aligned}
$$

where for both the inequalities we have used $a > 0$ and $b - d > 0$, which is needed in order the equilibrium point to be positive, namely physically feasible. The first condition implies thus:

$$\rho < \gamma \frac{c+a}{b-d} \tag{25}$$

$$\rho > \gamma \frac{c+a}{b-d} - \frac{2a}{b-d} \tag{26}$$

The second condition implies instead:

$$\begin{aligned}
&\left(2 - \frac{\gamma}{a}(c+a) + \frac{\rho}{a}(b-d)\right)^2 - \left(-2 + \frac{\gamma}{a}(c+a)\right)^2 > 0 \\
&\Rightarrow \left(2 - \frac{\gamma}{a}(c+a)\right)^2 + \left(\frac{\rho}{a}(b-d)\right)^2 + \left(2 - \frac{\gamma}{a}(c+a)\right)\left(\frac{\rho}{a}(b-d)\right) \\
&\quad - \left(-2 + \frac{\gamma}{a}(c+a)\right)^2 > 0 \\
&\Rightarrow \frac{\rho^2(b-d)^2}{a^2} + \frac{2}{a^2}\left(2a - \gamma(c+a)\right)\rho(b-d) > 0
\end{aligned}$$

thus, since, again, $b > d$ and $a > 0$,

$$\frac{\rho}{2} > \gamma \frac{c+a}{b-d} - \frac{2a}{b-d}. \tag{27}$$

Notice that this last equation implies, since $\rho > 0$, the second in Equation (25), which can be thus discarded. The two remaining conditions are thus equivalent to the stability of the system. We can moreover notice that if $\gamma < \frac{2a}{c+a}$, then the Right Hand Side of Equation (27) is negative, thus the condition is always satisfied. Referring to Fig. 13, this means taking $\gamma < \frac{2a}{c+a}$. A couple of conditions which are only sufficient is thus:

$$\begin{cases} \gamma < \frac{2a}{c+a} \\ \rho < \gamma \frac{c+a}{b-d} \end{cases}, \tag{28}$$

and the Theorem is proved.

4.1 A numerical example

To give an example of application of the previous theorem, we can consider the following example. We set $c(x) = \frac{1}{2}x^2$ and $v(x) = 20\log(5x)$ for simplicity, which yield $\dot{c}(x) = x$ and $\dot{v}(x) = \frac{20}{x}$. The equilibrium point is $x^* = \sqrt{20}$,

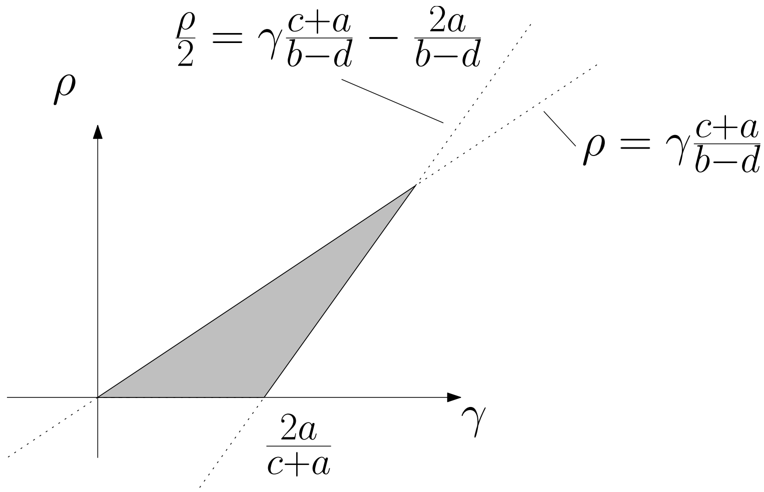

Figure 13: Conditions for the asymptotic stability of the equilibrium point in terms of ρ and γ. In grey the stability region. The point $(\frac{2a}{c+a}, 0)$ is relative to the sufficient condition.

for which we have $a = 1$, $c = \sqrt{20}$, $b = 1$ and $d = 0$. If we set $\gamma = 0.2$ and $\rho = 0.01$ it can be seen that the conditions are satisfied, and in fact the system is stable and converges to the equilibrium point, as it can be seen in Fig. 15. In Fig. 16 it is depicted the same trajectory zoomed around the equilibrium point, while in Fig. 14 we present the behavior of the ISO revenue in time. As it can be seen, at first, due to the small value of ρ, the influence of the ISO revenue is negligible, and the algorithm works in the usual way just trying to steer the retail price and the wholesale price to the same value. Since the initial energy consumption is on the right with respect to the equilibrium point, the retail price chosen by the ISO is too small, and it has to pay to the producers more than what it receives from the consumers, making up a shortfall. This grows fast, and despite the fact that the region around the equilibrium for the prices is reached in a few steps, the ISO maintains the retail prices higher than what it should in order to repay its deficit. This policy is kept for more than what it should (the integral control is slow), so that the ISO accrues a profit, which is slowly given back to the consumers. This explains the "overshoot" in Fig. 14 and the fact that, in Fig. 16, we can notice that the price is at first higher than the equilibrium and than slightly lower.

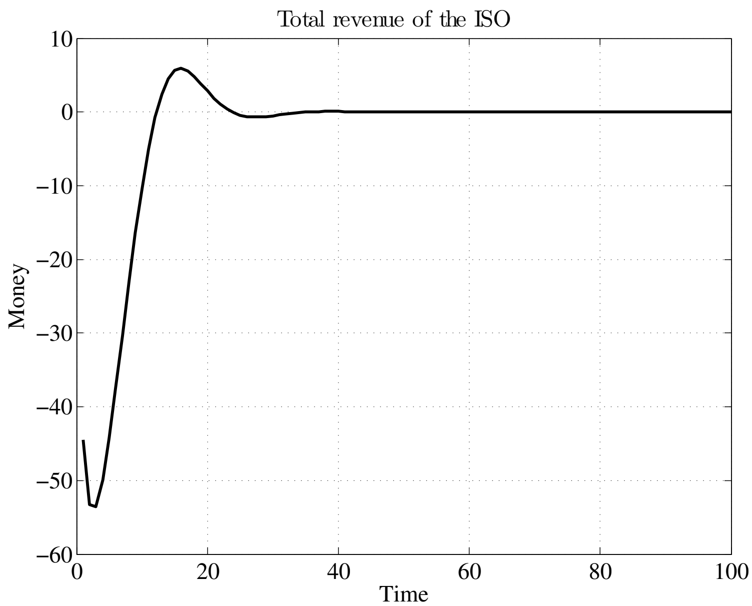

Figure 14: Trajectory of the total income/shortfall of the ISO.

5 Stabilization with noise

In the previous sections we carried on the analysis of the behavior of the Price Market Mechanism under the strong assumption that the cost function $c(x)$ for the producers and the utility function $v(x)$ for the consumers are both known by the two aggregate agents and constant in time. This assumption is fairly reasonable in the case of large power plants such as a nuclear power plant, in which not only the production should be kept as uniform as possible, but moreover there is no reason to assume that the cost function changes. Something similar, on the side of the consumers, happens for large factories, in which the demand never drops (at least on a large time horizon) and the value function relates in a known and fixed way the amount of energy with the amount of goods produced. However, in many other cases this time–invariance is a too strong assumption. In particular, and this is the case we are going to analyze in this section, the consumer could be unable to perfectly estimate the value of the energy unit at each instant, or could be forced by external reasons to change its value function. In other words, there exists a nominal value function for the consumer, which he would follow in the ideal case. However, this is not the actual case, and,

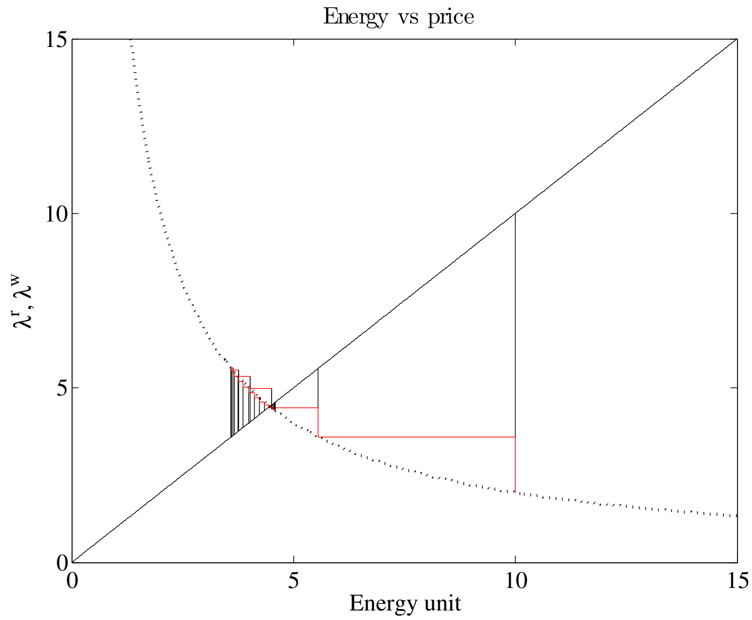

Figure 15: Trajectory of the energy request vs the retail price.

given the price λ_t^r decided by the ISO, the customer chooses to consume an amount of energy which is a corrupted version of the nominal $\dot{v}^{-1}(\lambda_t^r)$. This corruption could be, of course, of any type. For sake of simplicity, we consider the easy case in which the corruption is modeled by an additive noise

$$d_t = \dot{v}^{-1}(\lambda_t^r) + n_t$$

where the additive noise process $\{n_t\}_{t \geq 0}$ is a i.i.d. process with zero mean and variance $q < \infty$. By reducing to the essential the equations, it is easy to see that the system is governed by

$$\begin{cases} \lambda_{t+1}^r = \lambda_t^r + \gamma(c\dot{v}^{-1}(\lambda_t^r) + cn_t - \lambda_t^r) - \rho I_t \\ I_{t+1} = I_t + (\dot{v}^{-1}(\lambda_t^r) + n_t)(\lambda_t^r - c\dot{v}^{-1}(\lambda_t^r) + cn_t) \end{cases} . \quad (29)$$

A second order analysis to prove mean–square convergence is in this case quite difficult even in the case $\dot{c}(x) = cx$, since in the equation for I_{t+1} appear both a term $\lambda_t n_t$ and n_t^2, which pose quite difficult problems for the computation of the variance of the process. We limit to a simulative example which illustrates a typical trajectory of the system.

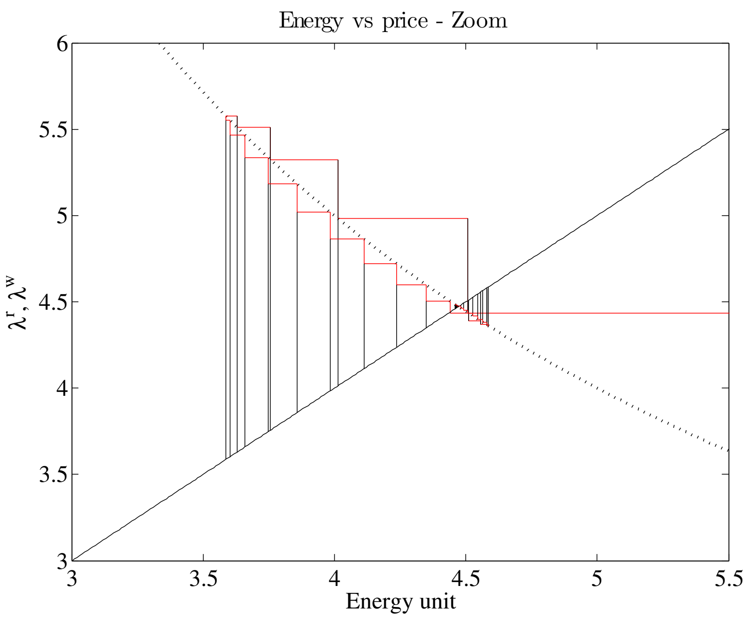

Figure 16: Trajectory of the energy request vs the retail price. Zoom around the equilibrium point.

A numerical example

The simulation has been carried on taking $c(x) = 0.5x^2$ and $v(x) = 20\ \log(5x+1)$, $\gamma = 0.6$, $\rho = 0.03$ and $q = 0.05$. In Figure 17 it is depicted a typical trajectory of the energy vs the prices, while Figure 18 presents the trend of the total ISO revenue. A remark which arises from this and several other simulations is that the nonlinearity has a nontrivial influence on the behavior of the price and the ISO revenue. If we consider, in fact, the sample expectation of the tail of both the processes $\{\lambda_t^r\}_{t\geq 0}$ and $\{I_t\}_{t\geq 0}$ (in order to "forget" the initial conditions), one sees that they are both slightly lower then the nominal equilibrium points λ^* and 0. However, this shift seems to be small and the algorithm is able to maintain the system in a neighborhood of the nominal equilibria.

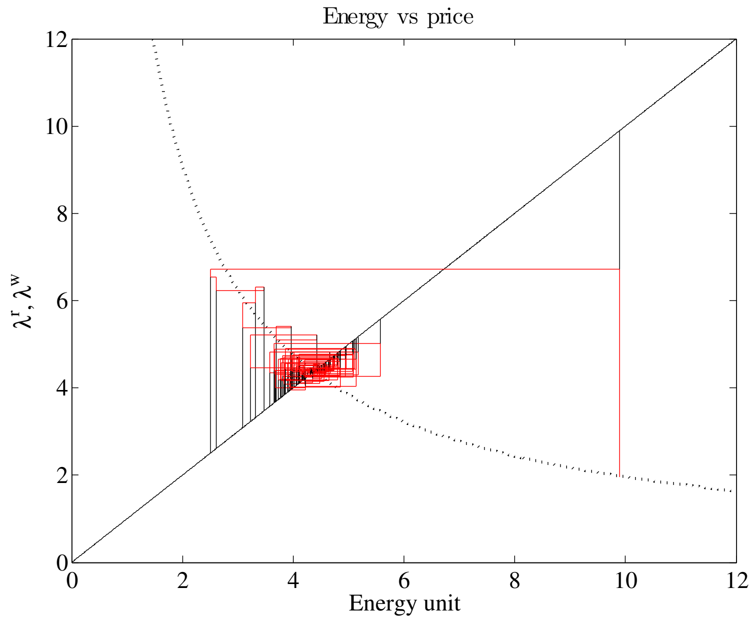

Figure 17: Trajectory of the energy request vs the retail price in the case with noise.

6 Customer's Dynamic Behaviour

In this section we want to address the problem of uncertainty or time–varying of customers' behaviour. We model this feature by considering different value functions taken within the same family, characterized by a gradual variation of a certain parameter. In particular, we want to test if the algorithms for price stabilization, presented in Section 3 of this article, are able to follow the consumers' behaviour evolution, by simply taking into account the available past data, and control the dynamics of prices with the goal to stabilize them. We have studied two different approaches to this problem, obtaining some nontrivial results.

Let's start with describing the dynamics of the problem: we have considered four different value functions $v_i(x), i \in [1, 2, 3, 4]$, corresponding to four different consumers' behaviours during different hours of the day. We divided one day (24 hours) in four periods, each made up of six hours, corresponding to early morning (4.00:10.00), midday (10.00:16.00), evening (16.00:22.00) and night (22.00:4.00). Consumers are in fact supposed to make different

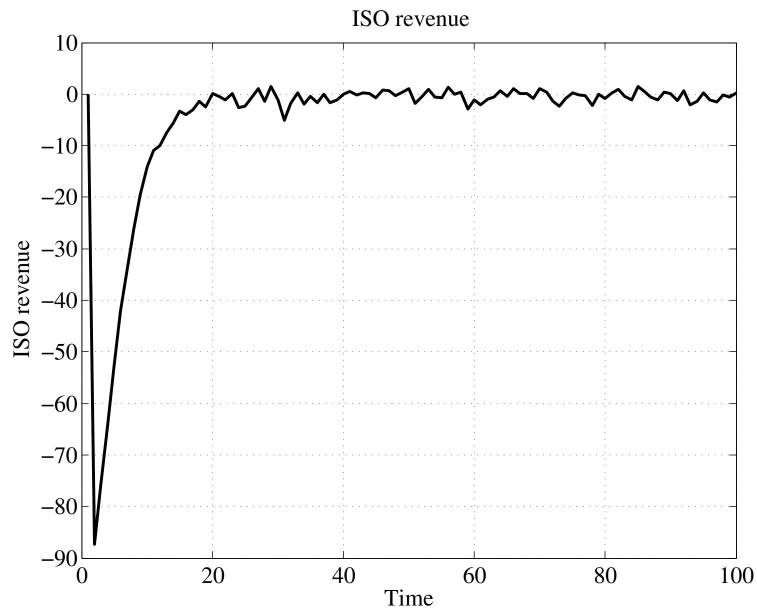

Figure 18: Trajectory of the total income/shortfall of the ISO with noise.

use of energy (electricity), with respect to different moments of the day.
The first approach employs the main result proposed in Section 3, i.e. the algorithm that calculates the retail price for the next time interval (hour) using a fixed parameter (γ) in order to stabilize the prices. From simulations we have obtained the algorithm convergence, both when starting far from each equilibrium point and when starting close to one of the four points of equilibrium. After each value function switch, the algorithm quickly settles on the new equilibrium point, until the next switch. This behaviour is shown in Figure 19. The evolution starts from the curve number 1. The arrows show the algorithm chasing the dynamics of the consumers after each slot switch. The trajectory of the retail prices is not shown here, as it emerges quite clearly from Fig. 19.

The second approach uses weighted least squares interpolation of the data recorded during the previous 24 hours, weighting them according to the time slots they belong to. The first problem we encountered was that we were considering points taken from four different curves, and the results were not satisfactory until we introduced a weighting scheme that matched current and past data belonging to the same time slots. Observations made dur-

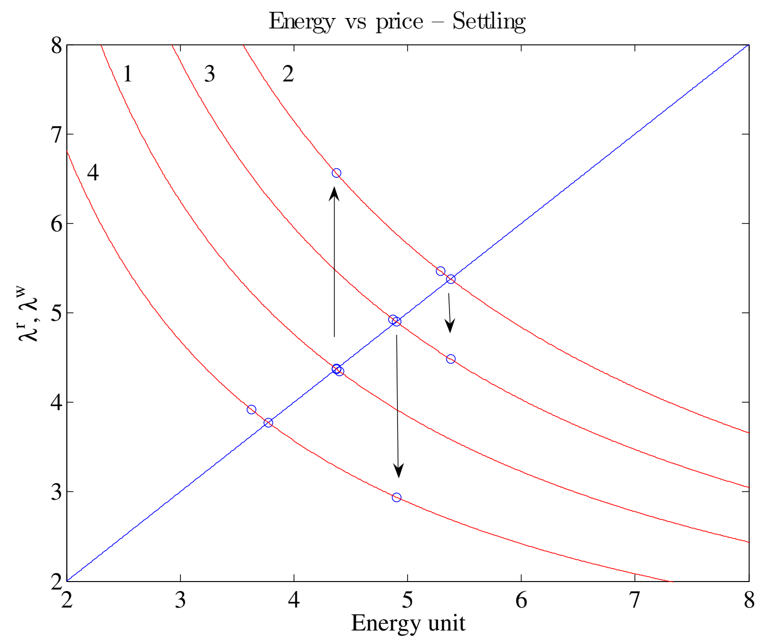

Figure 19: Behaviour of the algorithm while chasing $\dot{v}(x)$ with fixed γ.

ing the current time slot and during the same period of the previous day are preferred to the data corresponding to different slots. This approach is justified by the fact that we modeled the customers' behaviour evolution with a periodic function, which is a reasonable choice, as the energy request is similar at the same time during different days. A good weighting factor can be obtained by sampling the function $f(x) = \cos^{10} x$, suitably scaled in frequency (see Figure 20).
When starting close to one of the four equilibrium points (corresponding to the four different value functions), we interpolated the past 24 hours data (pairs of [energy, retail price]) with an hyperbola, which is a good approximation of the derivative of the chosen value functions. At first iteration, we used 24 data –from an execution of the previous algorithm– as a fictitious data history. The trajectory of the retail prices is shown in Figure 21. After each switch, the algorithm gradually moves towards the next equilibrium point, rather then reaching it in just one or two steps. This behaviour is due to the symmetric form of the weighting factor, which provides to the algorithm some information about the "future" evolution of the target. In other words, the algorithm is able to predict the customers' behaviour, and

thanks to this feature it starts moving towards the next equilibrium point even before the switching takes place. Such a trend could result very useful in a practical application, as we can assume that, in a real energy market, the value function varies in a continuous way.

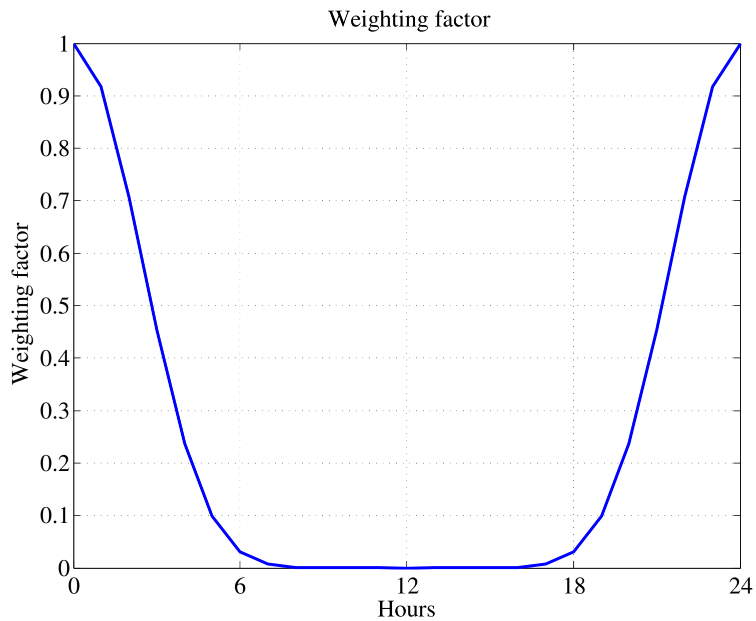

Figure 20: Weighting factor.

7 Conclusions and future work

In this article we presented an algorithm for the stabilization of retail and wholesale prices in the energy market, based on the main result provided in [2]. In particular, we obtained a superior bound to the control parameter (γ) which ensures local convergence.
In order to maximize the convergence rate, we tried some fitting functions to estimate the unknown $\dot{v}(x)$, and we compared them through numerical simulations. Afterwards, we modified the previous algorithm in order to steer to zero the ISO total revenue, according to the non–for–profit nature of the operator.
Adding some noise to the consumer behaviour, we noticed from simulations that the stability of the algorithm still holds.

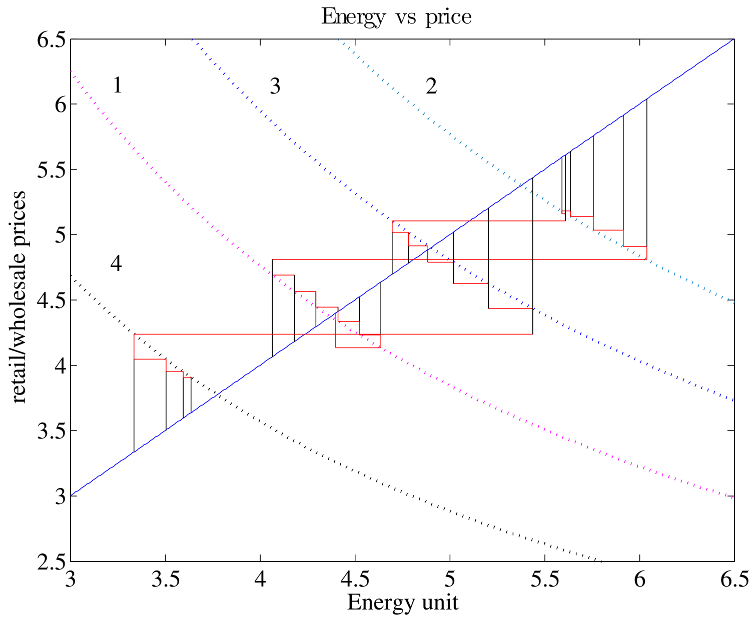

Figure 21: Trajectory of the retail prices while chasing $\dot{v}(x)$ with weighting factor.

Finally, in the last section, we tested the first algorithm varying the consumers' value functions, in accordance with the different moments of the day, and then introduced a weighting factor, to simulate a periodic demand. Future developments to the presented framework may involve effects of production uncertainties and more realistic models that take into account physical constraints, possibly testing different classes of cost and value functions. Other extending directions may include a more realistic variation of the consumers' value functions, and, in this context, the development of an algorithm to steer to zero the ISO revenue.

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