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Hard variants of stable marriage

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Abstract

The Stable Marriage Problem and its many variants have been widely studied in the literature (Gusfield and Irving, The Stable Marriage Problem: Structure and Algorithms, MIT Press, Cambridge, MA, 1989; Roth and Sotomayor, Two-sided matching: a study in game-theoretic modeling and analysis, Econometric Society Monographs, vol. 18, Cambridge University Press, Cambridge, 1990; Knuth, Stable Marriage and its Relation to Other Combinatorial Problems, CRM Proceedings and Lecture Notes, vol. 10, American Mathematical Society, Providence, RI, 1997), partly because of the inherent appeal of the problem, partly because of the elegance of the associated structures and algorithms, and partly because of important practical applications, such as the National Resident Matching Program (Roth, J. Political Economy 92(6) (1984) 991) and similar large-scale matching schemes. Here, we present the first comprehensive study of variants of the problem in which the preference lists of the participants are not necessarily complete and not necessarily totally ordered. We show that, under surprisingly restrictive assumptions, a number of these variants are hard, and hard to approximate. The key observation is that, in contrast to the case where preference lists are complete or strictly ordered (or both), a given problem instance may admit stable matchings of different sizes. In this setting, examples of problems that are hard are: finding a stable matching of maximum or minimum size, determining whether a given pair is stable—even if the indifference takes the form of ties on one side only, the ties are at the tails of lists, there is at most one tie per list, and each tie is of length 2; and finding, or approximating, both an ‘egalitarian’ and a ‘minimum regret’ stable matching. However, we give a 2-approximation algorithm for the problems of finding a stable matching of maximum or minimum size. We also discuss the significant implications of our results for practical matching schemes.

Keywords: Stable marriage problem; Indifference; Ties; NP-completeness; Approximation algorithms

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1. Introduction

An instance I of the classical Stable Marriage problem (SM) involves n men and n women, each of whom ranks all the members of the opposite sex in strict order of preference. A *matching* M in I is a one–one correspondence between the men and women. We say that a (man,woman) pair (m,w) *blocks* M, or is a *blocking pair* with respect to M, if m prefers w to $p_M(m)$, and w prefers m to $p_M(w)$, where $p_M(q)$ denotes the partner of q in M. A matching that admits no blocking pair is said to be *stable*. It is known that every instance of SM admits at least one stable matching, and that such a matching can be found in $O(n^2)$ time using the Gale/Shapley algorithm [3].

1.1. Incomplete preference lists

A generalisation of SM occurs when the preference lists of those involved can be incomplete. In this case, we say that person p is *acceptable* to person q if p appears on the preference list of q, and *unacceptable* otherwise. We use SMI to stand for this variant of SM where preference lists may be incomplete. A matching M in an instance I of SMI is a one–one correspondence between a subset of the men and a subset of the women, such that $(m,w)\in M$ implies that each of m,w is acceptable to the other. The revised notion of stability may be defined as follows: M is stable if there is no (man,woman) pair (m,w), each of whom is either unmatched in M and finds the other acceptable, or prefers the other to his/her partner in M.[3] (It follows from this definition that, from the point of view of finding stable matchings, we may assume, without loss of generality, that p is acceptable to q if and only if q is acceptable to p.) A stable matching in I need not be a complete matching. However, all stable matchings in I have the same size, and involve exactly the same men and exactly the same women [4]. It is a simple matter to extend the Gale/Shapley algorithm to cope with preference lists that may be incomplete (see [6, Section 1.4.2]).

We shall refer to the classical many–one generalisation of the (one–one) problem SMI, which is relevant in a number of important applications, as the Hospitals/Residents problem (HR) [6, 22]. An instance I of HR involves a set of *residents* and a set of *hospitals*, each resident seeking a post at one hospital, and the ith hospital having c_i posts. Each resident strictly ranks a subset of the hospitals, and each hospital strictly ranks its applicants. A matching M in I is an assignment of each resident to at most one hospital so that, for each i, at most c_i residents are assigned to the ith hospital. Matching M is stable if there is no (resident,hospital) pair (r,h) such that (i) r,h find each other acceptable, (ii) r is either unassigned or prefers h to his assigned hospital, and (iii) h either has an unfilled post or prefers r to at least one of the residents assigned to it. Again, the Gale/Shapley algorithm may be extended to find a stable

[3] Implicitly here, and henceforth for other stability definitions, such a pair (m,w) is defined to *block* M, or to be a *blocking pair* with respect to M, as for the SM case.

matching for a given instance of HR [6, Section 1.6.3]. Also, analogous to the SMI case, every stable matching in I has the same size, matches exactly the same set of residents, and fills exactly the same number of posts at each hospital. (This is known as the 'Rural Hospitals Theorem' [20, 4, 21].) Note that, in an instance of HR, it is not necessary for the numbers of residents and hospital posts to be equal; however, for simplicity we assume in this paper that the numbers of men and women are equal in an SMI instance.

1.2. Ties in the preference lists

An alternative natural extension of the original stable marriage problem arises when each person need not rank all members of the opposite sex in *strict* order. Some of those involved might be indifferent among certain members of the opposite sex, so that preference lists may involve *ties*.[4] We use SMT to stand for the variant of SM in which preference lists are complete but may include ties. In this context, a matching M is stable if there is no (man,woman) pair (m, w), each of whom *strictly* prefers the other to his/her partner in M. Note that this stability criterion is referred to as *weak stability* in [10], where two other notions of stability are formulated for SMT. However, of the three definitions, it is weak stability which has received the most attention in the literature [20, 18, 19, 14]. We are concerned exclusively with weak stability in this paper, and henceforth for brevity, the term *stability* will be used to indicate weak stability when ties are present.

By breaking the ties arbitrarily, an instance I of SMT becomes an instance I' of SM, and it is clear that a stable matching for I' is also a stable matching for I. Thus, a stable matching for I can be found using the Gale/Shapley algorithm. (Conversely, given a stable matching M in I, it is not difficult to see that there is an instance I_M of SM in which M is stable. Hence, a matching M is stable in I if and only if M is stable in some instance of SM obtained from I by breaking the ties.)

1.3. Ties and incomplete preference lists

In this paper, we focus on the variant of the stable marriage problem, denoted SMTI, which incorporates *both* extensions described above. Thus, an instance I of SMTI comprises preference lists, each of which may involve ties and/or be incomplete. A combination of the earlier definitions indicates that a matching M in I is stable if there is no (man,woman) pair (m, w), each of whom is either unmatched in M and finds the other acceptable, or strictly prefers the other to his/her partner in M.

As observed above, all stable matchings for a given instance of SMI are of the same size, and all stable matchings for a given instance of SMT are complete (and therefore

[4] In this paper, we restrict attention to the case where the indifference takes the form of ties in the preference lists, but it may be verified that all results are extendable to the general case where the preference lists are arbitrary partial orders.

of the same size). However, for a given instance of SMTI, it is no longer the case that all stable matchings need be of the same size. This fact does not appear to have been noted explicitly in the literature previously. We give a simple example to illustrate this in Section 2. Analogous observations apply if we introduce the possibility of ties into HR—we refer to this problem as Hospitals/Residents problem with Ties (HRT) [13]. The stability criterion for HRT may be defined by substituting 'strictly prefers' for 'prefers' in parts (ii) and (iii) of the stability criterion for HR. Clearly, SMTI is a special case of HRT (in which every hospital has one post, and the numbers of posts and residents are equal).

1.4. The practical setting

As stable matchings in an SMTI instance may be of different sizes, the question arises as to whether there exists an efficient algorithm to find a *maximum cardinality* stable matching for a given instance of SMTI and/or HRT. This question has particular significance within the context of matching residents to hospitals. As is current practice in the National Resident Matching Program [20] in the US and the Canadian Resident Matching Service [1], hospitals must rank a possibly large number of applicants in strict order of preference. However, it is unrealistic to expect large and popular hospitals to provide a strict ranking of all of their applicants; they might be happier, say, to rank their favourite applicants, and then group together the remainder at the tail of their list. In the recently introduced Scottish Pre-registration house officer Allocations (SPA) matching scheme [11, 16], any hospital may indeed include a tie at the tail of its preference list, but all ties are broken arbitrarily by the matching program so that the preference lists become strict. However, the previous observation indicates that breaking the ties in different ways can affect the sizes of the subsequent stable matchings. Since a prime objective is to match as many residents as possible, it would be desirable to have a strategy to break the ties so as to maximise the cardinality of the consequent stable matchings. In fact, as we shall show in this paper, the existence of a polynomial-time algorithm for this problem is unlikely, since a related decision problem turns out to be NP-complete, and the result holds for the restrictions corresponding to this practical setting that we have described. However, we give a 2-approximation algorithm for the maximisation problem.

1.5. Egalitarian and minimum regret stable matchings

Related stable matching problems which also have applications to centralised matching schemes involve finding 'fair' stable matchings which maximise the overall 'happiness' of the participants in some sense. To be more precise, let I be an instance of SMT and let M be a stable matching in I. For a person q in I, define $c_M(q)$, the *cost of M for q*, to be the ranking (possibly joint ranking, if ties are involved) of $p_M(q)$ in q's preference list. For example, if some woman w has preference list[5] m_2 (m_1 m_3) m_4 in I, then $c_M(w)=1,2,2,4$ if w's partner in M is m_2, m_1, m_3, m_4 respectively.[6] Let U and W denote the set of men and women in I respectively, and denote by $w(M)$ the *weight* of M, where $w(M)=\sum_{q\in U\cup W} c_M(q)$; similarly denote by $r(M)$ the *regret* of M, where $r(M)=\max_{q\in U\cup W} c_M(q)$. Define an *egalitarian* (resp. *minimum regret*) stable matching to be one whose weight (resp. regret) is minimum, taken over all stable matchings.

It is known that if I contains no ties (and is therefore an instance of SM), then each of the problems of finding an egalitarian stable matching and a minimum regret stable matching is polynomial-time solvable [12, 5]. However in this paper we show that, for an arbitrary instance of SMT, both of these problems are NP-hard, and are hard to approximate.

1.6. Related work

Ronn [18, 19] was possibly the first to study stable matching problems with ties in the preference lists from an algorithmic point of view. Among other things, he proved that the Stable Roommates problem (the non-bipartite extension of Stable Marriage), although solvable in polynomial time when all preference lists are strict [8], becomes NP-complete when ties are permitted. As previously mentioned, Irving [10] studied SMT, but primarily under two alternative definitions of stability to the one used here. Recently, Iwama et al. [14] have also investigated SMT and SMTI, and present two reductions proving the NP-completeness of the problem of deciding whether a given SMTI instance has a complete stable matching. However, both reductions introduce instances containing ties of length at least three, and ties on both sides. Also, it is shown that, for an SMT instance of size n, it is hard to approximate an egalitarian stable matching within a factor of $n^{1-\varepsilon}$, for any $\varepsilon>0$. But again, the constructed instance contains ties of length at least three, and ties on both sides.

1.7. Summary of results

In this section, we outline the organisation of the remainder of this paper. The following list indicates the main results that we establish, some of which have already been discussed in greater detail: (In what follows, the reader should bear in mind that SMT is a special case of SMTI, which in turn is a special case of HRT.)

(1) In contrast to the case where preference lists are strictly ordered or complete (or both), a single instance of SMTI may admit stable matchings of different sizes (Section 2).

[5] In a preference list throughout this paper, persons within round brackets are tied.

[6] For the egalitarian and minimum regret stable matching problems, the cost of a matching for a person q whose preference list is partially ordered may be defined as follows. Assume that $\prec_q$ denotes q's list, where $r \prec_q s$ if and only if q strictly prefers r to s. Then $c_M(q)$ is 1 plus the number of predecessors in $\prec_q$ of $p_M(q)$.

(2) For a given instance of SMTI, finding a stable matching of maximum, or minimum, size is NP-hard, even in the highly constrained case where the ties occur at the tails of lists and on one side only, there is at most one tie per list, and each tie is of length 2 (Section 2).
(3) There is a polynomial-time 2-approximation algorithm to find a stable matching of maximum, or minimum, size for a given instance of HRT; indeed, the maximum size cannot exceed the minimum size by more than a factor of 2 (Section 2).
(4) For a given instance of SMT, determining whether a given (man, woman) pair is stable, i.e. whether they can be paired in a stable matching, is NP-complete, even if the ties occur at the tails of lists and on one side only, there is at most one tie per list, and each tie is of length 2 (Section 3).
(5) For a given instance I of SMT, each of the problems of finding an egalitarian and a minimum regret stable matching is NP-hard, and not approximable within $n^{1-\varepsilon}$, for any $\varepsilon > 0$, unless P = NP, where n is the number of participants in I. Each of these results holds even if the ties occur on one side only and each tie is of length 2 (Section 4).

2. Cardinality of stable matchings in SMTI

As a simple illustration of the fact that an SMTI instance can have stable matchings of different sizes, consider the following instance involving two men, m_1, m_2, and two women, w_1, w_2:

$$\begin{array}{ll} m_1 : w_1 & w_1 : (m_1 \ m_2) \\ m_2 : w_1 \ w_2 & w_2 : m_2 \end{array}$$

There are two stable matchings for this instance, namely $\{(m_2, w_1)\}$, of cardinality 1, and $\{(m_1, w_1), (m_2, w_2)\}$, of cardinality 2.

In this section, we prove that the existence of algorithms to find a stable matching of maximum or minimum cardinality for a given instance of SMTI is unlikely, under several simultaneous restrictions. We also give an upper bound for how closely such matchings can be efficiently approximated for HRT, and remark that there is an interpolation result for stable matchings in SMTI.

Define the following decision problems:

Name: MAX (resp. MIN) CARDINALITY SMTI.

Instance: n men and n women, preference list of women for each man, preference list of men for each woman, and integer $K \in \mathbb{Z}^+$.

Question: Does the given instance admit a stable matching M with $|M| \geqslant K$ (resp. $|M| \leqslant K$)?

Name: MINIMUM (resp. EXACT) MAXIMAL MATCHING.
Instance: Graph $G=(V,E)$ and integer $K \in \mathbb{Z}^+$.
Question: Does G have a maximal matching M with $|M| \leqslant K$ (resp. $|M|=K$)?

MINIMUM MAXIMAL MATCHING is NP-complete,[7] even for subdivision graphs[8] [7].

2.1. Maximum cardinality stable matchings

We begin by proving that MAX CARDINALITY SMTI is hard when the ties are on one side only. The transformation begins from EXACT MAXIMAL MATCHING, the NP-completeness of which clearly follows from the corresponding result for MINIMUM MAXIMAL MATCHING.

Lemma 1. MAX CARDINALITY SMTI *is NP-complete, even if the ties occur on one side only.*

Proof. Clearly MAX CARDINALITY SMTI is in NP. To show NP-hardness, we transform from EXACT MAXIMAL MATCHING for subdivision graphs. Let $G=(V,E)$ and $K \in \mathbb{Z}^+$ be an instance of this problem. Then G is the subdivision graph of some graph $G'=(V',E')$, so that $V=V' \cup E'$ and

$$E = \{\{e,v\}\colon e \in E' \wedge v \in V' \wedge v \text{ is incident to } e \text{ in } G'\}.$$

Also G has a bipartition (U,W), where $U=E'$ and $W=V'$. Thus every vertex in U has degree 2 in G. Without loss of generality, we may assume that G' is connected and is not a forest, so that $|E'| \geqslant |V'|$, i.e., $|U| \geqslant |W|$. Again without loss of generality, we may assume that $|U|=|W|$. For if $|U|=|W|+r$ for some $r>0$, then we may add r vertices $a_1,\dots,a_r$ to U, and $2r$ vertices $b_1,\dots,b_r$, $c_1,\dots,c_r$ to W, where a_i is adjacent to b_i and c_i for each i $(1 \leqslant i \leqslant r)$. Clearly, every vertex in the new set U has degree 2 in the new graph, and G has a maximal matching of size K if and only if the transformed graph has a maximal matching of size $K+r$. Finally, without loss of generality, we may assume that $K \leqslant n$, where $n=|U|=|W|$.

Let $U=\{m_1,m_2,\dots,m_n\}$ and $W=\{w_1,w_2,\dots,w_n\}$. We construct an instance I of MAX CARDINALITY SMTI as follows: let $U \cup U' \cup X$ be the set of men, and let $W \cup Y \cup Z$ be the set of women, where $U'=\{m'_1,m'_2,\dots,m'_n\}$, $X=\{x_1,x_2,\dots,x_{n-K}\}$, $Y=\{y_1,y_2,\dots,y_n\}$, and $Z=\{z_1,z_2,\dots,z_{n-K}\}$. Assume that j_i and k_i are two sequences such that $j_i<k_i$, $\{m_i,w_{j_i}\} \in E$ and $\{m_i,w_{k_i}\} \in E$ $(1 \leqslant i \leqslant n)$. For any w_j $(1 \leqslant j \leqslant n)$, let M_j contain the men m_i such that $\{m_i,w_j\} \in E$, and let M'_j contain the men m'_i such that

[7] In fact, Horton and Kilakos proved that MINIMUM EDGE DOMINATING SET is NP-complete for this class of graphs. The MINIMUM EDGE DOMINATING SET problem is to determine, given a graph $G=(V,E)$ and an integer $K \in \mathbb{Z}^+$, whether G contains an *edge dominating set* of size at most K. A set of edges S is an edge dominating set in G if every edge in $E \backslash S$ is adjacent to some edge in S. It is known that MINIMUM MAXIMAL MATCHING and MINIMUM EDGE DOMINATING SET are polynomially equivalent; indeed the size of a minimum maximal matching of a given graph G is equal to the size of a minimum edge dominating set of G [23].

[8] A subdivision graph G is a graph obtained from another graph G' by replacing every edge in G' by a path of length 2 in G.

$\{m_i, w_j\} \in E$ and $j = k_i$. Create a preference list for each person as follows:

$$
\begin{array}{ll}
m_i : y_i \; w_{j_i} \; w_{k_i} \text{ [women in } Z] & (1 \leqslant i \leqslant n) \\
m_i' : y_i \; w_{k_i} & (1 \leqslant i \leqslant n) \\
x_i : \text{[women in } W] & (1 \leqslant i \leqslant n-K) \\
w_j : (\text{men in } M_j \cup M_j') \; (x_1 \ldots x_{n-K}) & (1 \leqslant j \leqslant n) \\
y_j : (m_j \; m_j') & (1 \leqslant j \leqslant n) \\
z_j : (m_1 \ldots m_n) & (1 \leqslant j \leqslant n-K)
\end{array}
$$

In a preference list throughout this paper, persons within square brackets are listed in arbitrary strict order at the point where the symbol appears. Clearly the ties occur in the women's preference lists only. To complete the construction of the instance, we set the target value to be $K' = 3n - K$. Clearly, the maximum size of stable matching for this instance is at most K'. We claim that G has a maximal matching of size exactly K if and only if the stable marriage instance admits a stable matching of size K'.

For, suppose that G has a maximal matching M, where $|M| = K$. We construct a matching M' in I as follows. For each edge $\{m_i, w_j\}$ in M, if $j = j_i$, then we add (m_i, w_{j_i}) and (m_i', y_i) to M'.[9] If $j = k_i$, then we add (m_i', w_{k_i}) and (m_i, y_i) to M'. There remain $2(n-K)$ men of the form m_{p_i}, m_{p_i}' $(1 \leqslant i \leqslant n-K)$ who are as yet unmatched. Add (m_{p_i}, z_i) and (m_{p_i}', y_{p_i}) to M' $(1 \leqslant i \leqslant n-K)$. Similarly there remain $n-K$ women of the form w_{q_i} $(1 \leqslant i \leqslant n-K)$ who are as yet unmatched. Add (x_i, w_{q_i}) to M' $(1 \leqslant i \leqslant n-K)$. Clearly M' is a matching of size $2K + 2(n-K) + (n-K) = K'$.

It is straightforward to verify that no person in $X \cup Y \cup Z$, can be involved in a blocking pair of M'. Similarly, neither can any man in U', since y_i's list is a single tie $(1 \leqslant i \leqslant n)$. Also, no unmatched pair (m_i, w_j) blocks M'. For if this occurs, then $(m_i, z_k) \in M'$ for some $z_k \in Z$, and $(x_l, w_j) \in M'$ for some $x_l \in X$. Thus no edge of M is incident to m_i or w_j in G. Hence $M \cup \{\{m_i, w_j\}\}$ is a matching in G, contradicting the maximality of M. Thus M' is stable.

Conversely, suppose that M' is a stable matching for I, where $|M'| = K'$. Then everybody has a partner in M'. For each i $(1 \leqslant i \leqslant n)$, at most one of m_i and m_i' is matched in M' to a woman in W, for otherwise y_i is unmatched, a contradiction. Thus,

$$M = \{\{m_i, w_j\} \in E \colon 1 \leqslant i, j \leqslant n \wedge ((m_i, w_j) \in M' \vee (m_i', w_j) \in M')\}$$

is a matching in G. There are exactly $n-K$ men m_{r_i} $(1 \leqslant i \leqslant n-K)$ who have a partner from Z in M'. Since $p_{M'}(m_{r_i}') = y_{r_i}$ $(1 \leqslant i \leqslant n-K)$, then $|M| = K$.

To complete the proof, it remains to show that M is maximal. For, suppose not. Then there is some edge $\{m_i, w_j\}$ in G such that no edge of M is incident to either m_i or w_j. Thus $(m_i, z_k) \in M'$ for some $z_k \in Z$, and $(x_l, w_j) \in M'$ for some $x_l \in X$. But then (m_i, w_j) blocks M', for m_i strictly prefers w_j to z_k, and w_j strictly prefers m_i to x_l. This contradiction to the stability of M' implies that M is indeed maximal. $\square$

[9] Note that, in this paper, we use (m, w) to denote a (man,woman) pair in a stable marriage instance, and $\{m, w\}$ to denote an edge connecting vertices m and w in a graph.

Thus, MAX CARDINALITY SMTI is NP-complete if each man's preference list contains no ties, and each woman's preference list comprises either one tie or two ties. As a by-product, the proof establishes NP-completeness for the question of whether a *complete* stable matching exists in the constructed instance of SMTI (this fact will be used henceforth). We now demonstrate NP-completeness for MAX CARDINALITY SMTI in the case that the ties are at the tails of the lists and on one side only, there is at most one tie per list, and each tie is of length 2. Our exposition is made simpler if we transform from MAX CARDINALITY SMTI when restricted to the case that each man's preference list contains no ties, and each woman's preference list comprises two ties (a 'tie' can be of length 1 for this purpose). To see that the problem remains NP-complete for this restriction, consider the instance of MAX CARDINALITY SMTI as constructed in the proof of Lemma 1. Clearly, the preference list of each woman in W comprises two ties, and the preference list of each woman in $Y \cup Z$ comprises one tie. If e_j is any woman in $Y \cup Z$, then we append a new man a_j to her list. Create, in addition, a new man b_j and two new women c_j, d_j. The preference lists of the new persons are as follows:

$$\begin{array}{llll} a_j : c_j \ d_j \ e_j & & c_j : a_j \ b_j \\ b_j : d_j \ c_j & & d_j : b_j \ a_j \end{array}$$

Clearly $(a_j, c_j) \in M$ and $(b_j, d_j) \in M$, for any stable matching M in the transformed instance. In addition, every woman's preference list in the transformed instance comprises two ties (where a tie can be of length 1 in this case).

Theorem 2. MAX CARDINALITY SMTI *is* NP-*complete, even if the ties are at the tails of the lists and on one side only, there is at most one tie per list, and each tie is of length* 2.

Proof. Membership in NP was established in Lemma 1. To show NP-hardness, we transform from the restricted version of MAX CARDINALITY SMTI as discussed above, in which each man's preference list contains no ties, each woman's preference list comprises two ties (where a tie can be of length 1), and the target value is equal to the number of men. Let I be an instance of this problem, in which $U = \{m_1, m_2, \ldots, m_n\}$ is the set of men, and $W = \{w_1, w_2, \ldots, w_n\}$ is the set of women. For each woman $w_j \in W$, let M_j^h (resp. M_j^t) be the set of men tied at the head (resp. tail) of w_j's list. Assume that

$$M_j^h = \{m_{k_{j,1}}, m_{k_{j,2}}, \ldots, m_{k_{j,h_j}}\} \quad \text{and} \quad M_j^t = \{m_{l_{j,1}}, m_{l_{j,2}}, \ldots, m_{l_{j,t_j}}\}$$

for some $h_j > 0$ and $t_j > 0$. We form an instance I' of MAX CARDINALITY SMTI as follows. Let $U \cup (\bigcup_{i=1}^{i=n} X_i)$ be the set of men in I', and let $(\bigcup_{j=1}^{j=n} W_j) \cup Y$ be the set of women in I', where

$$W_j = \{w_{j,r}\colon 1 \leqslant r \leqslant h_j + t_j\} \quad (1 \leqslant j \leqslant n),$$

$$X_i = \{x_{i,r}\colon 1 \leqslant r \leqslant h_i + t_i\} \quad (1 \leqslant i \leqslant n)$$

and

$$Y = \{y_j\colon 1 \leqslant j \leqslant n\}.$$

For each j $(1 \leqslant j \leqslant n)$, let $W_j^t = \bigcup_{r=h_j+1}^{r=h_j+t_j} \{w_{j,r}\}$. We form the preference lists of the persons in I' as follows. Each man in U initially has the same preference list in I' as in I. Let m_i $(1 \leqslant i \leqslant n)$ be given, and let w_j be any woman who appears in m_i's list in I. If $m_i \in M_j^h$, then $m_i = m_{k_{j,a}}$ for some a $(1 \leqslant a \leqslant h_j)$; we replace w_j by the women in $\{w_{j,a}\} \cup W_j^t$ in any strict order in m_i's preference list in I'. Otherwise, $m_i \in M_j^t$, and $m_i = m_{l_{j,b}}$ for some b $(1 \leqslant b \leqslant t_j)$; we replace w_j by $w_{j,b+h_j}$ in m_i's preference list in I'. The other preference lists in I' are as follows:

$$\begin{array}{rll} x_{i,r} : & (w_{i,r} \;\; y_i) & (1 \leqslant i \leqslant n, 1 \leqslant r \leqslant h_i + t_i) \\ w_{j,r} : & x_{j,r} \;\; m_{k_{j,r}} & (1 \leqslant j \leqslant n, 1 \leqslant r \leqslant h_j) \\ w_{j,r+h_j} : & x_{j,r+h_j} \text{ [men in } M_j^h\text{] } m_{l_{j,r}} & (1 \leqslant j \leqslant n, 1 \leqslant r \leqslant t_j) \\ y_j : & \text{[men in } X_j\text{]} & (1 \leqslant j \leqslant n) \end{array}$$

Clearly the ties occur in the men's preference lists only, any tie forms the whole of the list in which it appears, and each tie is of length 2. We claim that I has a stable matching in which everybody is matched if and only if I' does (implicitly we set the target value in I' to be the number of men in I').

For, suppose that I has such a matching M. Let m_i $(1 \leqslant i \leqslant n)$ be given, and let $w_j = p_M(m_i)$. If $m_i \in M_j^h$, then $m_i = m_{k_{j,a}}$ for some a $(1 \leqslant a \leqslant h_j)$. If $m_i \in M_j^t$, then $m_i = m_{l_{j,b}}$ for some b $(1 \leqslant b \leqslant t_j)$; let $a = b + h_j$. In both cases, add the pairs $(m_i, w_{j,a})$, $(x_{j,r}, w_{j,r})$ (for $1 \leqslant r \leqslant h_j + t_j$, $r \neq a$), and $(x_{j,a}, y_j)$ to M'. Clearly M' is a complete matching in I'.

It is straightforward to verify that no man in X_i $(1 \leqslant i \leqslant n)$, and consequently no woman in Y, can be involved in a blocking pair of M' in I'. Now suppose that $(m_i, w_{j,a})$ blocks M' in I'. Then $a > h_j$ and $m_i \in M_j^h$. Let $m_p = p_{M'}(w_{j,a})$; then $m_p = m_{l_{j,b}}$, where $b = a - h_j$. Clearly $(m_i, w_{j,a}) \notin M'$, and also $(m_i, w_{j,r}) \notin M'$ (for $1 \leqslant r \leqslant h_j + t_j$, $r \neq a$), since $(x_{j,r}, w_{j,r}) \in M'$. Thus $p_{M'}(m_i) \notin W_j$, so that in I, m_i strictly prefers w_j to $p_M(m_i)$. Also, in I, w_j strictly prefers m_i to m_p. Hence (m_i, w_j) blocks M in I, a contradiction. Thus M' is stable in I'.

Conversely, suppose that M' is a stable matching in I' in which everybody is matched. Let j $(1 \leqslant j \leqslant n)$ be given. Then $p_{M'}(y_j) = x_{j,a}$ for some a $(1 \leqslant a \leqslant h_j + t_j)$, and hence $p_{M'}(w_{j,a}) = m_i$, for some $m_i \in U$. Since $p_{M'}(x_{j,r}) = w_{j,r}$ (for $1 \leqslant r \leqslant h_j + t_j$, $r \neq a$), then $M' \cap (U \times W_j) = \{(m_i, w_{j,a})\}$. Let m_i be the partner of w_j in M. Clearly M is a complete matching in I.

Suppose that (m_i, w_j) blocks M in I. Let $m_p = p_M(w_j)$. Then in I, w_j strictly prefers m_i to m_p, so that $m_i \in M_j^h$ and $m_p \in M_j^t$. Thus $m_p = m_{l_{j,b}}$ for some b $(1 \leqslant b \leqslant t_j)$, so that $w_{j,a} = p_{M'}(m_p)$, where $a = b + h_j$. Now in I', $w_{j,a}$ strictly prefers m_i to m_p. Also in I', m_i strictly prefers $w_{j,a}$ to $p_{M'}(m_i)$ (since $p_M(m_i) \neq w_j$ implies that $p_{M'}(m_i) \notin W_j$). Thus $(m_i, w_{j,a})$ blocks M' in I', a contradiction. Hence M is stable in I. $\square$

2.2. Minimum cardinality stable matchings

It is also possible to establish the NP-completeness of MIN CARDINALITY SMTI, in the case that ties are at the tails of lists and on one side only, there is at most one tie per list, and each tie is of length 2. We begin by demonstrating NP-completeness for the first two restrictions holding simultaneously, using a similar transformation to the one in Lemma 1.

Lemma 3. MIN CARDINALITY SMTI *is* NP-*complete*, *even if the ties are at the tails of lists and on one side only*, *and there is at most one tie per list.*

Proof. Clearly MIN CARDINALITY SMTI is in NP. To show NP-hardness, we transform from MINIMUM MAXIMAL MATCHING for subdivision graphs. Let $G=(V,E)$ and $K\in\mathbb{Z}^+$ be an instance of this problem. Then G is the subdivision graph of some graph $G'=(V',E')$, so that $V=V'\cup E'$ and

$$E=\{\{e,v\}\colon\ e\in E'\wedge v\in V'\wedge\ v \text{ is incident to } e \text{ in } G'\}.$$

Also G has a bipartition (U,W), where $U=E'$ and $W=V'$. Thus every vertex in U has degree 2 in G. As in Lemma 1, we may assume, without loss of generality, that $|U|=|W|=n$, and $K\leqslant n$.

Let $U=\{m_1,m_2,\ldots,m_n\}$ and $W=\{w_1,w_2,\ldots,w_n\}$. We construct an instance I of MIN CARDINALITY SMTI as follows: let $U\cup U'$ be the set of men, and let $W\cup Y$ be the set of women, where $U'=\{m_1',m_2',\ldots,m_n'\}$ and $Y=\{y_1,y_2,\ldots,y_n\}$. Assume that j_i and k_i are two sequences such that $j_i<k_i$, $\{m_i,w_{j_i}\}\in E$ and $\{m_i,w_{k_i}\}\in E$ $(1\leqslant i\leqslant n)$. For any w_j $(1\leqslant j\leqslant n)$, let M_j (resp. M_j') contain the men m_i (resp. m_i') such that $\{m_i,w_j\}\in E$. Create a preference list for each person as follows:

$$\begin{array}{ll} m_i:\ y_i\ w_{j_i}\ w_{k_i} & (1\leqslant i\leqslant n)\\ m_i':\ y_i\ w_{k_i}\ w_{j_i} & (1\leqslant i\leqslant n)\\ w_j:\ (\text{members of } M_j \text{ and } M_j') & (1\leqslant j\leqslant n)\\ y_j:\ (m_j\ m_j') & (1\leqslant j\leqslant n) \end{array}$$

Clearly, the ties occur in the women's preference lists only, and any tie forms the whole of the list in which it appears. To complete the construction of I, we set the target value to be $K'=n+K$. We claim that G has a maximal matching of size at most K if and only if I admits a stable matching of size at most K'.

For, suppose that G has a maximal matching M, where $|M|=k\leqslant K$. We construct a matching M' in I as follows. For each edge $\{m_i,w_j\}$ in M, if $j=j_i$, then we add (m_i,w_{j_i}) and (m_i',y_i) to M'. If $j=k_i$, then we add (m_i',w_{k_i}) and (m_i,y_i) to M'. There remain $n-k$ men of the form m_{p_i}' $(1\leqslant i\leqslant n-k)$ who are as yet unmatched. Add (m_{p_i}',y_{p_i}) to M' $(1\leqslant i\leqslant n-k)$. Clearly M' is a matching of size $2k+(n-k)\leqslant K'$. It remains to show that M' is stable.

No woman in Y can be involved in a blocking pair of M', since every such woman is matched in M'. Additionally, no unmatched pair (m_i, w_j) blocks M'. For if this occurs, then each of m_i and w_j is unmatched in M', and thus no edge of M is incident to either vertex in G. Hence $M \cup \{\{m_i, w_j\}\}$ is a matching in G, contradicting the maximality of M. Finally, no unmatched pair (m'_i, w_j) blocks M', for either $(m'_i, y_i) \in M'$ or $(m'_i, w_{k_i}) \in M'$ holds. Thus M' is stable.

Conversely, suppose that M' is a stable matching for I, where $|M'| = k' \leqslant K'$. For each i $(1 \leqslant i \leqslant n)$, y_i is matched in M', for otherwise (m_i, y_i) blocks M', a contradiction. Thus at most one of m_i and m'_i is matched in M' to a woman in W. Hence

$$M = \{\{m_i, w_j\} \in E\colon\ 1 \leqslant i, j \leqslant n \wedge ((m_i, w_j) \in M' \vee (m'_i, w_j) \in M')\}$$

is a matching in G, and $|M| = |M'| - n = k' - n \leqslant K$.

To complete the proof, it remains to show that M is maximal. For, suppose not. Then there is some edge $\{m_i, w_j\}$ in G such that no edge of M is incident to either m_i or w_j. Thus m_i^* and w_j are both unmatched in M', where $m_i^* \in \{m_i, m'_i\}$. Since each of m_i^*, w_j finds the other acceptable, then (m_i^*, w_j) blocks M'. This contradiction to the stability of M' implies that M is indeed maximal. □

By transforming from the NP-complete problem MINIMUM MAXIMAL MATCHING for the subdivision graphs of graphs of maximum degree 3 [7], it may be verified that the length of any tie in the instance of MIN CARDINALITY SMTI constructed in Lemma 3 is either 2, 4 or 6. We now show how to eliminate the ties of length greater than 2.

Theorem 4. MIN CARDINALITY SMTI *is* NP-*complete*, *even if the ties occur at the tails of lists and on one side only*, *there is at most one tie per list*, *and each tie is of length* 2.

Proof. Membership in NP was established in Lemma 3. To show NP-hardness, we transform from the restricted version of MIN CARDINALITY SMTI as discussed above, in which each man's preference list contains no ties, and each woman's preference list comprises a tie of length either 2, 4 or 6. Let I be an instance of this problem, in which $U = \{m_1, m_2, \ldots, m_n\}$ is the set of men, $W = \{w_1, w_2, \ldots, w_n\}$ is the set of women, and $K \in \mathbb{Z}^+$ is the target value. Without loss of generality, suppose that $W' = \{w_1, w_2, \ldots, w_t\}$ is the set of women, each of whom has a preference list comprising a tie of length 6. Let $W'' = W \backslash W'$. For each woman $w_j \in W'$, let M_j be the set of men tied in w_j's list. Assume that

$$M_j = \{m_{k_{j,1}}, m_{k_{j,2}}, \ldots, m_{k_{j,6}}\}.$$

We now form an instance I' of MIN CARDINALITY SMTI. Let $U \cup (\bigcup_{j=1}^{j=t} (P_j \cup R_j))$ be the set of men in I', and let $W'' \cup (\bigcup_{j=1}^{j=t} (W_j \cup Q_j \cup \{s_j\}))$ be the set of women in I', where $P_j = \{p_{j,l}\colon\ 1 \leqslant l \leqslant 6\}$, $Q_j = \{q_{j,l}\colon\ 1 \leqslant l \leqslant 5\}$, $R_j = \{r_{j,l}\colon\ 1 \leqslant l \leqslant 5\}$, and $W_j = \{w_{j,l} : 1 \leqslant l \leqslant 6\}$. We form the preference lists of the persons in I' as follows. Each woman in W'' has the same preference list in I' as in I. Each man in U *initially* has the same preference list in I' as in I. Now let m_i $(1 \leqslant i \leqslant n)$ be given, and suppose that some woman $w_j \in W'$ appears in m_i's list in I. Replace w_j by all the women in W_j in arbitrary (strict) order in m_i's preference list in I'. The other preference lists in I' are as follows, for each j $(1 \leqslant j \leqslant t)$:

$$
\begin{array}{rll}
p_{j,l} : & q_{j,l}\ w_{j,l} & (1 \leqslant l \leqslant 5) \\
p_{j,6} : & [\text{women in } Q_j]\ w_{j,6} & \\
q_{j,l} : & r_{j,l}\ (p_{j,l}\ \ p_{j,6}) & (1 \leqslant l \leqslant 5) \\
r_{j,l} : & s_j\ q_{j,l} & (1 \leqslant l \leqslant 5) \\
s_j : & (r_{j,1}\ r_{j,2}\ r_{j,3}\ r_{j,4}\ r_{j,5}) & \\
w_{j,l} : & p_{j,l}\ m_{k_{j,l}}\ [\text{men in } M_j \backslash \{m_{k_{j,l}}\}] & (1 \leqslant l \leqslant 6)
\end{array}
$$

Clearly, the ties occur in the women's preference lists only, any tie is at the tail of some woman's list, there is at most one tie per list, and each tie has length 2, 4 or 5 (we discuss in due course how to eliminate ties of length 4 and 5). Set $K' = K + 11t$. We claim that I has a stable matching of size at most K if and only if I' has a stable matching of size at most K'.

For, suppose that M is a stable matching in I, where $|M| = k \leqslant K$. We construct a matching M' in I' as follows. Each woman in W'' is unmatched in M' if she is unmatched in M, otherwise she is given the same partner in M' as in M. Now let $w_j \in W'$. If w_j is unmatched in M, then add the pairs $(p_{j,l}, w_{j,l})$ $(1 \leqslant l \leqslant 5)$, $(p_{j,6}, q_{j,5})$, $(r_{j,l}, q_{j,l})$ $(1 \leqslant l \leqslant 4)$ and $(r_{j,5}, s_j)$ to M'. Now suppose that w_j is matched in M, to m_i say. Then $m_i = m_{k_{j,a}}$ for some a $(1 \leqslant a \leqslant 6)$. Add $(m_i, w_{j,a})$ to M'. For each l $(1 \leqslant l \neq a \leqslant 6)$, add the pair $(p_{j,l}, w_{j,l})$ to M'. Add the pair $(p_{j,a}, q_{j,b})$ to M', where $b = \min\{a, 5\}$. For each l $(1 \leqslant l \neq b \leqslant 5)$, add the pairs $(r_{j,l}, q_{j,l})$ to M'. Finally, add the pair $(r_{j,b}, s_j)$ to M'. Clearly M' is a matching in I', and $|M'| = k + 11t \leqslant K'$.

It may be verified that, for each j $(1 \leqslant j \leqslant t)$, every person $z \in P_j \cup Q_j \cup R_j \cup \{s_j\}$ is matched in M'. Hence, by inspection of z's preference list in I', we may deduce that z cannot be involved in a blocking pair of M' in I'. Clearly, if (m_i, w_j) blocks M' in I', where $w_j \in W''$, then (m_i, w_j) blocks M in I, a contradiction. Now suppose that $(m_i, w_{j,l})$ blocks M' in I', where $1 \leqslant l \leqslant 6$. Then $w_{j,l}$ is unmatched in M', so that $l = 6$ and w_j is unmatched in M. Since m_i is not matched to any member of W_j in M', then (m_i, w_j) blocks M in I, a contradiction. Hence M' is stable in I'.

Conversely, suppose that M' is a stable matching in I', where $M' = k' \leqslant K'$. It is easy to see that, for every j $(1 \leqslant j \leqslant t)$, each member of $P_j \cup Q_j \cup R_j \cup \{s_j\}$ must be matched in M'. We construct a matching M in I as follows. Each woman in W'' is unmatched in M if she is unmatched in M', otherwise she is given the same partner in M as in M'. Now let $w_j \in W'$. If every member of W_j is matched in M', then some woman $w_{j,l}$ $(1 \leqslant l \leqslant 6)$ has a man $m_i \in M_j$ as her partner in M': let m_i be the partner of w_j in M. Otherwise, let w_j be unmatched in M. Clearly M is a matching in I, and $|M| = k' - 11t \leqslant K$.

Now suppose that (m_i, w_j) blocks M in I. If $w_j \in W''$, then (m_i, w_j) blocks M' in I', a contradiction, so suppose that $w_j \in W'$. Since w_j's list in I comprises a single tie, then w_j is unmatched in M. Thus some $w_{j,l}$ $(1 \leqslant l \leqslant 6)$ is unmatched in M'. But then m_i is not matched to any member of W_j in M'. Thus $(m_i, w_{j,l})$ blocks M' in I', a contradiction. Hence M is stable in I.

Clearly I' contains ties of length 4 and 5. However, the construction of I' does not rely on any special properties of the ties of length 6, and thus a similar reduction can be used in order to replace the ties of length 5 in I' by ties of length 2 and 4 occupying different women's lists. (This method is applicable, since a woman in I' who has a tie of length 5 in her list does not find any other men acceptable apart from the five tied men.) Similarly, a further iteration will replace the ties of length 4 by ties of length 2 and 3 occupying different women's lists, and a final iteration will give an instance with ties of length 2 only. Since the additional ties of length 2 that are generated at each stage appear only at the tails of women's lists, and there is at most one tie per list, then it is clear that we end up with an instance which satisfies the restrictions of the statement of the theorem, and furthermore, this instance can be constructed in polynomial time from I. □

2.3. Approximability results

It turns out that, for a given instance I of HRT, each of the problems of finding stable matching of minimum or maximum size is approximable within a factor of 2. This follows immediately by choosing an arbitrary stable matching, and from the following result:

Theorem 5. *For an arbitrary instance of HRT, the size of the largest stable matching is at most twice the size of the smallest.*

Proof. Let I be an instance of HRT, and let M be a stable matching in I of maximum cardinality. Suppose that M' is any stable matching in I, and suppose that $|M'| < |M|/2$. Then there is a set $r_1, \ldots, r_p$ of residents in I such that, for each j $(1 \leqslant j \leqslant p)$, r_j is unmatched in M' but matched in M, where $p > |M'|$. As r_j is matched in M $(1 \leqslant j \leqslant p)$, then there are p hospitals $h_{i_1}, \ldots, h_{i_p}$ (not necessarily distinct), such that r_j is assigned to h_{i_j} in M $(1 \leqslant j \leqslant p)$. Let $k = |\{h_{i_1}, \ldots, h_{i_p}\}|$ and let t be the sum of the capacities of the k distinct hospitals. Each of the hospitals must be fully subscribed in M', for otherwise some hospital h_{i_j} $(1 \leqslant j \leqslant p)$ has a vacancy in M', so that (r_j, h_{i_j}) blocks M', a contradiction. Thus $|M'| \geqslant t$. But $t \geqslant p$, so $|M'| \geqslant p$, a contradiction. □

2.4. Stable matchings interpolate

Given that the stable matchings in an instance of SMTI can be of different sizes, the question arises as to whether there exist stable matchings of all sizes between the minimum and maximum, i.e. whether stability in SMTI is an *interpolating invariant*. We remark that the answer is in the affirmative; furthermore, given an SMTI instance and stable matchings of sizes i, j, we may find in polynomial time a stable matching of size k, for each $i < k < j$. The proof is omitted here; see [17] for further details.

3. Testing whether a (man,woman) pair is stable in SMT

In this section, we show that the problem of determining whether a given (man, woman) pair (m, w) is stable in an instance I of SMT (i.e. whether there exists a stable matching M in I such that $(m, w) \in M$) is NP-complete. Note that, when ties are absent, this problem is polynomial-time solvable [5].

Theorem 6. *For a given instance of SMT and a given* (*man,woman*) *pair* (m, w), *the problem of determining whether* (m, w) *is a stable pair is* NP-*complete*, *even if the ties occur at the tails of lists and on one side only*, *there is at most one tie per list*, *and each tie is of length* 2.

Proof. Clearly this problem is in NP. To show NP-hardness, we transform from the NP-complete problem MAX CARDINALITY SMTI when ties are at the tails of lists and on the women's side only, there is at most one tie per list, and each tie is of length 2: let $U = \{m_1, m_2, \ldots, m_n\}$ be the set of men and let $W = \{w_1, w_2, \ldots, w_n\}$ be the set of women in a given instance I of this problem. We may assume that the given target value in I is equal to n. Let P_i (resp. Q_i) denote the preference list of man m_i (resp. woman w_i) for $1 \leqslant i \leqslant n$ (as mentioned in Section 1, without loss of generality, we may assume that for any man $m_i \in U$ and for any woman $w_j \in W$, $w_j \in P_i$ if and only if $m_i \in Q_j$). We construct an instance I' of SMT as follows: let $\{m_0\} \cup U$ be the set of men, and let $\{w_0\} \cup W$ be the set of women. Create a preference list in I' for each person as follows:

$$\begin{array}{ll} m_0 : w_0 \ \ldots & w_0 : \ldots \ m_0 \\ m_i : P_i \ w_0 \ \ldots \ (1 \leqslant i \leqslant n) & w_i : \ldots \ Q_i \ (1 \leqslant i \leqslant n) \end{array}$$

The symbol '…' in a person p's preference list denotes all people in I' of the opposite sex to p who are not explicitly listed elsewhere in p's preference list, listed in arbitrary strict order at the point where the symbol appears. Clearly, the ties in I' occur at the tails of lists and on the women's side only, there is at most one tie per list, each tie is of length 2, and all lists are complete. We claim that I has a complete stable matching if and only if (m_0, w_0) is a stable pair in I'.

For, suppose that M is a complete stable matching for I. Let $M' = M \cup \{(m_0, w_0)\}$. We claim that M' is stable in I'. For, if some pair (m, w) blocks M' then either $m = m_0$ or $w = w_0$, as M is stable in I. But $m \neq m_0$, since m_0 has his first-choice partner in M'. Hence $w = w_0$, so that $m = m_i$ for some i $(1 \leqslant i \leqslant n)$, and m_i strictly prefers w_0 to $p_M(m_i)$. But $p_M(m_i)$ appears on the list P_i, a contradiction.

Conversely, suppose that M' is a stable matching for I', such that $(m_0, w_0) \in M'$. Clearly M' is a complete matching for I'. Also, for any i $(1 \leqslant i \leqslant n)$, we claim that

$p_{M'}(m_i)$ appears on the list P_i. For if not, then m_i strictly prefers w_0 to $p_{M'}(m_i)$. Since w_0 strictly prefers m_i to $m_0 = p_{M'}(w_0)$, then (m_i, w_0) blocks M', a contradiction. Hence, for any i $(1 \leqslant i \leqslant n)$, $p_{M'}(w_i)$ appears on the list Q_i. Thus $M = M' \backslash \{(m_0, w_0)\}$ is a complete matching in I. Clearly M is stable in I. □

It is straightforward to alter the above transformation in order to prove that determining whether a given person has a stable partner (i.e. has a partner in some stable matching) in a given SMTI instance is also NP-complete.

4. Egalitarian and minimum regret stable matchings in SMT

In this section, we prove that each of the problems of finding an egalitarian and a minimum regret stable matching for a given instance of SMT is NP-hard and difficult to approximate.

Given an SMT instance I, denote by $w(I)$ the weight of an egalitarian stable matching in I, and denote by $r(I)$ the regret of a minimum regret stable matching in I. Let EGALITARIAN SMT OPT (resp. MINIMUM REGRET SMT OPT) denote the optimisation problem which, given an instance I of SMT as its input, requires to output $w(I)$ (resp. $r(I)$) as a solution. We begin by proving that EGALITARIAN SMT OPT is NP-hard and hard to approximate.

Theorem 7. EGALITARIAN SMT OPT *is not approximable within $N^{1-\varepsilon}$, for any $\varepsilon > 0$, unless* P = NP, *where N is the number of men in a given instance of the problem, even if the ties are on one side only, there is at most one tie per list, and each tie is of length* 2.

Proof. Let $\varepsilon > 0$ be given, and let $c = 3/\varepsilon$. We consider the NP-complete problem MAX CARDINALITY SMTI when ties occur on the women's side only, and each tie is of length 2: let $U = \{m_1, m_2, \dots, m_n\}$ be the set of men and let $W = \{w_1, w_2, \dots, w_n\}$ be the set of women in a given instance I of this problem. We may assume that the given target value in I is equal to n. Let P_i (resp. Q_i) denote the preference list of man m_i (resp. woman w_i) for $1 \leqslant i \leqslant n$. We construct an instance I' of SMT as follows: let $U^0 \cup (\bigcup_{j=1}^{C} U^j)$ be the set of men, and let $W^0 \cup (\bigcup_{j=1}^{C} W^j)$ be the set of women, where $C = n^{c-1}$, $U^0 = \{m_1^0, m_2^0, \dots, m_{n^c}^0\}$, $U^j = \{m_1^j, m_2^j, \dots, m_n^j\}$ $(1 \leqslant j \leqslant C)$, $W^0 = \{w_1^0, w_2^0, \dots, w_{n^c}^0\}$, and $W^j = \{w_1^j, w_2^j, \dots, w_n^j\}$ $(1 \leqslant j \leqslant C)$. Thus I' comprises $2n^c$ men and $2n^c$ women, so that $N = 2n^c$. For each i $(1 \leqslant i \leqslant n)$ and j $(1 \leqslant j \leqslant C)$, let P_i^j denote the preference list that is obtained from P_i by replacing woman w_k in P_i by the corresponding woman w_k^j, for any k $(1 \leqslant k \leqslant n)$. Let us refer to the women in P_i^j as the *proper women* for m_i^j. Define Q_i^j and the *proper men* for w_i^j similarly. Create a preference list in I' for each person as follows:

$$m_i^0 : w_i^0 \ \dots \qquad (1 \leqslant i \leqslant n^c)$$

$$m_i^j : P_i^j \ [\text{women in } W^0] \ \dots \ (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant C)$$

$$w_i^0 : m_i^0 \ldots \qquad (1 \leqslant i \leqslant n^c)$$

$$w_i^j : Q_i^j \text{ [men in } U^0] \ \ldots \ (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant C)$$

Note that the symbol '...' in the above preference lists has a similar meaning to its usage in Theorem 6. Clearly, the only ties featuring in I' occur in the preference lists of women of the form w_i^j, there is at most one tie per list, and each tie is of length 2.

Suppose that M is a stable matching in I, where $|M| = n$. We form a matching M' in I' as follows: for each i $(1 \leqslant i \leqslant n^c)$, add the pair (m_i^0, w_i^0) to M', and for each i $(1 \leqslant i \leqslant n)$, add the pair (m_i^j, w_k^j) to M' $(1 \leqslant j \leqslant C)$, where $(m_i, w_k) \in M$. Clearly M' is stable in I', and it may be verified that

$$w(M') \leqslant 2(n^c + n^{c-1}n^2) \leqslant 2\left(\frac{n^{c+2}}{2}\right)$$

since we may choose $n \geqslant 3$, without loss of generality. Hence $w(I') \leqslant n^{c+2}$.

Now suppose that I does not have a stable matching of cardinality n. Let M' be any stable matching in I'. Then it may be verified that, for each j $(1 \leqslant j \leqslant C)$, there is some i $(1 \leqslant i \leqslant n)$ such that m_i^j cannot be matched in M' to one of his proper women. But in M', m_k^0 and w_k^0 must be partners, for each k $(1 \leqslant k \leqslant n^c)$, and hence $c_{M'}(m_i^j) > n^c$. Similarly, for each j $(1 \leqslant j \leqslant C)$, there is some i $(1 \leqslant i \leqslant n)$ such that w_i^j cannot be matched in M' to one of her proper men, and hence $c_{M'}(w_i^j) > n^c$. Thus $w(M') > 2n^{2c-1}$, so that $w(I') > 2n^{2c-1}$.

Hence the existence of a polynomial-time approximation algorithm for EGALITARIAN SMT OPT whose approximation ratio is as good as $(2n^{2c-1})/n^{c+2} = 2n^{c-3}$ would give a polynomial-time algorithm for determining whether I has a stable matching in which everybody is matched. Finally, $2n^{c-3} = (2/2^{1-3/c})N^{1-3/c} > N^{1-3/c} = N^{1-\varepsilon}$, which concludes the proof. □

Note that in general, the hardness of finding an egalitarian stable matching in no way implies the hardness of finding a minimum regret stable matching: for example, in the case of stable roommates, although the problem of finding an egalitarian stable matching is NP-hard [2], the problem of finding a minimum regret stable matching is polynomial-time solvable [9]. Nevertheless, it turns out that MINIMUM REGRET SMT OPT has similar approximability behaviour to EGALITARIAN SMT OPT. In fact, the transformation of Theorem 7 can be adapted in a straightforward fashion to prove a result analogous to Theorem 7 for MINIMUM REGRET SMT OPT, as we now demonstrate.

Theorem 8. MINIMUM REGRET SMT OPT *is not approximable within* $N^{1-\varepsilon}$, *for any* $\varepsilon > 0$, *unless* P = NP, *where* N *is the number of men in a given instance of the problem, even if the ties are on one side only, there is at most one tie per list, and each tie is of length* 2.

Proof. Let $\varepsilon > 0$ be given. We consider the SMT instance I' constructed in Theorem 7 from the MAX CARDINALITY SMTI instance I. In order to prove the inapproximability result for MINIMUM REGRET SMT OPT, it suffices to take $c = 2/\varepsilon$ and $C = 1$ in the construction of I'. Hence there are $n + n^c$ men and $n + n^c$ women in I', so that $N = n + n^c$.

Suppose that M is a stable matching in I, where $|M| = n$. We construct a matching M' in I' as follows: for each i $(1 \leqslant i \leqslant n^c)$, add the pair (m_i^0, w_i^0) to M', and for each i $(1 \leqslant i \leqslant n)$, add the pair (m_i^1, w_k^1) to M', where $(m_i, w_k) \in M$. Clearly M' is stable in I', and $r(M') \leqslant n$. Hence $r(I') \leqslant n$.

Now suppose that I does not have a stable matching of cardinality n. Let M' be any stable matching in I'. Then as in Theorem 7, there is some i $(1 \leqslant i \leqslant n)$ such that $c_{M'}(m_i^1) > n^c$. Thus $r(M') > n^c$, so that $r(I') > n^c$.

Hence the existence of a polynomial-time approximation algorithm for MINIMUM REGRET SMT OPT whose approximation ratio is as good as $n^c/n = n^{c-1}$ would give a polynomial-time algorithm for determining whether I has a stable matching in which everybody is matched. Finally, $n^{c-1} > n^{(c+1)(1-\varepsilon)} \geqslant (n+n^c)^{(1-\varepsilon)} = N^{1-\varepsilon}$ (without loss of generality, $n \geqslant 2$), which concludes the proof. □

Note that, for a given instance I of HRT and a stable matching M in I, a more suitable definition of $c_M(h)$ for a hospital h might be the sum of the (possibly joint) rankings of each resident designated to h in h's list, divided by the capacity of h. Clearly, each of the inapproximability results of Theorems 7 and 8 carries over to this revised definition, by considering the restriction of HRT in which each hospital has capacity one, and the numbers of posts and residents are equal.

5. Conclusion and open problems

In this paper, we have established the hardness of various problems involving stable matchings in the case where the preference lists of the participants may be incomplete and may involve ties. Among these is the important practical problem of finding a stable matching of maximum cardinality for an HRT instance in which all of the ties are on one side (the hospitals' side), and are at the tails of lists, and there is at most one tie per list (and even if each tie is of length 2).

A number of interesting open problems remain. These include:

- Is there an approximation algorithm for finding a stable matching of maximum cardinality in SMTI (and HRT) with a guarantee better than 2? Perhaps some special cases, say with restrictions on the positions or size of ties, may be more accessible.
- Is the problem of finding a stable matching of maximum size APX-complete?
- Is there a reasonable algorithm to generate all of the stable matchings for a given instance of SMTI (and HRT)? For each of SM and HR, such an algorithm can be derived by exploiting the underlying lattice structure [5], but there appears to be no such useful mathematical structure present in SMTI or HRT: Roth [20] constructs an instance of SMT, comprising three men and three women, which admits no man-optimal or woman-optimal stable matching.

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