On the Stability of Wholesale Electricity Markets under Real-Time Pricing

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Abstract— **The paper proposes a mathematical model for describing the dynamic evolution of supply, demand, and market clearing prices under real-time pricing mechanisms that pass on–either directly or a (static) function of–the real-time wholesale electricity market prices to the end consumers. The effects that such mechanisms could pose on the stability and efficiency of the entire system is investigated and several stability criteria are presented. It is shown, under some reasonable assumptions, that relaying the real-time wholesale market prices to the end consumers could create an unstable closed loop feedback system. Finally, a result is presented which characterizes the efficiency losses incurred when, in order to achieve stability, the wholesale market prices are modified through a (static) pricing function before they are passed on to the end consumers.**

I. Introduction and Motivation

The increasing demand for energy and growing environmental concerns have created a need for a more efficient and modern power grid that will incorporate a large number of renewable resources and better storage and real-time demand response technologies. This paper focuses on the analysis, and to a lesser extent, the design of dynamic pricing mechanisms that are expected to be one of the characteristic features of modern power grids.

The rationale for using demand response and real-time pricing in electricity networks is manyfold, of which we mention two. First, is to reduce the annual demand peak. This helps the system designers and system operators reduce the maximum required capacity, and minimize or eliminate the need for expensive high-carbon emission mega power plants that are brought online only a couple of hours per year to meet the peak demand, and delay or suppress capacity expansion that is driven by peak demand. The second rationale is to minimize the amount of daily reserve capacity required to meet the demand in the face of contingencies. As the contribution of renewable resources to the supply of electricity grows in magnitude and extent, increased stochastic uncertainty challenges the system operators. A contingency may no longer be considered a discrete 0-1 event corresponding to the tripping of a generator or loss of a transmission line, rather, it would be characterized by a continuum of probabilistically quantified scenarios ranging from maximum to zero production. System reliability constraints would then require the system operator to hold more reserve capacity to deal with the increased uncertainty. Dynamic pricing can be used as a mechanism for stochastic matching of supply and demand and hence, minimize the required reserve capacity by exploiting the production capacity of renewable resources to a greater extent. In this paper we are not concerned with the stochastic matching or other promises of dynamic pricing, but with system stability and efficiency questions that arise.

The idea of using time-varying pricing mechanisms over a network to achieve certain objectives–similar in nature to the abovementioned in the sense of regulating the behavior of agents that compete for a resource–has been discussed in the economics, operations research, and engineering literature. The literature covering the topic of time-varying prices to control the flow rates in communication networks or in transportation networks is extensive. See for instance [6] and [5] for communication networks and [1] for transportation networks. However, the specific characteristics of power grids arising from physics, the safety-critical nature of the system, and market operation and economics pose new and unique challenges that need to be addressed.

Borenstein et. al. [2] study both the theoretical and the practical implications of various forms of dynamic pricing such at *Critical Peak Pricing*, *Time-of-Use Pricing*, and *Real-Time Pricing*. They argue in favor of real-time pricing, characterized by passing on a price that reflects the wholesale market prices to the end consumers, and conclude that it delivers the most benefits in the sense of reducing the peak and flattening the load curve. A similar conclusion is reached in a study conducted by Energy Futures Australia as part of a larger scope research conducted by the International Energy Agency Demand Side Management Programme [3]. In California, the state's Public Utility Commission (CPUC) enacted a series of new regulations in July 2008, with the objective of enhancing California's energy efficiency standards and renewable energy production [11]. In particular, CPUC set a deadline of 2011 for the state utility PG&E to propose a new *dynamic pricing* rate structure. CPUC defines dynamic pricing as electric rates that reflect actual wholesale market conditions, such as critical peak pricing, or real-time pricing. It regards the real-time price a rate linked to actual hourly wholesale energy prices [11]. In this paper we show that

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directly linking the consumer prices to the wholesale market prices may create an unstable close-loop feedback system in which the prices oscillate or diverge to unacceptable limits. We are not arguing against the idea of real-time pricing in power grids in its entirety, nor are we suggesting that it cannot fulfill its purposes. The message that we intend to deliver is that the design of a real-time pricing mechanism must take system stability issues into consideration, and that successful design and implementation of such a mechanism entails careful analysis of consumer behavior in response to price signals. Whether or not directly linking consumer prices to the wholesale market prices will cause instability depends on the utility functions of consumers and producers.

We propose a simple model for the consumers' response to price signals based on the maximization of a utility function. We assume that at each instant of time, supply and demand match exactly. Moreover, supply always follows the demand in the sense that at each instant of time any amount of the resource requested by the consumers must be matched by the producers and the price per unit for this exchange is the marginal cost of production. This is consistent with the current practice in electricity markets in most of the United States where Locational Marginal Pricing [8] (LMP) is implemented by an Independent System Operator (ISO). Though the actual pricing algorithms may be more detailed, incorporate some heuristics, and exhibit slight variations across different ISOs, the simple model used in this paper serves well as a good first order approximation of reality. We assume that the Locational Marginal Prices of the wholesale market, or a function of these prices, are passed to the consumers and the consumers adjust their consumption according to a curve which maximizes their utility function. Their adjusted demand is then a feedback signal to the wholesale market and affects the prices for the next time step. We analyze the properties of the closed loop system arising from this setup.

II. Preliminaries

A. Notation

The set of nonnegative real numbers is denoted by $\mathbb{R}_+$, positive real numbers by $\mathbb{R}_{++}$, and natural numbers by $\mathbb{N}$. The class of real-valued functions with a continuous n-th derivative on $X \subset \mathbb{R}$ is denoted by $\mathcal{C}^n X$. The identity function is denoted by I. For a function $f \in \mathcal{C}^1 X$, when convenient, we use $\dot{f}$ to denote the derivative of f with respect to its argument: $\dot{f}(x) = df(x)/dx$. Throughout the paper, $c_i \in \mathcal{C}^2[0,\infty)$ (or just c) denotes the cost function of producer i, and $v_j \in \mathcal{C}^2[0,\infty)$ (or just v) the value function of consumer j. Finally for a measurable set $X \subset \mathbb{R}$, $\mu_L(\{X\})$ is the Lebesgue measure of X.

B. Market Participants

We start with developing a simple electricity market model with three participants: 1. The suppliers, 2. The consumers, and 3. An independent system operator (ISO). The objective of the participating suppliers and consumers is to maximize the net benefit –to be defined precisely in the sequel– that they can draw from their engagement in the market. The ISO is an independent non-for-profit player in charge of clearing the market, that is, matching supply and demand subject to the network constraints with the objective of maximizing the social welfare (the aggregate surplus of consumers and producers). Below we describe the characteristics of the market participants in detail.

1) The Consumers and the Producers: Given $n_s \in \mathbb{N}$ and $n_d \in \mathbb{N}$, let $S := \{1, ..., n_s\}$ and $D := \{1, ..., n_d\}$ denote the index sets of suppliers and consumers respectively. We associate a value function $v_j(x)$ to each consumer $j \in D$, where $v_j(x)$ can be thought of as the dollar value that consumer j derives from consuming x units of a particular resource, electricity in this case. Similarly, to each producer $i \in S$, we associate a cost function $c_i(x)$ representing the dollar cost of producing x units of the resource.

Assumption I: For all $i \in S$, the cost functions $c_i(\cdot)$ are in $\mathcal{C}^2[0,\infty)$, strictly increasing, and strictly convex. For all $j \in D$, the value functions $v_j(\cdot)$ are in $\mathcal{C}^2[0,\infty)$, strictly increasing, and strictly concave.

It is assumed that the utility functions are quasi-linear. Hence, given a clearing price λ, the utility function of supplier $i \in S$ is given by $u_{\lambda i}(x) := \lambda x - c_i(x)$. Similarly, the utility function of consumer $j \in D$ is given by $u_{\lambda j}(x) := v_j(x) - \lambda x$. The utility function represents the net benefit that an agent derives from consuming or producing x units of electricity when the market clearing price (or the settlement price) is λ per unit. Let $d_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j \in D$, and $s_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i \in S$ denote $\mathcal{C}^1$ functions mapping price to consumption and production respectively. In the framework of utility maximizing agents, each agent maximizes the net benefit that they can derive from the market, therefore,

$$d_j(\lambda) = \arg\max_{x \in \mathbb{R}_+} \; v_j(x) - \lambda x, \qquad j \in D \qquad (1)$$

$$s_i(\lambda) = \arg\max_{x \in \mathbb{R}_+} \; \lambda x - c_i(x), \qquad i \in S \qquad (2)$$

Remark 1: Under Assumption I, when $\lambda \in [0,\infty)$, the maximization problems defined in (1) and (2) have a unique solution in $\mathbb{R}_+$ and the functions $d_j(\cdot)$ and $s_i(\cdot)$ are well-defined. Furthermore,

$$d_j(\lambda) = \max\{0, \arg\{\dot{v}_j(x) = \lambda\}\} = \max\{0, \dot{v}_j^{-1}(\lambda)\}$$

$$s_i(\lambda) = \max\{0, \arg\{\dot{c}_i(x) = \lambda\}\} = \max\left\{0, \dot{c}_i^{-1}(\lambda)\right\}$$

In the interest of simplicity and to avoid distracting details, for the rest of this paper, we assume that $d_j(\lambda) = \dot{v}^{-1}(\lambda)$ and $s_i(\lambda) = \dot{c}^{-1}(\lambda)$. This can be mathematically justified by adding the assumptions $\dot{v}(0) = \infty$, and $\dot{c}(0) = 0$ to *Assumption I* (in which case an adjustment must be made that $v \in \mathcal{C}^1(0,\infty)$), or by assuming that $\lambda \in [\dot{c}(0), \dot{v}(0)]$.

Remark 2: The demand of consumer $j \in D$ is inelastic when $d_j(\lambda)$ is constant for all $\lambda \in \mathbb{R}_+$. In order to treat inelastic demand within the same framework, we allow one exception to Assumption I. We allow the following value function:

$$\breve{v}_j(d_j) = \begin{cases} -\infty & d_j \neq \bar{d}_j \\ 0 & d_j \end{cases} \quad (3)$$

and define $\breve{v}_j^{-1}(\lambda) = \bar{d}_j$, $\forall \lambda \in [0, \infty)$.

Definition 1: The social welfare is defined as the aggregate benefit of the producers and the consumers:

$$\mathcal{S} = \sum_{j \in D} u_{\lambda j}(d_j) - \sum_{i \in S} u_{\lambda i}(s_i)$$

When the system is at the equilibrium in the sense that the total supply equals the total demand and there is a unique clearing price λ for the entire system, then:

$$\mathcal{S} = \sum_{j \in D} v_j(d_j) - \sum_{i \in S} c_i(s_i)$$

2) The Independent System Operator: The system operator is an independent non-for-profit entity responsible for optimal matching of supply and demand subject to network constraints. The objective function in the ISO optimization problem is the value function of the demand less the production cost [1], adjusted for reserve requirements. The network constraints include power flow constraints (Kirchhoff's Current Law and Kirchhoff's Voltage Law), transmission line constraints, generator capacity constraints, and local and system wide-reserve capacity requirements and potentially some other ISO-specific constraints. We refer the interested reader to [9], [7], [10], [12] for more details on the specificity of the constraints and the objective function in the ISO optimization problem. We note in passing that for real-time system and market operation, the constraints are linearized near the steady state operating point and the ISO optimization problem is reduced to a linear program, usually referred to as the *Economic Dispatch Problem*. A set of Locational Marginal Prices emerge as the dual variables corresponding to the nodal power balance constraints of this optimization problem. These prices vary from location to location and they represent the marginal cost of supplying electricity at a particular location. Again we refer the reader to [8], [9], [12] for more details. However, we would like to mention an important and intuitive property of the LMP algorithm. The spatial variation in the LMPs is a consequence of congestion in the transmission lines. When there is sufficient capacity in the network so that no transmission line is congested, one uniform price will materialize for the entire network. With this observation in sight, in order to develop simple mathematical models around which we can build a framework and improve our intuition, we make the following simplifying assumptions:

1) Resistive losses in transmission/distribution lines are negligible.
2) The line capacities are high enough, so as no congestion will occur.
3) There are no capacity constraints on the generators.
4) There are no reserve capacity requirements.

Under the first two assumptions the network parameters become irrelevant in the supply-demand optimal matching problem. The 3rd and 4th assumptions are made in the interest of keeping the development in this paper focused. They could, otherwise, be relaxed at the expense of a rather involved technical analysis.

The following problem then characterizes the ISO's optimization problem:

$$\begin{aligned} \max \quad & \sum_{j \in D} v_j(d_j) - \sum_{i \in S} c_i(s_i) \\ \text{s.t.} \quad & \sum_{j \in D} d_j = \sum_{i \in S} s_i \end{aligned} \quad (4)$$

The following lemma is adopted from [6].

Lemma 1: Let $d^* = [d_1^*, \cdots, d_{n_d}^*]$, and $s^* = [s_1^*, \cdots, s_{n_s}^*]$ where $d_j^*, \ j \in D$ and $s_i^*, \ i \in S$, solve (4). There exists a price $\lambda^* \in (0, \infty)$, such that (d^*, s^*) solves (1) and (2). Furthermore, λ^* is the Lagrangian multiplier corresponding to the balance constraint.

Proof: The proof is based on Lagrangian duality and is omitted here for brevity. The proof in [6] would be applicable here with some minor adjustments. ■

The implication of Lemma 1 is that by setting the market price to λ^*, the Lagrangian multiplier corresponding to (4), the system operator creates an environment in which, the collective selfish behavior of the participating agents results in a system-wide optimal condition. In other words, the aggregate surplus is maximized while each agent maximizes his own net benefit.

The uniform clearing price λ^* in Lemma 1, is the would-be Locational Marginal Price in the wholesale market. Since (4) does not include network constraints, the price is uniform for all agents. Consider the special case of inelastic demand with value function defined in (3). Let $\bar{d} := \sum_{j \in D} d_j$ denote the aggregate inelastic demand. Problem (4) is then equivalent to minimization of $\sum_{i \in S} c_i(s_i)$ subject to $\sum_{i \in S} s_i = \bar{d}$. Therefore, the ISO has to find the minimum cost solution to production of $\bar{d}$ units of electricity and dispatch each generator $i \in S$ accordingly. As usual, the corresponding wholesale market price would be the marginal cost of that production.

3) Representative Agent Model: In this section, we develop an abstraction of the model in (4) with only one producer agent and one consumer agent, representing the entire group of producers and consumers respectively. The rationale is that from a wholesale market stability point of view, it is the aggregate demand or supply that influences the system. In particular, when there is one uniform system price, the system operator would not be interested in how an individual consumer/producer reacts to real-time prices, rather, the aggregate response is the quantity of interest. In multi-agent systems, especially in the context of economics, a representative agent is a fictitious agent whose decisions

[1] When the demand is fixed, the objective becomes to minimize the total production cost.

and responses to signals and events is mathematically equivalent to the aggregate decision of a group of agents. Whether it is possible/realistic or not, to abstract the decision of a group of agents by a representative agent is a well-studied subject in economics [4]. For the purpose of this paper, such construction is always possible, though explicit formulae for the representative agent may sometimes be very complicated or impossible to find.

Lemma 2: Let functions v_j, $j \in D$, and c_i, $i \in S$, satisfying Assumption I, and $\dot{v}_j(0) = \infty$, $\forall j$, and $\dot{c}_i(0) = 0$, $\forall i$. Suppose that there exists functions $\bar{v}$ and $\bar{c}$ satisfying assumption I, such that

$$\lambda = \dot{\bar{v}}\left(\sum_{i=1}^{n_d} \dot{v}_i^{-1}(\lambda)\right), \ \forall \lambda \in \mathbb{R}_+ \tag{5}$$

and

$$\lambda = \dot{\bar{c}}\left(\sum_{i=1}^{n_s} \dot{c}_i^{-1}(\lambda)\right), \ \forall \lambda \in \mathbb{R}_+ \tag{6}$$

Then:

1) If (d^*, s^*) solves (4), then $\bar{d}^* := \sum \bar{d}_j^*$ and $\bar{s}^* := \sum_i^* \bar{s}_i^*$ satisfy:
$$\bar{d}^* = \bar{s}^* = x^*$$
where x^* solves:
$$\max_x \quad \bar{v}(x) - \bar{c}(x) \tag{7}$$

2) If λ^* and $\bar{\lambda}^*$ are the optimal clearing prices corresponding to (4) and (7) respectively, then $\lambda^* = \bar{\lambda}^* = \dot{\bar{v}}(x^*) = \dot{\bar{c}}(x^*)$.

Lemma 2 presents a construction for the representative agent model applicable to the development in this paper.

Example 1: Consider the case where all agents are identical: $v_{i_1} = v_{i_2}, \forall i_1, i_2$. Then $\bar{v}(x) := n_d v_1\left(n_d^{-1} x\right)$ satisfies (5). As another example, consider $v_i(x) = \alpha_i \log(1 + x)$, and define $\bar{v}(x) := \bar{\alpha} \log(n_d + x)$, where $\bar{\alpha} = \sum \alpha_i$. Then $\bar{v}$ satisfies (5). However, since $\dot{v}_i(0) = \alpha_i < \infty$, the response to a price λ of the representative agent with value function $\bar{\alpha}\log(n_d + x)$ is equal the sum of the responses of the individual agents only when $\lambda \leq \min_i \alpha_i$.

C. Dynamic Supply-Demand Model

In this section we develop a dynamical system model for the interaction of wholesale supply and retail demand in electricity markets. The model is based on the current practice in real-time balancing markets in the United States, except that it assumes that the consumers adjust their usage based on the real-time wholesale market prices. We will use representative agent models with cost and value functions $c(\cdot)$ and $v(\cdot)$ to represent supply and demand respectively. Studying the stability properties and some mechanisms by which a stable equilibrium can be reached is the subject of Section III.

In a power grid, the aggregate supply has to exactly match the aggregate demand at each instant of time. Therefore, in real-time, the supply always follows the demand. The real-time market is cleared at discrete time intervals and the prices are calculated and announced for each interval[2]. When the price announced at time $t = k$ corresponds to the time interval $[k-1, k]$, it is called expost pricing. In expost pricing the demand is subject to some price uncertainty as the actual price will be revealed after the consumption has materialized. When the announced price corresponds to the next time interval ahead (based on predicted demand), it is called exanté pricing. In exanté pricing the ISO faces price uncertainty as it will have to pay the generators based on the actual marginal cost of production (that is, the expost price), while it can charge the demand only based on the exanté price, which is only a prediction of the actual price.

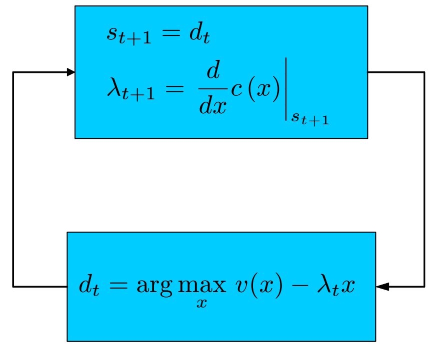

Fig. 1. Exanté Priced Supply/Demand Feedback

1) Price Dynamic under Exanté Pricing: Let λ_t denote the exanté price announced by the ISO corresponding to consumption of one unit of electricity in the time interval $[t, t+1]$. Let d_t be the actual consumption that will occur during this time interval, then: $d_t = \arg\max_{x \in \mathbb{R}_+} \ v(x) - \lambda_t x$. (Since $v(\cdot)$ is known only to the consumer, at time t, the ISO has only an estimate of d_t, based on which it has calculated and announced the price λ_t). At time $t+1$, the ISO needs to announce λ_{t+1}, which will be the marginal cost of predicted production during the next time interval. We assume that the ISO's predicted production for each time interval is equal to the demand at the previous time interval: $s_{t+1} = d_t$. The following equations describe the dynamics of the market:

$$\begin{aligned} \lambda_{t+1} &= \dot{c}(s_{t+1}) \\ s_{t+1} &= d_t \\ d_t &= \arg\max_{x \in \mathbb{R}_+} \ v(x) - \lambda_t x \end{aligned}$$

Combining the above equations the price dynamics is expressed in the following way:

$$\lambda_{t+1} = \dot{c}\left(\dot{v}^{-1}(\lambda_t)\right) \tag{8}$$

Remark 3: ISO's Risk: The system operator commits to a price of $\lambda_t \equiv \dot{c}(d_{t-1})$ for the consumers, while he has to pay

[2] In most regions of the United States, such as New England, California, or PJM, the real-time market is operated in five-minutes intervals.

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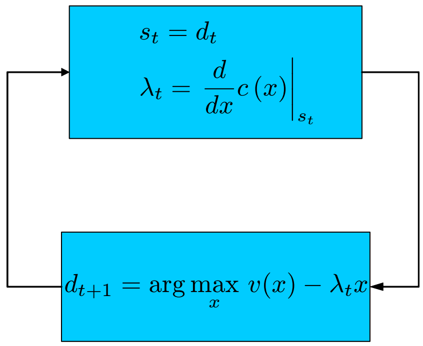

Fig. 2. Expost Priced Supply/Demand Feedback

the generators $\lambda_{t+1} \equiv \dot{c}(d_t)$. The ISO's revenue differential (either excess or shortfall) is therefore, given by:

$$\Delta_t = [\dot{c}(d_t) - \dot{c}(d_{t-1})] d_t \simeq \ddot{c}(d_t) d_t (d_{t+1} - d_t)$$

2) Price Dynamic under Expost Pricing: Under expost pricing, the price charged for consumption of one unit of electricity during the interval $[t, t+1]$ is calculated and declared only at the end of the interval, when the total consumption has materialized. In this case, the ISO bears no risk and the price uncertainty and the associated risks are all bore by the consumer. In order to decide on his consumption during the time interval $[t, t+1]$ the consumer needs to make a prediction about the price for the next interval. We assume that the consumer's predicted price for the next time interval is equal to the price at the previous interval: $\hat{\lambda}_{t+1} = \lambda_t$. Therefore,

$$\begin{aligned} \lambda_{t+1} &= \dot{c}(d_{t+1}) \\ \hat{\lambda}_{t+1} &= \lambda_t \\ d_{t+1} &= \arg\max_x v(x) - \hat{\lambda}_{t+1} x \end{aligned}$$

It can be easily seen that the price dynamics is the same as the case with exanté pricing: $\lambda_{t+1} = \dot{c}\left(\dot{v}^{-1}(\lambda_t)\right)$. The only difference is that the price uncertainty affects the consumer. The consumer's risk is:

$$\Delta_t = [\dot{c}(d_{t+1}) - \dot{c}(d_t)] d_t \simeq \ddot{c}(d_t) d_t (d_{t+1} - d_t)$$

Note that in this case, the consumer has an incentive to make his/her usage curve flat to minimize his/her risk.

III. Theoretical Statements

A. Stability Analysis

In this section we present several stability analysis Propositions based on Lyapunov techniques, and examine the properties of the market price dynamics (8) within the framework. The following Theorem provides an analysis tool based on a specific system decomposition which can be conveniently applied to (8).

Theorem 1: Consider a nonnegative sequence $\{x_k\}$ satisfying

$$\begin{aligned} x_0 &\in X_0 \subset \mathbb{R}_+ \\ x_{k+1} &= \psi(x_k) \end{aligned} \tag{9}$$

for some function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$. Then, there exists a function $x^* : X_0 \to \mathbb{R}_+$, satisfying $x^* = \psi \circ x^*$, such that

$$\lim_{k\to\infty} x_k = x^*(x_0)$$

if either of the following conditions hold:

1) $\psi(x) \leq x$, $\forall x \in \mathbb{R}_+$.
2) $\psi \in \mathcal{C}^1[0, \infty)$, and the following two conditions hold:

$$\text{(i)} \ \left|\dot{\psi}(x)\right| \leq 1. \qquad \mu_L\left(\{x \mid \dot{\psi}(x) = 1\}\right) = 0$$
$$\text{(ii)} \ \lim_{x\to\infty}\{\psi(x) - x\} < 0$$

3) $\psi \in \mathcal{C}^1[0, \infty)$, and there exist functions f and g mapping $\mathbb{R}_+$ to $\mathbb{R}_+$ satisfying:

$$g(x_{k+1}) = f(x_k) \tag{10}$$

and

$$\begin{aligned} &\text{(i)} \ \left|\dot{f}(x)\right| \leq |\dot{g}(x)|, \quad \mu_L(\{x \mid \dot{f}(x) = \dot{g}(x)\}) = 0 \\ &\text{(ii)} \ \lim_{x\to\infty}\{f(x) - g(x)\} < 0 \\ &\text{(iii)} \ \text{either } \dot{g}(x) \geq 0, \ \text{ or } \dot{g}(x) \leq 0, \ \forall x \in \mathbb{R}_+ \end{aligned} \tag{11}$$

Proof: To prove 1, define $V(x) = \psi(x)$, then V is a Lyapunov function for (9) in the sense that $V(x_{k+1}) \leq V(x_k)$. Consider the iteration given in (9) and define

$$\underline{k} = \inf\left\{k \mid V\left(\psi^{k+1}(x_0)\right) = V\left(\psi^k(x_0)\right)\right\}$$

If $\underline{k} < \infty$, then define $x^*(x_0) := \lim_{k\to\infty} x_k = \psi^{\underline{k}}(x_0)$. If $\underline{k} = \infty$, then $\{V(x_k)\}$ is a strictly decreasing bounded sequence and must converge to a limit ψ^*. Then take $x^*(x_0) = \lim_{k\to\infty} x_k = \psi^*$. Only 3 needs to be proven as 2 is a special case of 3 with $g = I$, and $f = \psi$. Let $V(x) = |f(x) - g(x)|$. Then

$$\begin{aligned} \forall x, y &\in \mathbb{R}_+, \ x \neq y : \\ |f(x) - f(y)| &\leq \left|\int_y^x \left|\dot{f}(\tau)\right| d\tau\right| \\ &< \left|\int_y^x |\dot{g}(\tau)| \, d\tau\right| = |g(x) - g(y)| \end{aligned} \tag{12}$$

(The last equality holds under the assumption of sign-invariance property of $\dot{g}$, though the inequality is still valid in the absence of this property). Then

$$\begin{aligned} & V(x_{k+1}) - V(x_k) \\ = \ & |f(x_{k+1}) - g(x_{k+1})| - |f(x_k) - g(x_k)| \\ = \ & |f(x_{k+1}) - f(x_k)| - |g(x_{k+1}) - g(x_k)| \\ < \ & 0 \end{aligned} \tag{13}$$

Therefore, $\{V(x_k)\}$ is a strictly decreasing bounded sequence and converges to a limit $c \geq 0$. We show that $c > 0$ is not possible. Note that the sequence x_k is bounded from below since ψ is nonnegative. It can be shown, using an argument similar to that of proof of statement 1, that the condition $\lim_{x\to\infty}\{f(x) - g(x)\} < 0$ implies that x_k is bounded from above too, and resides in a compact subset $D \subset \mathbb{R}_+$. Therefore, $\{x_k\}$ has a subsequence $\{x_{k_i}\}$ which converges to a limit x^*. Then

$$\begin{aligned}\lim_{k\to\infty} V(x_k) &= \lim_{i\to\infty} V(x_{k_i}) = \left|\lim_{i\to\infty}\{f(x_{k_i}) - g(x_{k_i})\}\right| \\ &= |f(x^*) - g(x^*)|\end{aligned}$$

If $g(x^*) = g(\psi(x^*))$ then $c = |f(x^*) - g(\psi(x^*))| = 0$ (due to (10)). If $g(x^*) \neq g(\psi(x^*))$ then

$$\exists\delta, \varepsilon > 0, \text{ s.t. } |g(\psi(x)) - g(x)| \geq \varepsilon, \ \forall x \in \mathcal{B}(x^*, \delta)$$

Consider the function $\theta : \mathcal{B}(x^*, \delta) \to \mathbb{R}_+$, where

$$\theta : x \mapsto \frac{|f(\psi(x)) - f(x)|}{|g(\psi(x)) - g(x)|}$$

Then $\theta(x) < 1$, $\forall x \in \mathcal{B}(x^*, \delta)$ (cf. 13). Furthermore, the function is continuous over the compact set $\mathcal{B}(x^*, \delta)$ and achieves its supremum $\bar{\theta}$, where $\bar{\theta} < 1$. Since x_{k_i} converges to x^* there exists $\hat{k} \in \mathbb{N}$, such that $x_{\hat{k}} \in \mathcal{B}(x^*, \delta)$. Then

$$\begin{aligned}&V(x_{k+1}) - \bar{\theta}V(x_k) = \\ &|f(x_{k+1}) - f(x_k)| - \bar{\theta}\,|g(x_{k+1}) - g(x_k)| \leq 0, \ \forall k \geq \hat{k}\end{aligned}$$

Since $\bar{\theta} < 1$, this proves that $c = 0$. Finally,

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} g(x_k) = g(x^*) = f(x^*)$$

$$x^* = g^{-1}(\lim_{k\to\infty} f(x_k)) = \lim_{k\to\infty} g^{-1} \circ f(x_k) = \lim_{k\to\infty} x_k$$

This completes the proof. ■

Remark 4: If the assumption $\dot{g}(x) \neq 0$ is relaxed, then the core of the proof of the theorem remains correct. Only the final step needs to change and the conclusion would be that $\{g(x_k)\}$ is convergent. Also, if condition (i) of 3 is changed to the more conservative condition:

$$\exists\theta \in (0, 1) : \forall x \in \mathbb{R}_+ : \left|\dot{f}(x)\right| \leq \theta\,|\dot{g}(x)|$$

then condition (ii) is not needed, and if it is changed to:

$$\exists\theta \in (-1, 1) : \forall x \in \mathbb{R}_+ : \left|\dot{f}(x)\right| \leq \theta\dot{g}(x) \qquad (14)$$

then (iii) is automatically satisfied. In order to present our results is a more concise form and with less technical details, we will use (14) to replace conditions (i)–(iii) of 3 in Theorem 1.

There exist situations in which, a natural decomposition of system (9) via functions f and g satisfying (10) is readily available. This is for instance the case for the price dynamics defined in (8), where $\psi = \dot{c} \circ \dot{v}^{-1}$, and the decomposition is obtained with $g = \dot{c}^{-1}$, and $f = \dot{v}^{-1}$. However, f and g obtained in this way may not readily satisfy (11). We present the following corollaries.

Corollary 1: Consider the dynamical system (9) and suppose that functions f and g satisfying (10) are given. Then, there exists a function $x^* : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$\lim_{k\to\infty} x_k = x^*(x_0)$$

if there exists $\theta \in (-1, 1)$, and a continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\left|\rho(f(x))\,\dot{f}(x)\right| \leq \theta\rho(g(x))\,\dot{g}(x), \ \forall x \in \mathbb{R}_+$$

Proof: If f and g satisfy (10) then so do $r \circ f$ and $r \circ g$ for any $r \in \mathcal{C}^1[0, \infty)$. The result then follows from Theorem 1 and Remark 4 and defining $\rho = \dot{r}$. ■

Corollary 2: The market price dynamics (8) is convergent if there exists $\theta \in (-1, 1)$ and a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$|\rho(\upsilon)\,\dot{\upsilon}| \leq \theta\rho(\sigma)\,\dot{\sigma} \qquad (15)$$

where

$$\upsilon = \dot{v}^{-1} \text{ and } \sigma = \dot{c}^{-1} \qquad (16)$$

Furthermore, if

$$|\ddot{c}| \leq \theta\ddot{v}$$

then (8) converges.

Proof: The first statement follows immediately from Corollary 1. The second statement is proven by taking $\rho = \ddot{c}$ and using the inverse derivative formulae

$$\frac{d}{dx}h^{-1}(x) = \left[\dot{h}\left(h^{-1}(x)\right)\right]^{-1}$$

■

Example 2: Consider (8) with $c(x) = x^\beta$, and $v(x) = x^{1/\alpha}$, where $\alpha, \beta > 1$. Then

$$\lambda_{t+1} = \beta\,(\alpha\lambda_t)^{\frac{\alpha\beta-\alpha}{1-\alpha}}$$

$$\upsilon(\lambda) = \dot{v}^{-1}(\lambda) = (\alpha\lambda)^{\frac{\alpha}{1-\alpha}} \ \Rightarrow \dot{\upsilon}(\lambda) = \frac{\alpha^2(\alpha\lambda)^{\frac{2\alpha-1}{1-\alpha}}}{1-\alpha}$$

$$\sigma(\lambda) = \dot{c}^{-1}(\lambda) = \left(\beta^{-1}\lambda\right)^{\frac{1}{\beta-1}} \ \Rightarrow \dot{\sigma}(\lambda) = \frac{\left(\beta^{-1}\lambda\right)^{\frac{2-\beta}{\beta-1}}}{\beta^2-\beta}$$

It can be verified that there does not exists a constant $\theta \in \mathbb{R}$ for which $|\dot{\upsilon}(\lambda)| \leq \theta\dot{\sigma}(\lambda)$, $\forall\lambda \in \mathbb{R}$. However, with $\rho(\upsilon) = \upsilon^{-1}$, we have:

$$|\rho(\upsilon(\lambda))\,\dot{\upsilon}(\lambda)| = \frac{\alpha\lambda^{-1}}{\alpha-1}, \quad \rho(\sigma(\lambda))\,\dot{\sigma}(\lambda) = \frac{\lambda^{-1}}{\beta-1}$$

Therefore, (15) is satisfied with

$$\theta = \frac{\alpha(\beta-1)}{\alpha-1}, \qquad \theta < 1 \text{ if } \beta < 2 - \alpha^{-1}$$

Hence the system is convergent for $\beta < 2 - \alpha^{-1}$. It can be verified that the condition is also necessary and the system diverges for $\beta > 2 - \alpha^{-1}$.

The following result uses Corollary 2 and can be proven by inspection.

Corollary 3: Suppose that (15) holds for a monotonically increasing function ρ. Then (8) is stable for all value functions v_a and cost functions c_a whenever:

$$\begin{array}{lcl} (i)\ \dot{c} \text{ and } \dot{c}_a \text{ are concave} & & (i)'\ \dot{v} \text{ and } \dot{v}_a \text{ are convex} \\ & \text{or} & \\ (ii)\ \dot{c}_a \leq \dot{c} & & (ii)'\ \dot{v}_a \geq \dot{v} \end{array}$$

Similarly, if ρ is monotonically decreasing, then (8) is stable for all value functions v_a and cost functions c_a whenever:

$$\begin{array}{lcl} (i)\ \dot{c} \text{ and } \dot{c}_a \text{ are convex} & & (i)'\ \dot{v} \text{ and } \dot{v}_a \text{ are concave} \\ & \text{or} & \\ (ii)\ \dot{c}_a \geq \dot{c} & & (ii)'\ \dot{v}_a \leq \dot{v} \end{array}$$

The qualitative interpretation of the above Corollary is that when the marginal cost is a convex function, higher costs have a stabilizing effect, while when it is concave, higher costs have a destabilizing effect. Similar statements can be made about the value functions.

B. Periodic Demand with Adjustment

The model that we have used so far in this paper assumes that the entire demand makes adjustments in response to price signals and that the response is completely characterized by the value function of the consumer. In this section we study a more generic model in which the demand is comprised of two components, one component is a periodic function of time which is insensitive to price variations, and the other is a price-sensitive component which is, as before, determined by a concave value function. More specifically,

$$\begin{aligned} d_t &= (1-\mu)\, p_t + \mu \dot{v}^{-1}(\lambda_t) \\ \lambda_{t+1} &= \dot{c}(d_t) \end{aligned} \qquad (17)$$

where $p_t : \mathbb{Z} \to \mathbb{R}_+$ is a periodic function representing the natural fluctuation of demand and $\mu \in [0,1]$ is a parameter. An interpretation of (17) is as follows: p_t represents the total population's demand in the absence of dynamic pricing and $\dot{v}^{-1}(\cdot)$ represents the demand response when the entire population is responsive. The parameter μ in this case represents the percentage of population subscribing to a real-time pricing program and $(1-\mu)\, p_t + \mu \dot{v}^{-1}(\lambda_t)$ is the entire demand.

Definition 2: Given a periodic function $p_t : \mathbb{Z}_+ \to \mathbb{R}_+$ satisfying $p_{t+T} = p_t$, a periodic orbit of (17) is a function $\bar{\lambda}_t : \mathbb{Z}_+ \to \mathbb{R}_+$, satisfying

$$\begin{aligned} \dot{c}^{-1}\left(\bar{\lambda}_{t+1}\right) &= (1-\mu)\, p_t + \mu \dot{v}^{-1}\left(\bar{\lambda}_t\right), \qquad \forall t \in \mathbb{Z}_+ \\ \bar{\lambda}_t &= \bar{\lambda}_{t+T}, \qquad \forall t \in \mathbb{Z}_+. \end{aligned}$$

Corollary 4: Consider system (17) and assume that the function $p_t : \mathbb{Z}_+ \to \mathbb{R}_+$ satisfies $p_{t+T} = p_t$. If there exists $\theta \in (-1,1)$ and a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\mu \left|\rho(\upsilon)\, \dot{\upsilon}\right| \leq \theta \rho(\sigma)\, \dot{\sigma} \qquad (18)$$

where $\dot{\upsilon}$ and $\dot{\sigma}$ are defined as in (16), then (17) has a periodic orbit $\bar{\lambda}_t$ with period T. Furthermore, all solutions converge to the periodic orbit in the sense that

$$\lim_{t\to\infty} \left|\lambda_t - \bar{\lambda}_t\right| = 0$$

for all functions λ_t satisfying (17) for some $\lambda_0 \in \mathbb{R}_+$.

Proof: The proof uses the same machinery of Theorem 1 and it omitted for brevity. ■

It is an immediate consequence of the above Corollary that participation of a small portion of the population in dynamic pricing programs will not destabilize the system as the left hand side of (18) goes to zero as μ goes to zero. System stability concerns should arise when a large portions of the population are exposed to real-time pricing. Consider for instance Example 2 with $\alpha = 2$ and $\beta = 2$. The system (17) would be unstable if $\mu = 1$ and would converge to a periodic orbit when $\mu < 0.5$.

C. Pricing for Stabilization and Loss of Efficiency

In this section we examine a pricing mechanism in which, the retail price is a static function of the wholesale market price. It is obvious that if we allow the retail market prices to be different than the wholesale market prices then achieving stability is not difficult, for instance, a constant retail market price is always stabilizing. We are interested in examining the effects of this type of pricing on the efficiency of the system. Suppose that the system has reached an equilibrium state with $\bar{\lambda}^r$ and $\bar{\lambda}^w$ as the retail and wholesale market prices respectively. Then:

$$\begin{aligned} \mathcal{S} &= v(x) - s(x) \\ &= v\left(\dot{v}^{-1}\left(\bar{\lambda}^r\right)\right) - c\left(\dot{c}^{-1}\left(\bar{\lambda}^w\right)\right) \end{aligned}$$

where $\mathcal{S}$ is the aggregate surplus. Let us denote by S_ϕ the surplus function corresponding to the case where $\lambda_t^r = \phi(\lambda_t^w)$ for some function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$. We present the following Theorem.

Theorem 2: Suppose that at any given time t, the wholesale price λ_t^w, and the consumer price λ_t^r satisfy $\lambda_t^w = \phi(\lambda_t^r)$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a $C^1[0,\infty)$ function. Then the wholesale market price dynamics is give by

$$\lambda_{t+1}^w = \dot{c}\left(\dot{v}^{-1}\left(\phi\left(\lambda_t^w\right)\right)\right) \qquad (19)$$

and converges to an equilibrium price $\bar{\lambda}^w$ satisfying $\phi\left(\bar{\lambda}^w\right) = \bar{\lambda}^r$ provided that there exists a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying:

$$\left|\rho(\upsilon(\phi))\, \dot{\upsilon}(\phi)\, \dot{\phi}\right| \leq \theta \rho(\sigma)\, \dot{\sigma}$$

where $\sigma = \dot{c}^{-1}$, and $\upsilon = \dot{v}^{-1}$. Furthermore, if for functions ϕ_1 and ϕ_2, either

$$0 < -\bar{\lambda}_1^w + \phi_1\left(\bar{\lambda}_1^w\right) < -\bar{\lambda}_2^w + \phi_2\left(\bar{\lambda}_2^w\right)$$

or

$$0 > -\bar{\lambda}_1^w + \phi_1\left(\bar{\lambda}_1^w\right) > -\bar{\lambda}_2^w + \phi_1\left(\bar{\lambda}_2^w\right)$$

Then

$$S_{\phi_2} < S_{\phi_1}$$

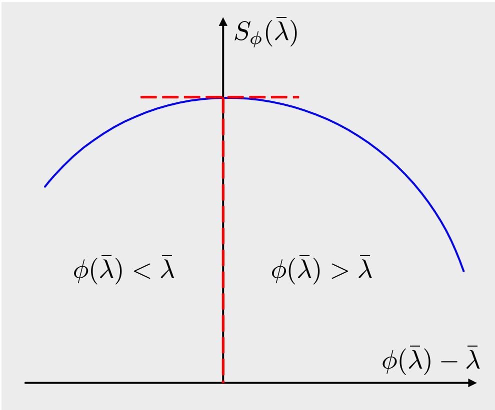

Fig. 3. The aggregate surplus as a function of the difference between the wholesale and the retail price.

Proof: The first statement is a corollary of Theorem 1. We present a proof for the second statement. Let $x_{\bar{\lambda}^w}$ denote the equilibrated supply and demand. Then:

$$x_{\bar{\lambda}^w} = \dot{c}^{-1}\left(\bar{\lambda}^w\right) = \dot{v}^{-1}\left(\phi\left(\bar{\lambda}^w\right)\right)$$

$$S\left(\bar{\lambda}^w\right) := v\left(x_{\bar{\lambda}^w}\right) - c\left(x_{\bar{\lambda}^w}\right) = v\left(\dot{c}^{-1}\left(\bar{\lambda}^w\right)\right) - c\left(\dot{c}^{-1}\left(\bar{\lambda}^w\right)\right)$$

$$\begin{aligned}\frac{dS\left(\bar{\lambda}^w\right)}{d\bar{\lambda}^w} &= \dot{v}\left(\dot{c}^{-1}\left(\bar{\lambda}^w\right)\right)\dot{\sigma}\left(\bar{\lambda}^w\right) - \bar{\lambda}^w\dot{\sigma}\left(\bar{\lambda}^w\right) \\ &= \left(\phi\left(\bar{\lambda}^w\right) - \bar{\lambda}^w\right)\dot{\sigma}\left(\bar{\lambda}^w\right)\end{aligned}$$

Since by assumption $c(\cdot)$ is convex, $\dot{c}^{-1}(\cdot)$ is increasing and $\dot{\sigma}\left(\bar{\lambda}^w\right) > 0$. Therefore, $dS\left(\bar{\lambda}^w\right)/d\bar{\lambda}^w$ is zero only when $\bar{\lambda}^w = \phi\left(\bar{\lambda}^w\right)$, which immediately implies that there is a loss of efficiency when the wholesale price and the consumer price at the equilibrium are not identical. Furthermore,

$$\begin{aligned}\frac{d\left(S\left(\bar{\lambda}^w\right)\right)}{d\left(\phi\left(\bar{\lambda}^w\right) - \bar{\lambda}^w\right)} &= \frac{d\left(S\left(\bar{\lambda}^w\right)\right)/d\bar{\lambda}^w}{d\left(\bar{\lambda}^r - \bar{\lambda}^w\right)/d\bar{\lambda}^w} \\ &= \frac{\left(\phi\left(\bar{\lambda}^w\right) - \bar{\lambda}^w\right)\dot{\sigma}\left(\bar{\lambda}^w\right)}{d\bar{\lambda}^r/d\bar{\lambda}^w - 1} \\ &= \frac{\left(\phi\left(\bar{\lambda}^w\right) - \bar{\lambda}^w\right)\dot{\sigma}\left(\bar{\lambda}^w\right)}{\ddot{v}\left(\sigma\left(\bar{\lambda}^w\right)\right)\dot{\sigma}\left(\bar{\lambda}^w\right) - 1}\end{aligned}$$

where the last equality follows from $\bar{\lambda}^r = \dot{v}\left(\dot{c}^{-1}\left(\bar{\lambda}^w\right)\right)$ and taking the derivative. The above derivation shows that $S\left(\bar{\lambda}^w\right)$ is in increasing function of $\phi\left(\bar{\lambda}^w\right) - \bar{\lambda}^w$ as long as $\phi\left(\bar{\lambda}^w\right) - \bar{\lambda}^w \leq 0$ (since $\ddot{v}\left(\sigma\left(\bar{\lambda}^w\right)\right)\dot{\sigma}\left(\bar{\lambda}^w\right) - 1 < 0$ and $\dot{\sigma}\left(\bar{\lambda}^w\right) > 0$), and a decreasing function of $\phi\left(\bar{\lambda}^w\right) - \bar{\lambda}^w$ as long as $\phi\left(\bar{\lambda}^w\right) - \bar{\lambda}^w \geq 0$. This proves the second statement. ■

The above Theorem indicates that when the consumer price is a (non-identity) function of the wholesale market price there is generally a loss of efficiency, and furthermore, the greater the discrepancy between the consumer price and the wholesale price, the greater the efficiency loss. Since the system is at the optimum if and only if $\phi\left(\bar{\lambda}^w\right) = \bar{\lambda}^w$, any function ϕ that results in an equilibrium with this property necessarily satisfies:

$$\bar{\lambda}^w = \dot{c}\left(\dot{v}^{-1}\left(\phi\left(\bar{\lambda}^w\right)\right)\right) = \dot{c}\left(\dot{v}^{-1}\left(\bar{\lambda}^w\right)\right)$$

Hence, any such $\bar{\lambda}^w$ should necessarily be the equilibrium of the original system under direct pricing.

IV. Conclusions and Future Work

We investigated the effects of dynamic pricing mechanisms on the stability and efficiency of electricity networks and showed that exposing the consumers to the real-time wholesale market prices could create an unstable closed loop feedback system, causing the prices to oscillate or diverge. We presented several stability criteria characterizing convergence based on the relation between the cost functions of the producers and the value functions of the consumers. The criteria extended very naturally to the case where the demand is combination of a periodic function and a price sensitive utility-maximizing component. It was shown that when the system operator uses a pricing strategy in which the consumer prices are a static function of the wholesale market prices, there is generally a loss of efficiency. The larger the difference between the wholesale market price and the consumer price, the farther is the system from an optimal equilibrium. Future works include extension of the results to the network case with power flow equations and transmission constraints in effect. Other directions for extension include analysis of non-static pricing mechanisms, that is, when the pricing function has memory. Analysis of the case of time-varying or stochastically fluctuating cost functions is also an important direction for future research.

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